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SEMI-SYMMETRIC CONTACT METRIC THREE-MANIFOLDS

By

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Abstract. We study three-dimensional semi-symmetric contact metric manifolds, obtaining several classification results.

1. Introduction

A *semi-symmetric space* is a Riemannian manifold (M, g) such that its curvature tensor R satisfies the condition

$$R(X, Y) \cdot R = 0, \tag{1.1}$$

for all vector fields X, Y on M, where $R(X, Y)$ acts as a derivation on R [14]. Such a space is called "semi-symmetric" since the curvature tensor of (M, g) at a point $p \in M$, R_p, is the same as the curvature tensor of a symmetric space (which may change with the point p). So, locally symmetric spaces are obviously semi-symmetric, but the converse is not true, as it was proved by H. Takagi [15]. Indeed, in any dimension greater than two do there exist examples of semi-symmetric space which are not locally symmetric. We can refer to [6] for a survey.

Even if in general a semi-symmetric space may be not locally symmetric, semi-symmetry implies local symmetry in several cases (see for example [3], [9]). So, given a class of Riemannian manifolds, it is worthwhile to investigate whether in such class semi-symmetry implies local symmetry or not. In this paper, we consider such problem in the class of contact metric manifolds $(M, \eta, g, \varphi, \xi)$.

Semi-symmetric contact metric manifolds have been studied by many authors. In particular, T. Takahashi [16] proved that semi-symmetric Sasakian manifolds have constant sectional curvature 1. In [10] and [11], B.J. Papantoniu and the second author classified semi-symmetric contact metric manifolds, of dimension greater than three, with ξ belonging to the (k, μ)-nullity distribution

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and $R(\xi,\cdot)\xi = -k\varphi^2$, respectively. In dimension three, in [11] it was proved that if M is semi-symmetric and h is ξ-parallel, then either M is flat or is of constant curvature 1. In [10], semi-symmetric contact metric three-manifolds with $\xi \in (k,\mu)$-nullity distribution were studied. On the other hand, D.E. Blair and R. Sharma [2] proved that every locally symmetric contact metric three-manifold has constant curvature 0 or 1.

The following theorems extend the results proved in dimension three in [11], [10] and [2].

THEOREM 1. *Let (M,η,g,φ,ξ) be a semi-symmetric contact metric three-manifold with Ricci curvature $\varrho(\xi,\xi)$ constant along the characteristic flow. Then, M^3 is locally symmetric. In particular, either M^3 is flat or it is Sasakian with constant curvature* 1.

THEOREM 2. *Let (M,η,g,φ,ξ) be a semi-symmetric contact metric three-manifold with non-vanishing vertical sectional curvatures at any point. Then M^3 is Sasakian with constant curvature* 1.

In the Section 2 we shall recall some basic facts about contact metric three-manifolds, in the Section 3 we shall consider semi-symmetric contact metric three-manifolds and we shall prove Theorems 1 and 2.

In the Section 4 we shall prove the following

THEOREM 3. *Let (M,G) be a Riemannian manifold of dimension two. Then the unit tangent sphere bundle T_1M, equipped with its standard contact metric structure, is semi-symmetric if and only if M is flat or has Gaussian curvature* 1.

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2. Preliminaries

A *contact manifold* is a $(2n+1)$-dimensional manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It has an underlying almost contact structure (η,φ,ξ) where ξ is a global vector field (called the *characteristic vector field*) and φ a global tensor of type (1.1) such that

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta\varphi = 0, \quad \varphi^2 = -I + \eta \otimes \xi .$$

A Riemannian metric g can be found such that

$$\eta = g(\xi,\cdot), \quad d\eta = g(\cdot,\varphi\cdot), \quad g(\cdot,\varphi\cdot) = -g(\varphi\cdot,\cdot) .$$

We refer to (M,η,g) or to (M,η,g,ξ,φ) as a contact metric (or Riemannian) manifold.

