YOKOHAMA MATHEMATICAL
JOURNAL VOL. 49, 2002

ON PROPERLY NESTED NORMAL IMMERSIONS OF THE CIRCLE INTO THE PLANE WITH TITUS CONDITION

By

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(Received August 25, 1999; Revised: December 12, 2001)

Abstract. For a properly nested normal immersion of the unit 1-sphere into the plane, Titus [3] gave a necessary and sufficient condition to extend to an immersion of the unit disk into the plane. Francis [1] also proved the uniqueness of the extension up to topological equivalence using the result of Blank [2]. In this paper we give an elementary proof of these two theorems by using a system of inequalities and the mean value theorem.

1. Preliminary and Main Theorem

We work in piecewise smooth category.

An immersion of a 1-sphere into the plane is said to be *normal* if the map is in general position. Hence the singularities of the map consist of double points.

For a smooth immersion f of a 1-sphere into the plane, the *tangent winding number* of the map f is the mapping degree of the map $f'(x)=\dfrac{(\mathrm{grad}\, f)_x}{||(\mathrm{grad}\, f)_x||}$. The tangent winding number is also defined for a piecewise smooth immersion by smoothing corners.

Let D denote a disk in the plane. Let S be the positively oriented boundary of D. We assume that the boundary S possesses a suitable parameter function δ of the closed interval $[0,2\pi]$ onto S such that the map δ is injective on the set $[0,2\pi)$ and that the orientation of S coincides with that of S induced from the standard orientation of $[0,2\pi]$ by δ.

Let f be a normal immersion of S into the plane. Let

$$f_*=f\circ\delta.$$

Any normal immersion in this paper is assumed to satisfy the following additional two conditions:

(A1) $f_*(0)$ is not a double point, and

2000 Mathematics Subject Classification: 57R42; Secondary 57Q35
Key words and phrases: immersion, extention, properly nested

(A2) $f_*(0)$ is a boundary point of the complementary unbounded region of the image of f.

Let $P_1, P_2, \cdots, P_n$ be the double points of the map f. For each double point P_i, let $0 < t_i < t_i' < 2\pi$ be the real numbers with $f_*(t_i) = f_*(t_i') = P_i$. Without loss of generality we can assume that $0 < t_1 < t_2 < \cdots < t_n < 2\pi$. There exists a positive real number ε such that

(B1) for each $i = 1, 2, \cdots, n$, two arcs $\alpha_i = f_*([t_i - \varepsilon, t_i + \varepsilon])$ and $\beta_i = f_*([t_i' - \varepsilon, t_i' + \varepsilon])$ are simple (see Figure 1),
(B2) both of the arcs α_i and β_i contain only one double point of the map f, which is P_i, and
(B3) $(\alpha_i \cup \beta_i) \cap (\alpha_j \cup \beta_j) = \emptyset \quad (i \neq j)$.

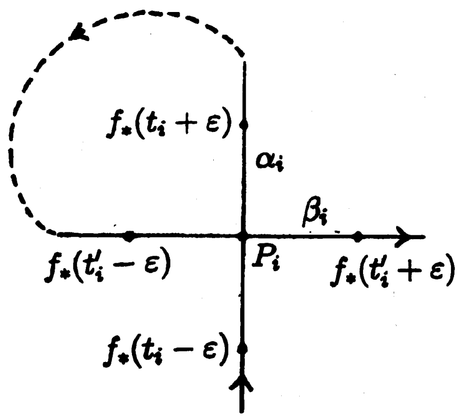


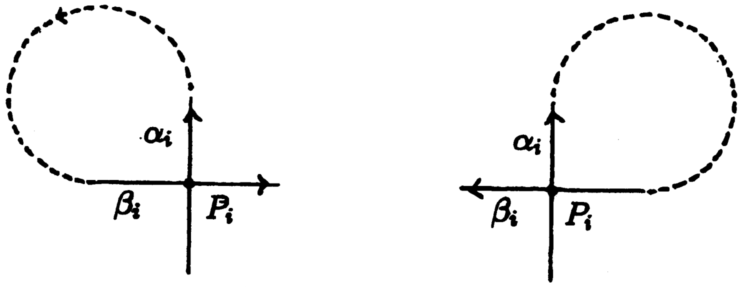
Figure 1

For each $j = 1, 2, \cdots, n$, let

$$I_j = \text{ the closed interval } [t_j, t_j'].$$

The map f is said to be *properly nested* provided that for each pair of integers $1 \leqq j < k \leqq n$, only one of following cases occurs: $I_j \supset I_k$, $I_j \subset I_k$, or $I_j \cap I_k = \emptyset$. In the words of graph theory, the map f is properly nested if each double point is a cut point of the graph $f_*([0, 2\pi])$. From now on we assume that all the maps are properly nested.

Now Titus defined the two kinds of functions $\mu(P_i, f)$ and $\lambda(P_i, f)$ for the immersion f as follows: The map f naturally induces the orientations of the arcs α_i and β_i from that of the 1-sphere S. If β_i crosses α_i from left to right with respect to the direction of α_i, we define $\mu(P_i, f) = +1$, otherwise $\mu(P_i, f) = -1$ (see Figure 2). The function $\lambda(P_i, f)$ is defined to be $\sum_{I_j \subset I_i} \mu(P_j, f)$.



