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EXACT ASYMPTOTIC BEHAVIOUR OF THE EXPECTED NUMBER OF PARTICLES OF AN AGE-DEPENDENT BRANCHING PROCESS

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Abstract. We study in detail the exact asymptotic behaviour of the expected number of particles of an age-dependent branching process. We take into account the influence of the characteristic equation on the asymptotic behaviour being investigated. Expansions for the renewal measure are also obtained and the exact asymptotic behaviour of the remainder terms is established.

1. Introduction

Let $\{Z(t), t \geq 0\}$ be an age-dependent branching process. The random variable $Z(t)$ is the number of particles at time t. The process is characterized by a generating function $f(s) = \sum_{n=0}^{\infty} p_n s^n$ governing particle production and by a distribution, say F, concentrated on $[0, \infty)$ of the lifetime of a single particle. The evolution of the process $\{Z(t), t \geq 0\}$ is an follows [2, Chapter IV]. At time $t = 0$ there is one particle which lives for time T_0 and after that it produces k particles with probability p_k. These particles have lifetimes $T_{11}, \ldots, T_{1k}$ and thereafter, in their turn, produce offspring according to the probability distribution $\{p_n\}$, and so on. All the lifetimes are independent random variables with distribution F. Particle production is independent of both the present state and past history of the process; moreover, the lifetimes and offspring numbers are independent. A mathematically rigorous construction of the process $\{Z(t), t \geq 0\}$ may be found in [8, Chapter VI].

The aim of the present paper is to investigate in detail the asymptotic behaviour of the expected number of particles $\mu(t) \stackrel{\text{def}}{=} EZ(t)$ as $t \to \infty$.

Recall some well-known facts. Suppose that the expected number, say m,

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of offspring produced by a single particle is finite: $m \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} kp_k < \infty$. The *Malthusian parameter* $\alpha = \alpha(m, F)$ is defined (when it exists) as the real root of the equation

$$m \int_0^\infty e^{-\alpha y} F(dy) = 1$$

and plays a central role in the asymptotic behaviour of $\mu(t)$. It is well known that $\mu(t) \equiv 1$ for $m = 1$ [2, Chapter IV, Theorem 3A]. If $m > 1$, then

$$\mu(t) \sim ce^{\alpha t}, \quad t \to \infty, \tag{1}$$

where α is the Mathusian parameter and

$$c = \frac{m-1}{\alpha m^2 \int_0^\infty y e^{-\alpha y} F(dy)}$$

If $m < 1$, if the Mathusian parameter α exists and if $\int_0^\infty y e^{-\alpha y} F(dy) < \infty$, then relation (1) holds; in this case $\alpha < 0$ [2, Chapter IV, Theorem 3A and 3B].

Consider now some known results concerning the asymptotic behaviour of $\mu(t)$ when the Malthusian parameter does not exist. If the distribution F belongs to the class $\mathcal{S}(\gamma)$, $\gamma \geq 0$, (see the definition in section 2) and if $m \int_0^\infty e^{\gamma x} F(dx) < 1$, then (see [4, Theorem 1])

$$\mu(t) \sim \frac{1-m}{[1 - m \int_0^\infty e^{\gamma x} F(dx)]^2} [1 - F(t)] \qquad \text{as} \quad t \to \infty. \tag{2}$$

In the *subexponential* case, i.e. when $F \in \mathcal{S}(0)$, relation (2) was obtained by Chistyakov [3, Theorem 5] under the assumption

$$m^{-1} > \sup_{t \geq 0} \int_0^t \frac{1 - F(t-y)}{1 - F(t)} F(dy),$$

and, more generally, Theorem 3B(ii) in [2, Chapter IV, Section 5] says that in the subexponential case $F \in \mathcal{S}(0)$ relation (2) holds for all $m < 1$.

In the present paper, we obtain refinements of the above results, when $m \neq 1$ and the tail of F can be compared with the tail of an arbitary distribution from the class $\mathcal{S}(\gamma), \gamma \geq 0$; in particular, when the disribution F itself belongs to $\mathcal{S}(\gamma)$. In the simplest situations, the results obtained are of the form

$$\mu(t) = ce^{\alpha t} + r(t),$$

where α is the Malthusian parameter, c is a well-defined constant and the limit as $t \to \infty$ of the ratio $r(t)/[1 - F(t)]$ is expressed explicitly in terms of the underlying lifetime distribution $F \in \mathcal{S}(\gamma)$ and the average number m of offspring.

It is well known [2, Chapter IV, Section 5, Theorem 1] that the function $\mu(t)$ is the unique solution of the renewal equation

$$\mu(t) = [1 - F(t)] + m \int_0^t \mu(t-y) F(dy). \tag{3}$$

This solution may be represented in the form

$$\mu(t) = [1 - F(t)] * U_m(t) \stackrel{\text{def}}{=} \int_0^t [1 - F(t-y)] U_m(dy),$$

where $U_m(t) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} m^k F^{k*}(t)$ is the renewal function; here F^{k*} is the k-fold convolution of F. Therefore, we shall spend a considerable amount of time in studying the asymptotic properties of the renewal measure $U_m = \sum_{k=0}^{\infty} m^k F^{k*}$. In this connection a special emphasis is laid on the influence of the roots of the *characteristic equation*

$$1 - m \int_0^\infty e^{sx} F(dx) = 0 \tag{4}$$

on the asymptotic behaviour of the renewal measure U_m. (Note that if the Malthusian parameter α exists, then $s = -\alpha$ is a root of equation (4).) The proofs of the renewal theorems will be based on the Banach algebraic techniques introduced in papers by Chover, Ney and Wainger [5], Essén [6] and Rogozin [10].

The further plan of the paper is as follows. In Section 2, we list some necessary facts from the theory of Banach algebras of measures and prove two results about the Laplace transforms of measures. Section 3 deals with the asymptotic properties of the renewal measure U_m. In Section 4, we study the asymptotic behaviour of the expected number of particles $\mu(t)$ as $t \to \infty$.

2. Preliminaries

DEFINITION 1. A probability distribution G concentrated on $[0, \infty)$ belongs to the class $\mathcal{S}(\gamma)$, $\gamma \geq 0$, if

(a) $G((x, \infty)) > 0 \qquad \forall x \geq 0,$

(b) $\displaystyle \lim_{x\to\infty} \frac{G((x+y, \infty))}{G((x, \infty))} = e^{-\gamma y} \qquad \forall y \in \boldsymbol{R},$

(c) $\displaystyle \lim_{x\to\infty} \frac{G * G((x, \infty))}{G((x, \infty))} = 2 \int_0^\infty e^{\gamma x} G(dx) < \infty.$

The class $\mathcal{S}(0)$ has been intoroduced by Chistyakov [3], and the classes $\mathcal{S}(\gamma)$ for positive γ were first considered by Chover, Ney and Wainger [4, 5]. These classes of probability distribution proved very useful in studying the exact asymptotic behaviour of various quantities of interest in numerous areas of probability theory (see, e.g. the references in [1]).