In what follows, we shall denote by ∇ the Levi Civita connection of M and by R the corresponding Riemannian curvature tensor given by

$$R_{X,Y}=\nabla_{[X,Y]}-[\nabla_X,\nabla_Y].$$

The Ricci tensor of type (0,2), the corresponding endomorphism field and the scalar curvature are respectively indicated by ϱ, Q and r. By $K(\xi,X)$ we denote the *vertical curvature*, that is, the sectional curvature of the plane spanned by ξ and $X\in\ker\eta$. The tensor

$$h=\frac{1}{2}L_\xi\varphi,$$

where L denotes the Lie derivative, is symmetric and satisfies

$$\nabla\xi=-\varphi-\varphi h,\quad \nabla_\xi\varphi=0,\quad h\varphi=-\varphi h,\quad h\xi=0. \tag{2.1}$$

A *K-contact* manifold is a contact metric manifold such that ξ is a Killing vector field with respect to g. Clearly, M is K-contact if and only if $h=0$. If the almost complex structure J on $M\times\mathbb{R}$ defined by

$$J\left(X,f\frac{d}{dt}\right)=\left(\varphi X-f\xi,\eta(X)\frac{d}{dt}\right)$$

is integrable, M is said to be *Sasakian*. Any Sasakian manifold is K-contact and the converse also holds for three-dimensional spaces. It easy to prove that if M is a contact metric three-manifold of constant sectional curvature 1, then M is necessarily Sasakian. We refer to [1] for more information about contact metric manifolds.

Next, let (M,η,g,ξ,φ) be a three-dimensional contact metric manifold. Let U be the open subset of M where $h\neq 0$ and V the open subset of points $m\in M$ such that $h=0$ in a neighborhood of m. Then, $U\cup V$ is an open dense subset of M. For any point $m\in U\cup V$ there exists a local orthonormal basis $\{\xi,e,\varphi e\}$ of smooth eigenvectors of h in a neighborhood of m. On U we put $he=\lambda e$, where λ is a non-vanishing smooth function which we suppose to be positive. From (2.1), we have $h\varphi e=-\lambda\varphi e$. We recall the following:

LEMMA 2.4 ([8]). *On U we have*

$$\nabla_\xi e = -a\varphi e, \qquad \nabla_\xi \varphi e = ae,$$

$$\nabla_e \xi = -(\lambda+1)\varphi e, \qquad \nabla_{\varphi e}\xi = -(\lambda-1)e,$$

$$(2.2) \qquad \nabla_e e = \frac{1}{2\lambda}\{(\varphi e)(\lambda) + A\}\varphi e, \qquad \nabla_{\varphi e}\varphi e = \frac{1}{2\lambda}\{e(\lambda)+B\}e,$$

$$\nabla_e \varphi e = -\frac{1}{2\lambda}\{(\varphi e)(\lambda) + A\}e + (\lambda+1)\xi,$$

$$\nabla_{\varphi e} e = -\frac{1}{2\lambda}\{e(\lambda)+B\}\varphi e + (\lambda-1)\xi,$$

$$(2.3) \qquad \nabla_\xi h = 2ah\varphi + \xi(\lambda)s,$$

where a is a smooth function, $A = \varrho(\xi, e)$, $B = \varrho(\xi, \varphi e)$ and s is the (1,1)-type tensor defined by $s\xi = 0$, $se = e$ and $s\varphi e = -\varphi e$.

Finally, we note that the components of the Ricci operator Q, with respect to $\{\xi, e, \varphi e\}$, are given by (see [12])

$$(2.4) \qquad \begin{cases} Q\xi = 2(1-\lambda^2)\xi + Ae + B\varphi e, \\ Qe = A\xi + \left(\frac{r}{2} - 1 + \lambda^2 + 2a\lambda\right)e + \xi(\lambda)\varphi e, \\ Q\varphi e = B\xi + \xi(\lambda)e + \left(\frac{r}{2} - 1 + \lambda^2 - 2a\lambda\right)\varphi e, \end{cases}$$

from which it follows easily (see also [7])

$$(\nabla_\xi Q)\xi = -4\lambda\xi(\lambda)\xi + \{\xi(A) + aB\}e + \{\xi(B) - aA\}\varphi e,$$