$\mu(P_i, f) = +1$ $\mu(P_i, f) = -1$

Figure 2

A double point P_i is said to be *maximal* if $I_i \cap I_j \neq \emptyset$ implies $I_i \supset I_j$. We say that a properly nested normal immersion f satisfies *Titus Condition* if the map satisfies the following two conditions:

(TC1) for any maximal double point P_i, $\lambda(P_i, f) = 0$, and
(TC2) for any double point P_j, $\lambda(P_j, f) \leqq 0$.

Let

$$I_0 = [0, 2\pi] \quad \textit{and}$$
$$C_i = f_*(Cl(I_i - \bigcup_{I_j \subsetneqq I_i} I_j)) \ (i = 0, 1, \cdots, n)$$

where $Cl(\cdots)$ means the closure of $(\cdots)$.

Each simple closed curve C_i possesses the orientation induced from the one of I_0 by the map f_*. We define $\tau(C_i) = +1$ if the orientaion of the circle C_i is positive, otherwise $\tau(C_i) = -1$. Then we have

$$\text{the tangent winding number of } f = \sum_{i=0}^{n} \tau(C_i).$$

Then we have the following three propositions.

PROPOSITION 1. *If the properly nested normal immersion f satisfies Condition* (A1) *and* (A2), *then we have*

$$\mu(P_i, f) = \tau(C_i) \quad (i = 1, 2, \cdots, n).$$

PROPOSITION 2. *If the properly nested normal immersion f satisfies Titus Condition, then for every double point P_i on the circle C_0, we have $\mu(P_i, f) = +1$.*

PROPOSITION 3. *If the properly nested normal immersion f satisfies Titus Condition, then we have*

$$\textit{the tangent winding number of } f = \tau(C_0).$$

The circle C_0 is called the *TWN circle* of the map f. By Proposition 3, the tangent winding number of the properly nested normal immersion f with Titus Condition is ± 1. Further if the map f extends to an orientation preserving immersion of the unit disk into the plane, the tangent winding number of the map f is $+1$. Therefore throughout this paper, we assume the following condition for the tangent winding number.

TWN Condition : *The tangent winding number of the map f is $+1$.*

Note. By Proposition 3 we have $\tau(C_0) = +1$. This means that if the map f extends to an immersion of the unit disk into the plane, then the extended map is orientation preserving.

THEOREM 1 (Titus [3]). *A properly nested normal immersion f of the unit 1-sphere into the plane extends to an orientation preserving immersion of the unit disk into the plane if and only if the map satisfies Titus Condition.*

THEOREM 2 (Francis [1]). *If a properly nested normal immersion f of the unit 1-sphere into the plane extends to an immersion of the unit disk into the plane, the extension is unique up to topological equivalence.*

If the properly nested normal immersion f does not have any double point, the above two theorems are true. Hence we assume that the map possesses a double point. Since the tangent winding number of the map f is $+1$, the map f must have a double point P with $\mu(P, f) = -1$. Let ν be the integer with $\mu(P_i, f) = +1$ $(0 < i < \nu)$ but $\mu(P_\nu, f) = -1$. The arc $f_*([0, t_\nu])$ is called the *principal arc*. Any double point P_i on the principal arc is said to be *principal* provided that $0 < i < \nu$ and $I_i \not\subset [0, t_\nu]$ (See Figure 3).

Let γ_f be an embedding of the unit interval $[0, 1]$ into the plane such that

(C1) $\gamma_f(0) = f_*(t'_\nu - \varepsilon)$ where ν is the number defined just above, and ε is the positive real number satisfies $(B1) \sim (B3)$,
(C2) $\gamma_f(1)$ lies in the complementary unbounded region of the image of f,
(C3) the image of γ_f is situated on the right side of the principal arc, and
(C4) the intersection of the image of γ_f and the image of f is equal to the set $\{f_*(t'_i + \varepsilon) \mid P_i \text{ is principal }\} \cup \{f_*(t'_\nu - \varepsilon)\}$.

The simple arc γ_f is called an *associated principal arc*. The set $\{P_1, P_2, \cdots, P_\nu, \gamma_f\}$ is called a *principal set* of the map f (see Figure 4)