In what follows we shall need some knowledge about Banach algebras of measures with preassigned tail behaviour.

Let $S(\gamma', \gamma)$, $\gamma' \leq 0 \leq \gamma$, be the collection of all complex-valued σ-finite measures ν defined on the σ-algebra $\mathcal{B}$ of Borel subsets of $\boldsymbol{R}$ such that

$$\|\nu\| \stackrel{\text{def}}{=} \int_{\boldsymbol{R}} \max(e^{\gamma' x}, e^{\gamma x})|\nu|(dx) < \infty,$$

where $|\nu|$ stands for the total variation of ν. Then $S(\gamma', \gamma)$ is a Banach algebra with respect to the norm $\|\nu\|$ and the usual operations of addition and scalar multiplication of measures; the product of two measures ν and κ from $S(\gamma', \gamma)$ is defined as their convolution $\nu * \kappa$. The unit element of the algebra $S(\gamma', \gamma)$ is the Dirac measure δ, i.e. the measure of unit mass concentrated at the origin. The Laplace transform $\widehat{\nu}(s)$ of any element $\nu \in S(\gamma', \gamma)$ is defined as

$$\widehat{\nu}(s) \stackrel{\text{def}}{=} \int_{\boldsymbol{R}} \exp(sx)\nu(dx).$$

This integral converges absolutely with respect to the measure $|\nu|$ for all s lying in the strip

$$\Pi(\gamma', \gamma) \stackrel{\text{def}}{=} \{s \in \boldsymbol{C} : \gamma' \leq \Re s \leq \gamma\},$$

where $\boldsymbol{C}$ is the field of complex numbers.

Let us state explicitly from the beginning that the paramenter $\gamma' \leq 0$ will play an auxiliary role and will be chosen so that all the roots of the charasteristic equation (4) lie in the strip $\Pi(\gamma', \gamma)$. Such arbitrary choice of γ' will not affect the final shape of the results themselves.

Now choose an arbitrary distribution $G \in \mathcal{S}(\gamma)$. Set $\tau(x) \stackrel{\text{def}}{=} G((x, \infty))$. Define a functional Q by the formula

$$Q(\nu) \stackrel{\text{def}}{=} \sup_{x \geq 0} \frac{|\nu|((x, \infty))}{\tau(x)}, \quad \nu \in S(0, \gamma).$$

Consider the collection $\mathfrak{S}\mathfrak{l}(\tau)$ of all measures $\nu \in S(0, \gamma)$ such that $Q(\nu) < \infty$ and there exists the limit

$$l(\nu) \stackrel{\text{def}}{=} \lim_{x \to \infty} \frac{\nu((x, \infty))}{\tau(x)} \in \boldsymbol{C}.$$

As shown in [13, Proposition 2], $\mathfrak{Sl}(\tau)$ is a Banach algebra with respect to some norm $\|\nu\|'$, equivalent to the norm $\|\nu\| + Q(\nu)$. The algebraic operations in $\mathfrak{Sl}(\tau)$ are the same as in $S(\gamma', \gamma)$, and the measure δ is the unit element of $\mathfrak{Sl}(\tau)$. Moreover, for any two elements $\nu, \kappa \in \mathfrak{Sl}(\tau)$ the following equality holds:

$$l(\nu * \kappa) = l(\nu)\widehat{\kappa}(\gamma) + l(\kappa)\widehat{\nu}(\gamma). \tag{5}$$

The collections of measures

$$\mathfrak{S}(\tau) \stackrel{\text{def}}{=} \{\nu \in S(0, \gamma) : Q(\nu) < \infty\},$$
$$\mathfrak{So}(\tau) \stackrel{\text{def}}{=} \left\{\nu \in \mathfrak{S}(\tau) : \lim_{x\to\infty} \frac{|\nu|((x, \infty))}{\tau(x)} = 0\right\}$$

are also Banach algebras with respect to the norm $\|\nu\|'$ and the usual operations of addition and scalar multiplication of measures [13, Proposition 2].

Let $\mathcal{A}$ be an arbitrary complex commutative Banach algebra with unit element e. The *spectrum* $\sigma(a)$ of an element $a \in \mathcal{A}$ is the set of all complex numbers λ such that the element $a - \lambda e$ does not have an inverse. If $f(z)$ is an analytic function in a domain containing the spectrum of the element $a \in \mathcal{A}$, then there exists an element $f(a) \in \mathcal{A}$ such that for each homomorphism $m : \mathcal{A} \to \boldsymbol{C}$ the following relation holds: $m(f(a)) = f(m(a))$ [15, Section 3]. The element $f(a)$ is called the *value of the analytic function $f(z)$ at the element $a \in \mathcal{A}$*.

We shall need the following result concerning the values of analytic functions at elements of the Banach algebras $\mathfrak{Sl}(\tau)$, $\mathfrak{S}(\tau)$ and $\mathfrak{So}(\tau)$ [13, Theorem 3].

THEOREM 1. *Let $f(z)$ be an analytic function in a domain containing the spectrum $\sigma(\nu)$ of an element $\nu \in S(\gamma', \gamma)$, and let $f(\nu)$ be the value of $f(z)$ at $\nu \in S(\gamma', \gamma)$. If $\nu \in \mathfrak{Sl}(\tau)$, then $f(\nu) \in \mathfrak{Sl}(\tau)$ and the following relation holds:*

$$l[f(\nu)] = f'[\widehat{\nu}(\gamma)] \cdot l(\nu).$$

Similar statements are also valid for the Banach algebras $\mathfrak{S}(\tau)$ and $\mathfrak{So}(\tau)$.

In studying the influence of the roots of the characteristic equation (4) on the asymptotic behaviour of the expected number of particles $\mu(t)$, we shall make repeated use of the following usefull property of the Laplace transforms of measures from $\mathfrak{Sl}(\tau)$.