$$(2.5) \qquad \begin{aligned} (\nabla_e Q)e = {} & \{e(A) + (\lambda+1)\xi(\lambda) - \frac{B}{2\lambda}[(\varphi e)(\lambda) + A]\}\xi + \\ & +\{e(\alpha + 2a\lambda) - \frac{\xi(\lambda)}{\lambda}[(\varphi e)(\lambda) + A]\}e + \\ & + \{e\xi(\lambda) + 2a(\varphi e)(\lambda) + (2a - \lambda - 1)A\}\varphi e, \end{aligned}$$

$$\begin{aligned} (\nabla_{\varphi e} Q)\varphi e = {} & \{(\varphi e)(B) + (\lambda-1)\xi(\lambda) - \frac{A}{2\lambda}[e(\lambda)+B]\}\xi + \\ & + \{(\varphi e)\xi\lambda - 2ae(\lambda) + (1 - \lambda - 2a)B\}e + \\ & + \{(\varphi e)(\alpha - 2a\lambda) - \frac{\xi(\lambda)}{\lambda}[e(\lambda)+B]\}\varphi e, \end{aligned}$$

where $\alpha = \frac{r}{2} - 1 + \lambda^2$.

3. Three-dimensional semi-symmetric contact metric manifolds

Let $(M, \eta, g, \varphi, \xi)$ be a contact metric three-manifold. If M is Sasakian and semi-symmetric, then it has constant sectional curvature 1 [16]. So, in what follows we shall assume M is not Sasakian. We now write down the conditions satisfied by a semi-symmetric contact metric three-manifold, proving the following

LEMMA 3.1. *Let $(M, \eta, g, \varphi, \xi)$ be a non-Sasakian contact metric three-manifold. Then M is semi-symmetric if and only if*

$$B(\lambda^2 - 1 + 2a\lambda) = A\xi(\lambda), \tag{3.1}$$

$$A(\lambda^2 - 1 - 2a\lambda) = B\xi(\lambda), \tag{3.2}$$

$$AB + \xi(\lambda)\left(\frac{r}{2} + 2\lambda^2 - 2\right) = 0, \tag{3.3}$$

$$B^2 - [\xi(\lambda)^2] + (\lambda^2 - 1 - 2a\lambda)\left(\frac{r}{2} + 3\lambda^2 - 3 + 2a\lambda\right) = 0, \tag{3.4}$$

$$A^2 - [\xi(\lambda)^2] + (\lambda^2 - 1 + 2a\lambda)\left(\frac{r}{2} + 3\lambda^2 - 3 - 2a\lambda\right) = 0. \tag{3.5}$$

Proof. Since $\dim M = 3$, we have the well-known formula

$$\begin{aligned} R(X,Y)Z = {} & g(X,Z)QY - g(Y,Z)QX + \varrho(X,Z)Y - \varrho(Y,Z)X + \\ & -\frac{r}{2}\{g(X,Z)Y - g(Y,z)X\}, \end{aligned} \tag{3.6}$$

for all X, Y, Z vector fields on M. Therefore, we can use (2.4) and (3.6) to compute the components of R with respect to the φ-basis $\{\xi, e, \varphi e\}$. We get

$$\begin{aligned}
R(\xi, e)\xi &= -(\lambda^2 - 1 - 2a\lambda)e + \xi(\lambda)\varphi e, \\
R(\xi, \varphi e)\xi &= \xi(\lambda)e - (\lambda^2 - 1 + 2a\lambda)\varphi e, \\
R(e, \varphi e)\xi &= -Be + A\varphi e, \\
R(\xi, e)e &= (\lambda^2 - 1 - 2a\lambda)\xi - B\varphi e, \\
R(\xi, \varphi e)e &= -\xi(\lambda)\xi + A\varphi e, \\
R(e, \varphi e)e &= B\xi + \left(\frac{r}{2} + 2\lambda^2 - 2\right)\varphi e, \\
R(\xi, e)\varphi e &= -\xi(\lambda)\xi + Be, \\
R(\xi, \varphi e)\varphi e &= (\lambda^2 - 1 + 2a\lambda)\xi - Ae, \\
R(e, \varphi e)\varphi e &= -A\xi - \left(\frac{r}{2} + 2\lambda^2 - 2\right)e.
\end{aligned} \tag{3.7}$$

These are all the possibly non-vanishing components of R, up to changes in the order of the vectors fields. (1.1) is equivalent to $R(X, \xi) \cdot R = 0$, for all vector field X on M, that is,

$$
\begin{aligned}
(3.8)\qquad & R(X,\xi)R(Y,Z)V - R(R(X,\xi)Y,Z)V - R(Y,R(X,\xi)Z)V + \\
& - R(Y,Z)R(X,\xi)V = 0,
\end{aligned}
$$

for all X, Y, Z, V vector fields on M. We now apply (3.8) taking $X = e$, $Y = \xi$, $Z = \varphi e$ and $V = \xi$. Using (3.7), we get

$$
\begin{aligned}
0 &= (R(e,\xi)\cdot R)(\xi,\varphi e)\xi \\
&= \{-A\xi(\lambda) + B(\lambda^2 - 1 + 2a\lambda)\}e + 2\{B\xi(\lambda) - A(\lambda^2 - 1 - 2a\lambda)\}\varphi e,
\end{aligned}
$$

from which (3.1) and (3.2) follow at once.

In the same way, we use (3.8) taking $X = e$, $Y = e$, $Z = \varphi e$ and $V = \xi$ and using (3.7). We obtain

$$
\begin{aligned}
0 = (R(e,\xi)\cdot R)(e,\varphi e)\xi &= \{-AB - \xi(\lambda)\left(\frac{r}{2} + 2\lambda^2 - 2\right)\}e + \\
&+ \{[\xi(\lambda)]^2 - B^2 - (\lambda^2 - 1 - 2a\lambda)\left(\frac{r}{2} + 3\lambda^2 - 3 + 2a\lambda\right)\}\varphi e
\end{aligned}
$$

and so, (3.3) and (3.4) hold. Finally, we put $X = \varphi e$, $Y = \varphi e$, $Z = e$ and $V = \xi$ in (3.8) and, making use of (3.7), we get

$$
\begin{aligned}
0 = (R(\varphi e,\xi)\cdot R)(\varphi e, e)\xi &= \{[\xi(\lambda)]^2 - A^2 - (\lambda^2 - 1 + 2a\lambda)\Big(\frac{r}{2} + 3\lambda^2 + \\
&- 3 - 2a\lambda\Big)\}e + \{-AB - \xi(\lambda)\left(\frac{r}{2} + 2\lambda^2 - 2\right)\}\varphi e,
\end{aligned}
$$

from which (3.5) follows.

Note that all the other possible choices of the vector fields in the φ-basis $\{\xi, e, \varphi e\}$, give again equations (3.1)-(3.5). So, if (3.1)-(3.5) hold, then (3.8) holds, that is, M is semi-symmetric. □

Proof of Theorem 1. If M is Sasakian then the result follows from [16]. So, let M be non-Sasakian. Then, (3.1)-(3.5) hold. Note that, because of (2.4), the constancy of $\varrho(\xi,\xi)$ along the characteristic flow means exactly $\xi(\lambda) = 0$. Hence, if $a = 0$, then $\nabla_\xi h = 0$, as it follows from (2.3), and the result then follows from [11]. To end the proof, we shall prove that the case $a \neq 0$ can not occur.

In fact, assume $a \neq 0$ and consider a point p at M such that $a(p) \neq 0$. Then, there exists a neighbourhood W of p such that $a \neq 0$ on W. We first multiply (3.1) by B and (3.2) by A and we get

$$
\begin{aligned}
(3.9)\qquad & B^2(\lambda^2 - 1 + 2a\lambda) = AB\xi(\lambda), \\
(3.10)\qquad & A^2(\lambda^2 - 1 - 2a\lambda) = AB\xi(\lambda).
\end{aligned}
$$

We substract (3.9) from (3.10) and we use (3.4) and (3.5) to express B^2 and A^2, respectively. We obtain

$$4a\lambda\{(\lambda^2-1)^2-4a^2\lambda^2-[\xi(\lambda)]^2\}=0,$$

from which, since $a\lambda\neq 0$, it follows

$$[\xi(\lambda)]^2=(\lambda^2-1)^2-4a^2\lambda^2. \tag{3.11}$$

Since $\xi(\lambda)=0$, (3.11) gives $\lambda^2-1\pm 2a\lambda=0$. Thus, suppose

$$\lambda^2-1+2a\lambda=0. \tag{3.12}$$

(If $\lambda^2-1-2a\lambda=0$, we proceed in the same way. Note that, since $a\neq 0$, the two conditions can not hold simultaneously).