THEOREM 2. *Let $\nu \in \mathfrak{Sl}(\tau) \cap S(\gamma', \gamma)$ and $\gamma' \le \Re\xi < \gamma$. If $\Re\xi = \gamma'$, then assume additionally that $\int_{-\infty}^{0} |x| e^{\gamma' x} |\nu|(dx) < \infty$. Then the function*

$$\frac{\widehat{\nu}(s) - \widehat{\nu}(\xi)}{s - \xi}, \qquad s \in \Pi(\gamma', \gamma),$$

is the Laplace transform of some measure $\kappa \in \mathfrak{S}\mathfrak{l}(\tau) \cap S(\gamma', \gamma)$*; moreover,* $l(\kappa) = l(\nu)/(\gamma - \xi)$.

Proof. Consider the measure κ with the density

$$\nu(x;\xi) \stackrel{\text{def}}{=} \begin{cases} -\int_{-\infty}^{x} e^{\xi(y-x)}\nu(dy) & \text{for} \quad x < 0, \\ \int_{x}^{\infty} e^{\xi(y-x)}\nu(dy) & \text{for} \quad x \geq 0. \end{cases} \tag{6}$$

A direct verification shows that $\kappa \in S(\gamma', \gamma)$ and

$$\widehat{\kappa}(s) = \frac{\widehat{\nu}(s) - \widehat{\nu}(\xi)}{s - \xi}, \qquad s \in \Pi(\gamma', \gamma)$$

(at point $s = \xi$ the value $\widehat{\kappa}(s)$ is difined by continuity as $\int_{\boldsymbol{R}} x e^{\xi x}\nu(dx)$). Let, for example, $\Re\xi = \gamma'$. Then

$$\int_{-\infty}^{0} e^{\gamma' x}|\kappa|(dx) = \int_{-\infty}^{0} e^{\gamma' x}|\nu(x;\xi)|dx \leq \int_{-\infty}^{0} |y|e^{\gamma' y}|\nu|(dy) < \infty.$$

Next,

$$\begin{aligned} \int_{0}^{\infty} e^{\gamma x}|\kappa|(dx) &\leq \int_{0}^{\infty} e^{(\gamma-\gamma')x} \int_{x}^{\infty} e^{\gamma' y}|\nu|(dy)dx \\ &= \frac{1}{\gamma - \gamma'} \int_{0}^{\infty} (e^{\gamma y} - e^{\gamma' y})|\nu|(dy) < \infty. \end{aligned}$$

We show that $\kappa \in \mathfrak{S}\mathfrak{l}(\tau)$. Interchanging the order of integration, we get

$$\begin{aligned} \kappa((t,\infty)) &= \int_{t}^{\infty} \int_{x}^{\infty} e^{\xi(y-x)}\nu(dy)dx \\ &= \frac{1}{\xi} \int_{t}^{\infty} e^{\xi(y-t)}\nu(dy) - \frac{\nu((t,\infty))}{\xi}. \end{aligned}$$

An integration by parts yeilds

$$\kappa((t,\infty)) = \int_{t}^{\infty} e^{\xi(y-t)}\nu((y,\infty))dy.$$

Hence

$$\frac{\kappa((t,\infty))}{\tau(t)} = \int_{0}^{\infty} e^{\xi y} \frac{\nu((t+y,\infty))}{\tau(t+y)} \frac{\tau(t+y)}{\tau(t)} dy. \tag{7}$$

Lemma 1. *Let* $G \in S(\gamma), \gamma > 0$*. Then, for each* $\beta < \gamma$*, there exists a constant* $C(\beta) \geq 1$ *such that*

$$\tau(y)e^{\beta y} \leq C(\beta)\tau(x)e^{\beta x} \qquad \forall y \geq x \geq 0. \tag{8}$$

A proof of Lemma 1 may be found in [12]. We return to the proof of Theorem 2. Let $\gamma > 0$. Choose $\beta \in (\Re\xi, \gamma)$. In view of Lemma 1, the integrand in (7) is bounded in absolute value by the integrable function

$$C(\beta)Q(\nu)\exp[(\Re\xi - \beta)y], \qquad y \geq 0.$$

In case $\gamma = 0$, one may take the function $Q(\nu)\exp(\Re\xi y), y \geq 0$, to bound the integrand in (7). Passing to the limit under the integral sign in (7), we get

$$l(\kappa) = l(\nu)\int_0^\infty e^{(\xi-\gamma)y}dy = \frac{l(\nu)}{\gamma - \xi}.$$

Finally, we show that $Q(\kappa) < \infty$. Let $\gamma > 0$. In virtue of (7) and (8), we have

$$\sup_{x\geq 0}\frac{|\kappa|((t,\infty))}{\tau(t)} \leq \frac{Q(\nu)C(\beta)}{\beta - \Re\xi} < \infty.$$

If $\gamma = 0$, then

$$\sup_{t\geq 0}\frac{|\kappa|((t,\infty))}{\tau(t)} \leq Q(\nu)\int_0^\infty e^{\Re\xi y}dy = \frac{Q(\nu)}{|\Re\xi|} < \infty.$$

The proof of Therom 2 is complete. ∎

In connection with Theorem 2 we introduce the following notation. Let $\xi \in \boldsymbol{C}$, and let ν be a σ-finite measure such that the measure $\int_A e^{\xi x}\nu(dx)$, $A \in \mathcal{B}$, is finite. Define a measure $T(\xi)\nu$ by the formula

$$T(\xi)\nu(A) = \int_A \nu(x;\xi)dx, \qquad A \in \mathcal{B},$$

where the function $\nu(x;\xi), x \in \boldsymbol{R}$, is given by (6). If

$$\int_{\boldsymbol{R}} |x|e^{\Re\xi x}|\nu|(dx) < \infty,$$

then the Laplace transform of the measure $T(\xi)\nu$ is of the following form:

$$[T(\xi)\nu]\hat{}(s) = \frac{\widehat{\nu}(s) - \widehat{\nu}(\xi)}{s - \xi}, \qquad \Re s = \Re\xi;$$

at point $s = \xi$ we set

$$[T(\xi)\nu]\hat{}(\xi) \stackrel{\text{def}}{=} \int_{\boldsymbol{R}} xe^{\xi x}\nu(dx).$$

In case $\xi = \gamma$ Theorem 2 is meaningless. However, the following result holds.

LEMMA 2. *Let $\nu \in S(\gamma', \gamma)$ be a non-negative measure such that*

$$\int_{R} |x| e^{\gamma x} \nu(dx) < \infty.$$

If $T(\gamma)\nu \in \mathfrak{Sl}(\tau)$, then $\nu \in \mathfrak{Sl}(\tau)$; moreover, $l(\nu) = 0$.