Since $\xi(\lambda)=0$, (3.3) also yields $AB=0$, that is, either $A=0$ or $B=0$ (locally). We assume $A=0$ and we prove that λ is constant and $B=0$ (if we assume $B=0$, we proceed in the same way) .

Differentiating (3.12) with respect to ξ, we get $\xi(a)=0$. Next, differentiating (3.12) with respect to e, we obtain

$$\lambda(e(\lambda)+e(a))=-ae(\lambda). \tag{3.13}$$

We recall the well-known formula

$$\frac{1}{2}X(r)=\sum_{i=1}^{n}g((\nabla_{e_i}Q)e_i,X), \tag{3.14}$$

which holds for any vector field X of a n-dimensional Riemannian manifold. Here, $\{e_i\}$ is an arbitrary orthonormal basis. With respect to our φ-basis, we can apply (3.14) and (2.5) to compute $\frac{1}{2}e(r)$ and $\frac{1}{2}(\varphi e)(r)$. We obtain

$$2\lambda\{e(\lambda)+e(a)\}=(\lambda+a-1)B,$$

that is, using (3.13),

$$2ae(\lambda)=(1-\lambda-a)B. \tag{3.15}$$

Differentiating (3.15) by ξ, since $\xi(\lambda)=\xi(a)=0$, we get

$$2a\xi e(\lambda)=(1-\lambda-a)\xi(B).$$

We now use (3.14) and (2.5) to compute $\frac{1}{2}(\varphi e)(r)$ and we obtain the following formula for $\xi(B)$:

$$\xi(B)=2\lambda\{(\varphi e)(a)-(\varphi e)(\lambda)\}. \tag{3.16}$$

Therefore, we get

$$a\xi e(\lambda) = \lambda(\lambda + a - 1)\{(\varphi e)(\lambda) - (\varphi e)(a)\}.$$

From $\xi(\lambda) = 0$ it then follows

$$a[\xi, e](\lambda) = a\xi e(\lambda) = \lambda(\lambda + a - 1)\{(\varphi e)(\lambda) - (\varphi e)(a)\}.$$

On the other hand, using (2.2) we also have

$$a[\xi, e](\lambda) = a(\nabla_\xi e - \nabla_e \xi)(\lambda) = a(\lambda - a + 1)(\varphi e)(\lambda).$$

Therefore,

$$a(\lambda - a + 1)(\varphi e)(\lambda) = \lambda(\lambda + a - 1)\{(\varphi e)(\lambda) - (\varphi e)(a)\}. \tag{3.17}$$

Differentiating (3.12) with respect to φe, we obtain

$$\lambda\{(\varphi e)(a) + (\varphi e)(\lambda)\} = -a(\varphi e)(\lambda)$$

and so, (3.17) becomes

$$(\lambda^2 + a^2 + a\lambda - \lambda - a)(\varphi e)(\lambda) = 0.$$

Moreover, from (3.12) we have $a = \frac{1-\lambda^2}{2\lambda}$ and so, the last formula becomes

$$(3\lambda^4 - 2\lambda^3 - 2\lambda + 1)(\varphi e)(\lambda) = 0.$$

It is now easy to conclude that $(\varphi e)(\lambda) = 0$. In fact, if we assume $(\varphi e)(\lambda) \neq 0$, then $3\lambda^4 - 2\lambda^3 - 2\lambda + 1 = 0$ and, differentiating by φe, we conclude again that $(\varphi e)(\lambda) = 0$, which contradicts our assumption. Hence, $(\varphi e)(\lambda) = 0$. Since $\xi(\lambda) = 0$, we also have

$$0 = [\xi, \varphi e](\lambda) = (\lambda + a - 1)e(\lambda)$$

and so, $e(\lambda) = 0$. In fact, if we assume $e(\lambda) \neq 0$, then $\lambda + a - 1 = 0$, which, taking into account (3.12) and differentiating by e, permits to conclude easily that $e(\lambda) = 0$, against our assumption. Thus, $e(\lambda) = (\varphi e)(\lambda) = 0$ and so, λ is constant. Moreover, $B = 0$. In fact, if $B \neq 0$, then from (3.15) we get $\lambda + a - 1 = 0$, which, together with (3.12), gives $a = 0$, which can not occur. Then, necessarily $B = 0$.

Finally, we compute $R(e, \varphi e)e$ using (2.2) and we compare with (3.7). Using (3.12), we then get

$$r = 4a(\lambda - 1). \tag{3.18}$$

Using (3.18) in (3.4), and taking into account (3.12), we can conclude that $a = 0$, against our assumption. □

REMARK 3.2. Note that formula (3.11) holds for any non-Sasakian semi-symmetric contact metric three-manifold. In fact, if $a = 0$ and $\xi(\lambda) \neq 0$ (otherwise $\nabla_\xi h = 0$ and the result follows from [11]), then either $AB \neq 0$ (in this case, (3.11) follows at once from (3.1) and (3.2)), or $AB = 0$. Suppose for example $A = 0$. Then (3.3) gives $r = 4(1-\lambda^2)$, which, together with (3.5), again gives (3.11). Note also that from (3.4), (3.5) and (3.11) it follows

$$A^2 + B^2 = r(1-\lambda^2) - 4(1-\lambda^2)^2. \tag{3.19}$$

The following Lemma has interesting applications and it will also be used in the proof of Theorem 3.

LEMMA 3.3. *A semi-symmetric contact metric manifold $(M^3, \eta, g, \varphi, \xi)$ satisfying $A = 0$ or $B = 0$, either is flat or has constant curvature* 1.

Proof. Without loss of generality, assume $A = 0$. Consider $W_1 = \{p \in M/\xi(\lambda) \neq 0 \text{ at p}\}$ and $W_2 = \{p \in M/\xi(\lambda) = 0 \text{ in a neighbourhood of } p\}$.

On W_1, from (3.3) we get

$$\frac{r}{2} = 2 - 2\lambda^2,$$

from which, differentiating by ξ, we obtain $\xi(r) = -8\lambda\xi(\lambda)$. On the other hand, Using (3.13) and (2.5) to compute $\xi(r)$, we also get $\xi(r) = -4\lambda\xi(\lambda)$. Therefore, $\xi(\lambda) = 0$ on W_1, which can not occur. Therefore, W_1 is empty.

In the open subset W_2, proceeding as in the proof of Theorem 1, we can show that $B = 0$. On W_2 we now have $\xi(\lambda) = A = B = 0$ and the conclusion follows as in the proof of Theorem 1. □

The class of contact metric manifolds for which the characteristic vector field is an eigenvector field of the Ricci tensor naturally appeared in many problems and examples in contact geometry (see also [8]). Taking into account (2.4), as an immediate consequence of Lemma 3.3, we have the following

COROLLARY 3.4. *A semi-symmetric contact metric manifold $(M^3, \eta, g, \varphi, \xi)$, whose characteristic field is an eigenvector of the Ricci tensor, either is flat or has constant curvature* 1.

REMARK 3.5. Theorem 1 extends and corrects Theorem 3.4, b) of [10]. In fact, if $\xi \in (k, \mu)$-nullity distribution, then necessarily $\xi(\lambda) = 0$.

Proof of Theorem 2.. If M is Sasakian, then the conclusion follows from [16]. Next, we assume M is not-Sasakian, that is, $U = \{p \in M : \lambda(p) \neq 0\}$ is not empty, and we prove that this case can not occur.

We first compute the eigenvalues of the Ricci operator Q on M. Using (2.4), we get the characteristic equation

$$x^3 - b_2x^2 + b_1x - b_0 = 0,$$

where $b_0 = \det Q$, $b_1 = \sum_i Q_{ii}$, $b_2 = \mathrm{tr}Q = r$, Q_{ii} being the algebraic complement of ϱ_{ii} in the matrix of Q. Using (3.4),(3.5) and (3.11) we obtain easily $b_0 = 0$, while using (3.11) and (3.19) we get $b_1 = \frac{r^2}{4}$. Thus, the eigenvalues of Q are $\lambda_1 = 0$ and $\lambda_2 = \frac{r}{2}$, of multiplicity 1 and 2, respectively (see also [6], Prop. 11.2). Next, let Y be the orthogonal projection of ξ on the distribution $V(\frac{r}{2}) = \{X \in TM : QX = \frac{r}{2}X\}$. If $Y = 0$ on U, then $Q\xi = 0$ and from (2.4) it follows that $\lambda = 1$ on U. This implies easily that U is flat and hence, the vertical curvatures all vanish at any point of U, against the hypothesis.