Proof. Let $\Delta(x) \stackrel{\text{def}}{=} T(\gamma)\nu((x,\infty)) - e^{\gamma} T(\gamma)\nu((x+1,\infty))$. Since $T(\gamma)\nu \in \mathfrak{Sl}(\tau)$, it is clear that $\Delta(x)/\tau(x) \to 0$ as $x \to \infty$. Further,

$$\begin{aligned} \Delta(x) &= \int_{x}^{\infty} e^{-\gamma y} \int_{y}^{y+1} e^{\gamma z} \nu(dz) dy \\ &\geq \int_{x+1}^{\infty} e^{\gamma z} \int_{z-1}^{z} e^{-\gamma y} dy\, \nu(dz) = \frac{(e^{\gamma} - 1)\nu((x+1,\infty))}{\gamma}. \end{aligned}$$

Hence $Q(\nu) < \infty$ and $\nu((x,\infty))/\tau(x) \to 0$ as $x \to \infty$, completing the proof of Lemma 2. ∎

3. Renewal theorems

Consider the renewal measure of the following type:

$$U_m \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} m^k F^{k*},$$

where $m > 0$ and F is a probability distribution concentrated on $[0, \infty)$. Let $\gamma \geq 0$ and $\widehat{F}(\gamma) < \infty$. Suppose that the set, say $\mathcal{Z}$, of the roots of the characteristic equation $1 - m\widehat{F}(s) = 0$ lying in the half-plane $\{\Re s < \gamma\}$ is finite. Denote the elements of $\mathcal{Z}$ by $s_1, s_2, \ldots, s_l$. Let n_j be the multiplicity of the root s_j; this means that $1 - m\widehat{F}(s) = (s - s_j)^{n_j} F_j(s)$, where $F_j(s_j) \neq 0$. If $s \in \mathcal{Z}$, then $\overline{s} \in \mathcal{Z}$ and the root $\overline{s}$ has the same multiplicity as s.

Let ν be a comlex-valued measure concentrated on $[0, \infty)$ such that $\int_0^{\infty} e^{\gamma x}$ $|\nu|(dx) < \infty$. Denote by $\sum_{k=1}^{n_j} (-1)^k B_{jk}^{(\nu)}/(s - s_j)^k$ the principal part of the Laurent series expansion of the analytic function

$$\widehat{U}_m(s)\widehat{\nu}(s) = \frac{\widehat{\nu}(s)}{1 - m\widehat{F}(s)}, \qquad s \in \{\Re s < \gamma\} \backslash \mathcal{Z},$$

about the isolated singular point $s = s_j \in \mathcal{Z}$. Note that the set $\mathcal{Z}$, the roots s_j and the coefficients $B_{jk}^{(\nu)}$ depend on m. For brevity, we shall not indicate the dependence on m in the notations $\mathcal{Z}$, s_j and $B_{jk}^{(\nu)}$. Denote by $\mathcal{E}_j$ the complex-valued measure with the density $\mathbf{1}_{(0,\infty)}(x) e^{-s_j x}$ ($\mathbf{1}_A(x)$ is the indicator of the set A); the Laplace transform of this measure is equal to $1/(s_j - s)$, $\Re(s - s_j) < 0$.

The absolutely continuous part of an arbitrary distribution F will be denote by F_c, and its singular component by $F_s : F_s = F - F_c$.

THEOREM 3. *Let a measure ν and a probability distribution F consentrated on $[0,\infty)$ be elements of the Banach algebra $\mathfrak{S}\mathfrak{l}(\tau)$. Suppose that $m^n(F^{n*})\widehat{_s}(\gamma) < 1$ for some integer $n \geq 1$ and $m\widehat{F}(s) \neq 1$ for $\Re s = \gamma$. Then the convolution $\nu * U_m$ admits the representation*

$$\nu * U_m = \sum_{j=1}^{l}\sum_{k=1}^{n_j} B_{jk}^{(\nu)}\mathcal{E}_j^{k*} + R_{\nu,m}, \tag{9}$$

where the remainder term $R_{\nu,m}$ belongs to the Banach algebra $\mathfrak{S}\mathfrak{l}(\tau)$;moreover,

$$l(R_{\nu,m}) = \frac{l(\nu)}{1-m\widehat{F}(\gamma)} + \frac{\widehat{\nu}(\gamma)ml(F)}{[1-m\widehat{F}(\gamma)]^2}. \tag{10}$$

Proof. First of all, we observe that the set $\mathcal{Z}$ is finite. This follows from the analyticity of $\widehat{F}(s)$ in the half-plane $\{\Re s < \gamma\}$ and the hypotheses $m^n(F^{n*})\widehat{_s}(\gamma) < 1$ for some $n \geq 1$ and $m\widehat{F}(s) \neq 1$ for $\Re s = \gamma$ [7]. Choose $r > \gamma$. Set $p \overset{\text{def}}{=} \sum_{j=1}^{l} n_j$ and

$$u(s) \overset{\text{def}}{=} \frac{[1-m\widehat{F}(s)](s-r)^p}{\prod_{j=1}^{l}(s-s_j)^{n_j}}, \qquad \Re s \leq \gamma,$$

defining the values of $u(s)$ at the points $s \in \mathcal{Z}$ by continuity. We show that the function $u(s)$ is the Laplace transform $\widehat{U}(s)$ of some measure $U \in \mathfrak{S}\mathfrak{l}(\tau)$. Representing a rational function as a sum of partial fractions, we have

$$u(s) = [1-m\widehat{F}(s)]\left[1+\sum_{j=1}^{l}\sum_{k=1}^{n_j}\frac{C_{jk}}{(s-s_j)^k}\right],$$

where C_{jk} are constants.

Choose $\gamma' < \Re s_j \forall j = 1,\dots,l$. It is clear that $F \in S(\gamma',\gamma)$. Consider the expression $[m\widehat{F}(s)-1]/(s-s_j)^k$ for $1 \leq k \leq n_j$. By Theorem 2, this expression is the Laplace transform of the measure $mT(s_j)^kF$ belonging to both $\mathfrak{S}\mathfrak{l}(\tau)$ and $S(\gamma',\gamma)$. Consequently, $u(s) = \widehat{U}(s)$, where $U \in \mathfrak{S}\mathfrak{l}(\tau)$. Moreover, $U \in S(\gamma',\gamma)$. By Theorem 2, we have

$$l(U) = -ml(F)\left[1+\sum_{j=1}^{l}\sum_{k=1}^{n_j}\frac{C_{jk}}{(\gamma-s_j)^k}\right] = \frac{-ml(F)(\gamma-r)^p}{\prod_{j=1}^{l}(\gamma-s_j)^{n_j}}.$$

By means of standard arguments it is not difficult to show that there exists an inverse element U^{-1} in the Banach algebra $S(\gamma',\gamma)$ (see, e.g. the proof of Theorem 1 in [14]). For completeness, we reproduce the corresponding reasoning.

Let $\mathcal{M}$ be the space of maximal ideals of the Banach algebra $S(\gamma',\gamma)$. Each maximal ideal $M \in \mathcal{M}$ induces a homomorphism, say h, of the Banach algebra $S(\gamma',\gamma)$ onto the filed of complex numbers $\boldsymbol{C}$, and M is the kernel of this homomorphism. For an arbitrary element $\nu \in S(\gamma',\gamma)$, denote by $\nu(M)$ the value of the homomorphism h at ν. An element $\nu \in S(\gamma',\gamma)$ has an inverse if and only if ν is not in any maximal ideal $M \in \mathcal{M}$. In other words, the element ν is invertible if and only if $\nu(M) \neq 0$ for every $M \in \mathcal{M}$.

The space $\mathcal{M}$ is split into two subsets: $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$, where $\mathcal{M}_1$ is the set of all maximal ideals which do not contain the collection $L(\gamma',\gamma)$ of all absolutely continuous measures from $S(\gamma',\gamma)$ and $\mathcal{M}_2 = \mathcal{M} \setminus \mathcal{M}_1$. If $M \in \mathcal{M}_1$, then the homomorphism $S(\gamma',\gamma) \to \boldsymbol{C}$ induced by M is of the form $\nu \to \widehat{\nu}(s_0)$, where $s_0 \in \Pi(\gamma',\gamma)$. In this case $M = \{\nu \in S(\gamma',\gamma) : \widehat{\nu}(s_0) = 0\}$ [9, Chapter IV, Section 4]. If $M \in \mathcal{M}_2$, then $\nu(M) = 0$ for each absolutely continuous measure $\nu \in S(\gamma',\gamma)$.

We show that $U(M) \neq 0$ for every $M \in \mathcal{M}$, thus we will prove the existence of an inverse element $U^{-1} \in S(\gamma',\gamma)$. Actually, if $M \in \mathcal{M}_1$, then we have $U(M) = \widehat{U}(s_0) \neq 0$ for some $s_0 \in \Pi(\gamma',\gamma)$. Let now $M \in \mathcal{M}_2$. Applying a theorem about the structure of homomorphisms of $S(\gamma',\gamma)$ onto $\boldsymbol{C}$ [11, Theorem 1], we have

$$h(\nu) = \int_{\boldsymbol{R}} \chi(x,\nu) \exp(\beta x)\, \nu(dx), \qquad \nu \in S(\gamma',\gamma), \tag{11}$$

where $\beta \in [\gamma',\gamma]$ and the function $\chi(x,\nu)$ of the two variables $x \in \boldsymbol{R}$ and $\nu \in S(\gamma',\gamma)$ is a generalized character; here we mention only one relevant property of a generalized character: $|\nu| - \operatorname{ess\,sup}_{x\in\boldsymbol{R}} |\chi(x,\nu)| \leq 1$.

The condition $m^n(F^{n*})\widehat{_s}(\gamma) < 1$ implies the inequality $m^n(F^{n*})\widehat{_s}(\beta) < 1$ for all $\beta \in [\gamma',\gamma]$. By(11), for some $\beta \in [\gamma',\gamma]$, we have

$$\begin{aligned} |mF(M)|^n &= |m^n F^{n*}(M)| = |m^n (F^{n*})_s(M)| \\ &= \left| m^n \int \chi(x, (m^n F^{n*})_s) \exp(\beta x)(F^{n*})_s(dx) \right| \\ &\leq\ m^n \int \exp(\beta x)(F^{n*})_s(dx) < 1. \end{aligned}$$

Since $T(s_j)^k F \in L(\gamma',\gamma) \forall j,k$, we conclude that $|U(M)| = |1 - mF(M)| > 0$. Thus, $U(M) \neq 0$ for all $M \in \mathcal{M}$. This means that there exists an inverse element $U^{-1} \in S(\gamma',\gamma)$ and, in particular, the function $1/u(s), \gamma' \leq \Re s \leq \gamma$, is the Laplace transform of the element U^{-1}. By Theorem 1 with $f(z) = 1/z$, we have $W \stackrel{\text{def}}{=} U^{-1} \in \mathfrak{Sl}(\tau)$ and

$$l(W) = -\frac{1}{[\widehat{U}(\gamma)]^2} \cdot l(U) = \frac{m\, l(F) \Pi_{j=1}^{l} (\gamma - s_j)^{n_j}}{[1 - m\widehat{F}(\gamma)]^2 (\gamma - r)^p}. \tag{12}$$

Obviously,

$$\widehat{U}_m(s)\widehat{\nu}(s) = \frac{\widehat{\nu}(s)\widehat{W}(s)(s-r)^p}{\Pi_{j=1}^l (s-s_j)^{n_j}} = \widehat{\nu}(s)\widehat{W}(s)\left[1+\sum_{j=1}^{l}\sum_{k=1}^{n_j}\frac{C_{jk}}{(s-s_j)^k}\right].$$

We write

$$\begin{aligned}\frac{\widehat{\nu}(s)\widehat{W}(s)}{(s-s_j)^k} &= \frac{\widehat{\nu}(s_j)\widehat{W}(s_j)}{(s-s_j)^k} + \frac{\widehat{\nu}(s)\widehat{W}(s)-\widehat{\nu}(s_j)\widehat{W}(s_j)}{(s-s_j)^k}\\ &= \sum_{i=0}^{k-1}\frac{w_{i,j}(s_j)}{(s-s_j)^{k-i}} + w_{k,j}(s),\end{aligned}$$

where $w_{0,j}(s) \stackrel{\text{def}}{=} \widehat{\nu}(s)\widehat{W}(s)$, $w_{p,j}(s) \stackrel{\text{def}}{=} [w_{p-1,j}(s) - w_{p-1,j}(s_j)]/(s-s_j)$, $p = 1,\dots,k$. By Theorem 2, w_p, $j(s)$ is the Laplace transform of the measure $W_{p,j} \stackrel{\text{def}}{=} T(s_j)^p(\nu * W) \in \mathfrak{Sl}(\tau)$ and $l(W_{p,j}) = l(\nu * W)/(\gamma - s_j)^p$. As a result, we have, by the uniqueness of the Laurent series expansion, that

$$\widehat{\nu}(s)\widehat{W}(s)\sum_{j=1}^{l}\sum_{k=1}^{n_j}\frac{C_jk}{(s-s_j)^k} = \sum_{j=1}^{l}\sum_{k=1}^{n_j}(-1)^k\frac{B_{jk}^{(\nu)}}{(s-s_j)^k} + \sum_{j=1}^{l}\sum_{k=1}^{n_j} C_{jk}w_{k,j}(s).$$