So, let p be a point of U such that $Y_p \neq 0$. Consider an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM, where $Qe_1 = 0$ and $e_2 = \frac{Y}{\|Y\|}$. Clearly, $\{e_i\}$ is a basis of eigenvectors of Q at p, with $Qe_1 = 0$, $Qe_2 = \frac{r}{2}e_2$ and $Qe_3 = \frac{r}{2}e_3$. Then, the sectional curvature K_{12} vanishes, where by K_{ij} we denote the sectional curvature of the plane spanned by (e_i, e_j). With respect to $\{e_i\}$, we have $\xi_p = \alpha e_1 + \beta e_2$. Put $W = \beta e_1 - \alpha e_2$. Since W is orthogonal to ξ_p, $W \in \ker\eta$. The sectional curvature of the plane spanned by ξ_p and W is

$$K(\xi_p, W) = R(\xi_p, W, \xi_p, W) = (\alpha^2 + \beta^2)K_{12} = K_{12} = 0.$$

Hence, the vertical curvature $K(\xi_p, W)$ vanishes at p and this contradicts the hypothesis. □

4. Three-dimensional semi-symmetric unit tangent sphere bundles

Let (M, G) be a Riemannian manifold and $\bar{\pi} : TM \to M$ its tangent bundle. If X is a vector field on M, we denote by X^h and X^v respectively the horizontal and the vertical lift of X on TM. The Sasaki metric g_s on TM is defined by

$$g_s(\bar{X}, \bar{Y}) = G(\bar{\pi}_*\bar{X}, \bar{\pi}_*\bar{Y}) + G(K\bar{X}, K\bar{Y})$$

where $\bar{X}, \bar{Y}$ are vector field on TM and K is the connection map corresponding to the Levi-Civita connection of (M, G). TM admits an almost complex structure J defined by $JX^h = X^v$ and $JX^v = -X^h$.

The tangent sphere bundle $\pi : T^1M \to M$ is considered as the hypersurface of TM defined by $\{(p, v) \in TM : G(v, v) = 1\}$. The vector field $N = v^i(\frac{\partial}{\partial v^i}) = v^i(\frac{\partial}{\partial x^i})^v$ is an unit normal, as well as the position vector for a point $z = (p, v)$.

Let g' be the metric on T^1M induced from g_s. On TM there is a horizontal vector field ξ' called *geodesic flow* of (M,G), which is defined by

$$\xi' = -JN = v^i\left(\frac{\partial}{\partial x^i}\right)^h.$$

Since ξ'_z, for $z \in T^1M$, is tangent to T^1M, ξ' can be considered as a vector field on T^1M. Let η' be the 1-form on T^1M dual to ξ' with respect to g', and φ' the (1,1) tensor given by $\varphi' X = JX - \eta'(X)N$. Then

$$(\eta,\xi,g,\varphi) = \left(\frac{1}{2}\eta', 2\xi', \frac{1}{4}g', \varphi'\right)$$

is the standard contact metric structure on T^1M.

We are now ready to give the

Proof of Theorem 3.. Using isothermal local coordinate (x^1,x^2) on M, the Riemannian metric G is given by

$$G = e^{2f}((dx^1)^2 + (dx^2)^2)$$

where f is a C^∞ function on M. The immersion of $T^1M = \{z = (p,v) \in TM : e^{2f}((v^1)^2 + (v^2)^2) = 1\}$ into TM is defined by

$$(y^1,y^2,\theta) \longrightarrow (x^1,x^2,v^1,v^2) = (y^1,y^2,e^{-f}cos\theta, e^{-f}sin\theta).$$