We set $R_{\nu,m} \stackrel{\text{def}}{=} \nu * W + \sum_{j=1}^{l}\sum_{k=1}^{n_j} C_{jk}T(s_j)^k(\nu * W)$. Then, by the above,

$$\widehat{U}_m(s)\widehat{\nu}(s) = \sum_{j=1}^{l}\sum_{k=1}^{n_j}(-1)^k\frac{B_{jk}^{(\nu)}}{(s-s_j)^k} + \widehat{R}_{\nu,m}(s); \tag{13}$$

moreover, $R_{\nu,m} \in \mathfrak{Sl}(\tau)$ and

$$l(R_{\nu,m}) = l(\nu * W)\left[1+\sum_{j=1}^{l}\sum_{k=1}^{n_j}\frac{C_jk}{(\gamma-s_j)^k}\right] = l(\nu * W)\frac{(\gamma-r)^p}{\Pi_{j=1}^l(\gamma-s_j)^{n_j}}. \tag{14}$$

In virtue of (5) and (12), we have

$$\begin{aligned}l(\nu * W) &= l(\nu)\widehat{W}(\gamma) + \widehat{\nu}(\gamma)l(W)\\ &= \frac{\Pi_{j=1}^l(\gamma-s_j)^{n_j}}{(\gamma-r)^p}\left[\frac{l(\nu)}{1-m\widehat{F}(\gamma)} + \frac{\widehat{\nu}(\gamma)ml(F)}{[1-m\widehat{F}(\gamma)^2]}\right].\end{aligned}$$

Passing in (13) from the Laplace transforms to the corresponding measures, we get the reprensentation (9). Substituting the value $l(\nu * W)$ into (14) gives (10). This completes the proof of Theorem 3. ■

Setting $\nu = \delta$ in Theorem 3, we obtain a representation for the renewal measure U_m.

COROLLARY 1. *Let $\sum_{k=1}^{n_j}(-1)^k B_{jk}/(s-s_j)^k$ be the principal part of the Laurent series expansion of the analytic function $\widehat{U}_m(s) = 1/[1-m\widehat{F}(s)]$ about the point $s = s_j \in \mathcal{Z}$, and let the hypotheses of Theorem 3 be satisfied. Then*

$$U_m = \sum_{j=1}^{l}\sum_{k=1}^{n_j} B_{jk}\mathcal{E}_j^{k*} + R_m, \tag{15}$$

where $R_m \in \mathfrak{Sl}(\tau)$ and $l(R_m) = m\,l(F)/[1-m\widehat{F}(\gamma)]^2$.

REMARK 1. The condition $m^n(F^{n*})\widehat{_s}(\gamma) < 1$, which we used in the proof of the invertibility of the element U, — *call it condition* $(\mathfrak{S})$ — is also necessary in order that the remainder term R_m in (15) satisfy the inequality $\int_0^\infty e^{\gamma x}\,|R_m|(dx) < \infty$ implied by $R_m \in \mathfrak{Sl}(\tau)$. Actually, suppose that $\int_0^\infty e^{\gamma x}\,|R_m|(dx) < \infty$, but condition $(\mathfrak{S})$ does not hold. Then there exist sets A_k of Lebesgue measure zero such that

$$m^k \int_{A_k} e^{\gamma x}(F^{k*})_s(dx) \geq 1, \qquad k = 1, 2, \ldots.$$

Put $A = \cup_{k=1}^{\infty} A_k$. The set A has Lebesgue measure zero. On the one hand,

$$\int_A e^{\gamma x} U_m(dx) \geq \sum_{k=1}^{\infty} m^k \int_{A_k} e^{\gamma x}(F^{k*})_s(dx) = \infty,$$

and on the other hand (see (15)),

$$\int_A e^{\gamma x} U_m(dx) = \int_A e^{\gamma x} R_m(dx) \leq \int_{\boldsymbol{R}} e^{\gamma x}|R_m|(dx) < \infty.$$

This contradiction shows that condition $(\mathfrak{S})$ is necessary for $R_m \in \mathfrak{Sl}(\tau)$.

The next theorem deals with the case $m\widehat{F}(\gamma) = 1$ for $\gamma > 0$. First we prove, however, the following simple lemma.

LEMMA 3. *Let μ and ν be some measures. Suppose that the integrals $\int_{\boldsymbol{R}}|x|\, e^{\gamma x}|\nu|(dx)$ and $\int_{\boldsymbol{R}}|x|e^{\gamma x}|\mu|(dx)$ are finite. Then*

$$T(\gamma)(\mu * \nu) = [T(\gamma)\mu] * \nu + \widehat{\mu}(\gamma)T(\gamma)\nu.$$

Proof. The assertion of the lemma follows from the equality

$$\frac{\widehat{\mu}(s)\widehat{\nu}(s) - \widehat{\mu}(\gamma)\widehat{\nu}(\gamma)}{s-\gamma} = \frac{[\widehat{\mu}(s) - \widehat{\mu}(\gamma)]\widehat{\nu}(s)}{s-\gamma} + \frac{\widehat{\mu}(\gamma)[\widehat{\nu}(s) - \widehat{\nu}(\gamma)]}{s-\gamma}. \qquad \blacksquare$$

Denote by $\mathcal{E}_\gamma$ the measure with the density $\mathbf{1}_{(0,\infty)}(x)e^{-\gamma x}$.

THEOREM 4. *Suppose that $m\widehat{F}(\gamma) = 1$ for $\gamma > 0$, $\int_0^\infty x^2 e^{\gamma x} F(dx) < \infty$ and $m^n (F^{n*})\widehat{}_s(\gamma) < 1$ for some $n \geq 1$. Let ν_1 and ν_2 be non-negative measures and $\nu = \nu_1 - \nu_2$. If the measures $T(\gamma)\nu_1$, $T(\gamma)\nu_2$ and $T(\gamma)^2 F$ belong to the Banach algebra $\mathfrak{S}\mathfrak{l}(\tau)$, then the convolution $\nu * U_m$ admits the representation*

$$\nu * U_m = B_\gamma^{(\nu)} \mathcal{E}_\gamma + R_{\nu,m}, \tag{16}$$

where $B_\gamma^{(\nu)} = \widehat{\nu}(\gamma)/[m\widehat{F}'(\gamma)]$ and $R_{\nu,m} \in \mathfrak{S}\mathfrak{l}(\tau)$; moreover,

$$l(R_{\nu,m}) = \frac{\widehat{\nu}(\gamma)\, l[T(\gamma)^2 F]}{m[\widehat{F}'(\gamma)]^2} - \frac{l[T(\gamma)\nu]}{m\widehat{F}'(\gamma)}. \tag{17}$$

Proof. Choose $r > \gamma$ and consider the function

$$u(s) \stackrel{\text{def}}{=} \frac{[1 - m\widehat{F}(s)](s - r)}{s - \gamma}, \qquad 0 \leq \Re s \leq \gamma.$$

The value $u(\gamma)$ is defined by continuity as $m\widehat{F}'(\gamma)(r - \gamma)$.We have $u(s) = 1 - m\widehat{F}(s) + (r - \gamma)m[T(\gamma)F]\widehat{}(s)$. By Lemma 2, both the measure $T(\gamma)F$ and the distribution F belong to the Banach algebra $\mathfrak{S}\mathfrak{l}(\tau)$; moreover, $l[T(\gamma)F] = l(F) = 0$. Hence $u(s)$ is the Laplace transform $\widehat{U}(s)$ of some measure $U \in \mathfrak{S}\mathfrak{l}(\tau)$ with $l(U) = 0$. Using standard arguments (see the proof of Theorem 3), we see that there exists an inverse element $W \stackrel{\text{def}}{=} U^{-1} \in S(0, \gamma)$ with Laplace transform equal to $1/u(s)$. By Theorem 1, $W \in \mathfrak{S}\mathfrak{l}(\tau)$ and $l(W) = 0$. Further,

$$\begin{aligned} \widehat{U}_m(s)\widehat{\nu}(s) &= \frac{\widehat{\nu}(s)}{u(s)} \frac{s - r}{s - \gamma} = \frac{\widehat{\nu}(s)}{u(s)} + \frac{\widehat{\nu}(s)}{u(s)} \frac{\gamma - r}{s - \gamma} \\ &= \frac{\widehat{\nu}(s)}{u(s)} + \frac{\widehat{\nu}(\gamma)}{u(\gamma)} \frac{\gamma - r}{s - \gamma} + \left[\frac{\widehat{\nu}(s)}{u(s)} - \frac{\widehat{\nu}(\gamma)}{u(\gamma)}\right] \frac{\gamma - r}{s - \gamma}. \end{aligned} \tag{18}$$

We set $R_{\nu,m} \stackrel{\text{def}}{=} \nu * W + T(\gamma)(\nu * W)(\gamma - r)$. Then relation (18) can be rewritten as

$$\widehat{U}_m(s)\widehat{\nu}(s) = -\frac{B_\gamma^{(\nu)}}{s - \gamma} + \widehat{R}_{\nu,m}(s).$$

Passing in this equality from the Laplace transforms to the corresponding measures, we establish the desired relation (16). It remains to show that $R_{\nu,m} \in \mathfrak{S}\mathfrak{l}(\tau)$ and verify the validity of (17). In view of Lemma 2 and the hypotheses of the theorem, we have

$$T(\gamma)U = -mT(\gamma)F + (r - \gamma)mT(\gamma)^2 F \in \mathfrak{S}\mathfrak{l}(\tau)$$

and $l[T(\gamma)U] = (r-\gamma)ml[T(\gamma)^2F]$. Note that $T(\gamma)W = -W * T(\gamma)U/u(\gamma)$. In fact,

$$\left[\frac{1}{u(s)} - \frac{1}{u(\gamma)}\right]\frac{1}{s-\gamma} = \frac{1}{u(s)u(\gamma)}\frac{u(\gamma)-u(s)}{s-\gamma} = -\frac{[W * T(\gamma)U]\widehat{\ }(s)}{u(\gamma)}.$$

By Lemma 3,

$$\begin{aligned} T(\gamma)(\nu * W) &= [T(\gamma)\nu] * W + \widehat{\nu}(\gamma)T(\gamma)W \\ &= [T(\gamma)\nu] * W - \widehat{\nu}(\gamma)W * T(\gamma)U/u(\gamma). \end{aligned}$$

Consequently, the hypotheses of the theorem and equality (5) imply that $T(\gamma)(\nu * W) \in \mathfrak{S}\mathfrak{l}(\tau)$ and

$$\begin{aligned} l[T(\gamma)(\nu * W)] &= \frac{l[T(\gamma)\nu]}{u(\gamma)} - \frac{\widehat{\nu}(\gamma)}{u(\gamma)}\frac{l[T(\gamma)U]}{u(\gamma)} \\ &= \frac{1}{u(\gamma)}\left\{l[T(\gamma)\nu] - m(r-\gamma)\frac{\widehat{\nu}(\gamma)}{u(\gamma)}l[T(\gamma)^2F]\right\}. \end{aligned}$$

Taking into account the equalities $l(W) = l(\nu) = 0$, we have $l(\nu * W) = 0$. As a result, we obtain

$$l(R_{\nu,m}) = \frac{\gamma - r}{u(\gamma)}\left\{l[T(\gamma)\nu] - m(r-\gamma)\frac{\widehat{\nu}(\gamma)}{u(\gamma)}l[T(\gamma)^2F]\right\}.$$

Recalling that $u(\gamma) = m\widehat{F}'(\gamma)(r-\gamma)$, we see that (17) holds. The proof of the theorem is complete. ■

4. The expected number of particles

Let $Z(t)$ be the number of particles at time t, and let F be the lifetime distribution of a single particle. The expectation $\mu(t) = EZ(t)$ satisfies the renewal equation (3). Recall that $m = f'(1)$, where $f(s) = \sum_{k=0}^{\infty} p_k s^k$ is the generating function of the offspring number generated by a single particle. The function $\mu(t) = \int_0^t [1 - F(t-y)]U_m(dy)$ is a solution of the equation (3). We shall assume that $m \neq 1$ since otherwise $\mu(t) \equiv 1$.