Setting $f_1 = \frac{\partial f}{\partial x^1}$ and $f_2 = \frac{\partial f}{\partial x^2}$, we find

$$\frac{\partial}{\partial\theta} = -v^2\frac{\partial}{\partial v^1} + v^1\frac{\partial}{\partial v^2},\ \frac{\partial}{\partial y^1} = \frac{\partial}{\partial x^1} - f_1N,\ \frac{\partial}{\partial y^2} = \frac{\partial}{\partial x^2} - f_2N,$$

and

$$\xi' = v^1\frac{\partial}{\partial y^1} + v^2\frac{\partial}{\partial y^2} + (v^1f_2 - v^2f_1)\frac{\partial}{\partial\theta}.$$

Moreover, we find $g'(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}) = g'(\xi',\xi') = 1$.
Since $\frac{\partial}{\partial\theta} = u^v$, and $\xi' = v^h$, where $u = -v^2\frac{\partial}{\partial x^1} + v^1\frac{\partial}{\partial x^2}$, and $v = v^1\frac{\partial}{\partial x^1} + v^2\frac{\partial}{\partial x^2}$, then

$$U = u^h = -v^2\frac{\partial}{\partial y^1} + v^1\frac{\partial}{\partial y^2} - (v_2f_2 + v_1f_1)\frac{\partial}{\partial\theta}$$

is g'-unitary and orthogonal to $\frac{\partial}{\partial\theta}, \xi'$ and N. Thus, $(\frac{\partial}{\partial\theta}, U, \xi')$ is a g'-orthogonal basis of vector fields tangent to T^1M. On the other hand, $\varphi U = \varphi u^h = u^v - G(u,v)N = \frac{\partial}{\partial\theta}$. So, $(\xi, e, \varphi e) = (2\xi', 2U, 2\frac{\partial}{\partial\theta})$ is an orthonormal φ-basis for (T^1M,η,g).

We now determine h. By (2.1), $h = \varphi\nabla\xi + \varphi^2$, where the covariant derivates $\nabla\xi$ are given by [B1]

$$(\nabla_W \xi)_z = -(R(\pi_* W, v)v)^v, \quad (\nabla_V \xi)_z = -2\phi V - (R(KV, v)v)^h.$$

Here W (resp. V) is a horizontal (resp. vertical) vector field tangent to T^1M and R is the curvature tensor of (M, G). Then, we get

$$hU = (k-1)U, \qquad h\left(\frac{\partial}{\partial\theta}\right) = (1-k)\left(\frac{\partial}{\partial\theta}\right), \tag{4.1}$$

where the Gaussian curvature k considered as a function on T^1M is defined by $k(p,v) := k(p)$. Hence, $\{\xi, e, \varphi e\}$ is a φ-basis of eigenvectors of h, with $\lambda = 1-k$.

The Ricci tensor of (T_1M, g') has been computed in [4]. Taking into account that g and g' are homothetic, it is easy to prove that the Ricci tensor $\bar{\varrho}$ of (T_1M, g) satisfies

$$\bar{\varrho}(p,v)(X^h, Y^h) = \varrho_p(X,Y) - \frac{1}{2}\sum_{i=1,2} G_p(R(v,E_i)X, R(v,E_i)Y),$$

where ϱ is the Ricci tensor of M and $\{E_i\}$ is an orthonormal basis of T_pM. In particular, taking $\{u, v\}$ as an orthonormal basis of T_pM, we obtain

$$\begin{aligned} A = \bar{\varrho}_{(p,v)}(\xi, e) &= \varrho_{(p,v)}(2v^h, 2u^h) = \\ &= 4\varrho_p(v,u) - 2G_p(R(v,u)v, R(v,u)u) = -k(p)^2 G_p(v,u) = 0. \end{aligned}$$

Therefore, if T_1M is semi-symmetric, then, since $A = 0$, Lemma 3.3 implies that T^1M has constant sectional curvature 0 or 1. Hence, (4.1) yields $\lambda = 1$ or $\lambda = 0$ respectively, and (M, G) has constant Gaussian curvature $k = 0$ or $k = 1$ respectively.

Conversely, if (M, G) has constant sectional curvature 0 or 1, then T^1M is locally symmetric (see for example [12]). In particular, it is semi-symmetric. □

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