THEOREM 5. *Suppose the distribution F belongs to the Banach algebra $\mathfrak{S}\mathfrak{l}(\tau)$. Assume that $m^n(F^{n*})\widehat{_s}(\gamma) < 1$ for some $n \geq 1$ and $m\widehat{F}(s) \neq 1$ for $\Re s = \gamma$. Then*

$$\mu(t) = -\sum_{j=1}^{l} e^{-s_j t}\sum_{k=1}^{n_j} B_{jk}^{(\delta - F)}\sum_{p=0}^{k-1}\frac{t^p}{p!\,s_j^{k-p}} + \rho(t), \tag{19}$$

where

$$\lim_{t\to\infty} \frac{\rho(t)}{\tau(t)} = \frac{(1-m)\, l(F)}{[1-m\widehat{F}(\gamma)]^2}. \tag{20}$$

Proof. We apply Theorem 3 with $\nu = \delta - F$. Put $\rho(t) = -R_{\delta-F,m}((t,\infty))$. We obtain

$$\mu(t) = (\delta - F) * U_m([0,t]) = \sum_{j=1}^{l}\sum_{k=1}^{n_j} B_{jk}^{(\delta-F)} \mathcal{E}_j^{k*}([0,t]) + R_{\delta-F,m}([0,\infty)) + \rho(t).$$

To complete the proof, it remains to note that

$$\begin{aligned} R_{\delta-F,m}([0,\infty)) &= \lim_{s\to 0}\left\{ \frac{1-\widehat{F}(s)}{1-m\widehat{F}(s)} - \sum_{j=1}^{l}\sum_{k=1}^{n_j} (-1)^k \frac{B_{jk}^{(\delta-F)}}{(s-s_j)^k} \right\} \\ &= -\sum_{j=1}^{l}\sum_{k=1}^{n_j} \frac{B_{jk}^{(\delta-F)}}{s_j^k} \end{aligned}$$

and

$$\mathcal{E}_j^{k*}([0,t]) = -e^{-s_j t} \sum_{p=0}^{k-1} \frac{t^p}{p!\, s_j^{k-p}} + \frac{1}{s_j^k}.$$

The proof of the theorem is complete. ■

Remark 2. If the Malthusian parameter $\alpha = \alpha(m, F)$ exists (this means that $\mathcal{Z} \neq \emptyset$), then Theorem 5 makes more precise the assertions of Theorems 3A and 3B in [2, Chapter IV] under the additional assumptions: (i) the tail of F can be compared with the tail of some distribution from the class $S(\gamma), \gamma \geq 0$, and (ii) $m^n (F^{n*})\widehat{_s}(\gamma) < 1$ for some $n \geq 1$. In this case $\alpha = -s_1$, $n_1 = 1$ and equality (19) may be rewritten in the form

$$\mu(t) = \frac{(m-1)e^{\alpha t}}{\alpha m^2 \int_0^\infty y e^{-\alpha y} F(dy)} - \sum_{j=2}^{l} e^{-s_j t} \sum_{k=1}^{n_j} B_{jk}^{(\delta-F)} \sum_{p=0}^{k-1} \frac{t^p}{p!\, s_j^{k-p}} + \rho(t).$$

Remark 3. When $F \in S(\gamma)$ and $m\widehat{F}(\gamma) < 1$, Theorem 5 contains Theorem 1 from [4] as a particular case. To see this, it suffices to note that in this case $\mathcal{Z} = \emptyset$, the condition $m^n (F^{n*})\widehat{_s}(\gamma) < 1$ is automatically fulfilled for $n = 1$, $\mu(t) = \rho(t)$ and relation (20) coincides with (2). Here we can take the distribution F itself as the "normalizing" distribution G, which implies $l(F) = 1$.

Consider the situation when the Malthusian parameter is equal to $-\gamma$.

THEOREM 6. *Suppose* $m\widehat{F}(\gamma) = 1$ *for* $\gamma > 0$, $\int_0^\infty x^2 e^{\gamma x} F(dx) < \infty$ *and* $m^n (F^{n*})\widehat{_s}(\gamma) < 1$ *for some* $n \geq 1$. *If the measure* $T(\gamma)^2 F$ *belongs to the Banach algebra* $\mathfrak{Sl}(\tau)$, *then*

$$\mu(t) = \frac{\widehat{F}(\gamma) - 1}{\gamma m \widehat{F}'(\gamma)} e^{-\gamma t} + \rho(t),$$

where

$$\lim_{t\to\infty} \frac{\rho(t)}{\tau(t)} = \frac{\widehat{F}(\gamma) - 1}{m[\widehat{F}'(\gamma)]^2} \cdot \lim_{t\to\infty} \frac{\int_t^\infty e^{-\gamma y} \int_y^\infty \int_u^\infty e^{\gamma v} F(dv)\,du\,dy}{\tau(t)}.$$

Proof. Apply Theorem 4 with $\nu = \delta - F$. Put $p(t) = -R_{\delta-F,m}((t,\infty))$. We get

$$\mu(t) = B_\gamma^{(\delta-F)} \mathcal{E}_\delta([0,t]) + R_{\delta-F,m}([0,\infty)) + \rho(t).$$

Further,

$$R_{\delta-F,m}([0,\infty)) = \lim_{s\to 0} \left\{ \frac{1-\widehat{F}(s)}{1-m\widehat{F}(s)} + \frac{B_\gamma^{(\delta-F)}}{s-\gamma} \right\} = \frac{B_\gamma^{(\delta-F)}}{-\gamma} = \frac{\widehat{F}(\gamma)-1}{\gamma m \widehat{F}'(\gamma)}.$$

Finally,

$$\lim_{t\to\infty} \frac{\rho(t)}{\tau(t)} = -l(R_{\delta-F,m}) = \frac{[\widehat{F}(\gamma)-1] l[T(\gamma)^2 F]}{m[\widehat{F}'(\gamma)]^2}$$

since, by Lemma 2, $l[T(\gamma)(\delta - F)] = -l[T(\gamma)F] = 0$. This completes the proof of the theorem. ■

REMARK 4. Theorem 6 makes more precise the assertions of Theorems 3A and 3B from [2, Chapter IV] in case the Malthusian parameter is equal to γ and $m^n(F^{n*})_s\widehat{\ }(\gamma) < 1$ for some $n \geq 1$.

REMARK 5. The results of the present paper will remain valid if in their statements we replace the Banach algebra $\mathfrak{Sl}(\tau)$ with $\mathfrak{S}(\tau)$ (or with $\mathfrak{So}(\tau)$). In this case the formula for $\rho(t)$ in Theorem 5 takes on the following form: $\rho(t) = O(\tau(t))$ (or $\rho(t) = o(\tau(t))$) as $t \to \infty$. Moreover, the total variation of the function $\rho(x)$ in the interval (t,∞) behaves like $O(\tau(t))$ (or $o(\tau(t))$) as $t \to \infty$ if $F \in \mathfrak{S}(\tau)$ (or $F \in \mathfrak{So}(\tau)$). A similar remark applies to Theorem 6 if we assume $T(\gamma)^2 F \in \mathfrak{S}(\tau)$ (or $T(\gamma)^2 F \in \mathfrak{So}(\tau)$).

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