If the map f satisfies Titus Condition, we have

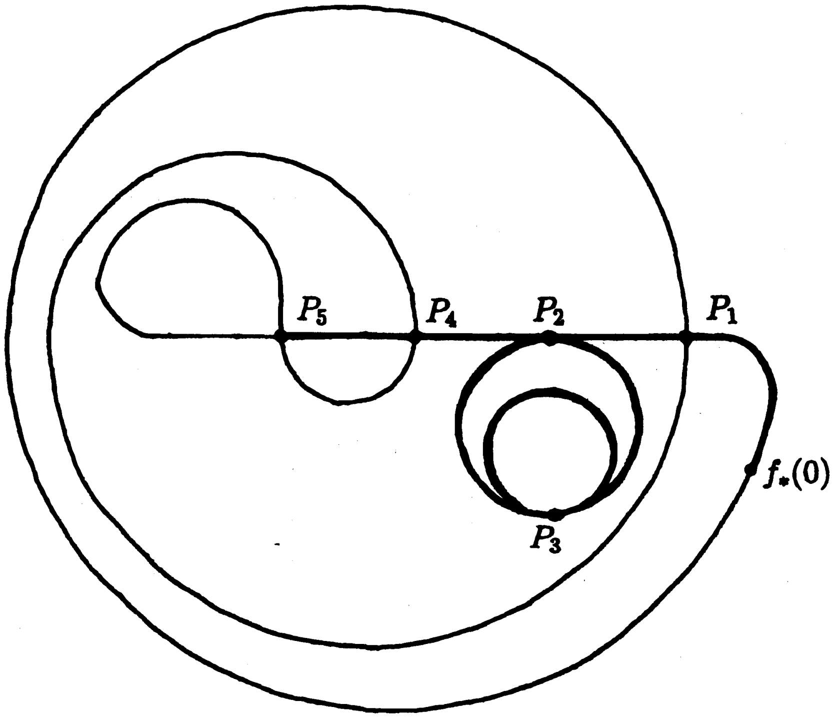


Figure 3 The heavier graph: the principal arc. P_1, P_4 are the principal but P_2, P_3 are not.

(D1) $\nu \geqq 2$,
(D2) $f_*([0, t_\nu])$ is a simple arc, and
(D3) all the double point P_i $(0 < i < \nu)$ is principal.

For, if $I_i \subset [0, t_\nu]$ for some $0 < i < \nu$, then $\mu(P_j, f) = +1$ for any double point P_j on $f_*(I_i)$. Hence $\lambda(P_i, f) > 0$. This contradicts to Condition (TC2). Hence all the double point P_i $(0 < i < \nu)$ is principal. Thus $f_*([0, t_\nu])$ is a simple arc.

Titus used two types of cuts to split an immersion (cf. [4]). A cut of Type I uses a double point of the immersion. A cut of Type II uses a simple curve. We do not use a cut of Type I, but a special curve for a cut of type II as follows. For each i with P_i principal, let

$$s_i = \gamma_f^{-1} \circ f_*(t_i' + \varepsilon),$$
$$G_i = \delta([t_\nu' - \varepsilon, t_i' + \varepsilon]), \textit{ and}$$
$$H_i = \text{the complementary arc of } G_i \text{ in the 1-sphere } S.$$

Let g_i and h_i be normal immersions of S into the plane such that

(E1) the maps g_i and f coincide on G_i,
(E2) the maps h_i and f coincide on H_i,

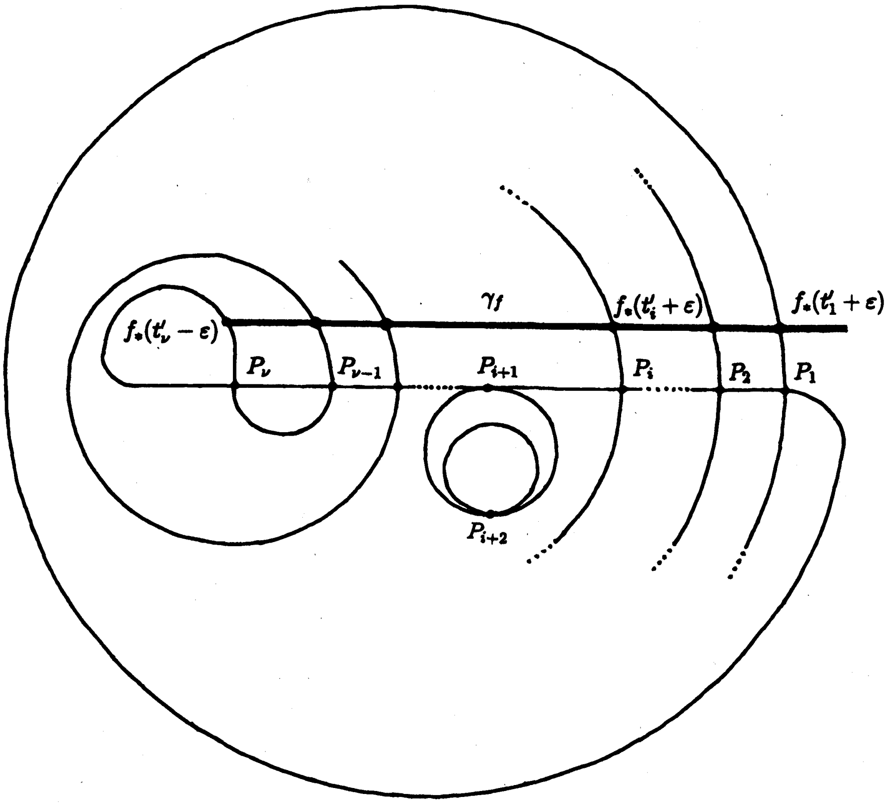


Figure 4 The heavier graph: an associated principal arc. If the map f satisfies Titus Condition, P_{i+1} and P_{i+2} do not exist.

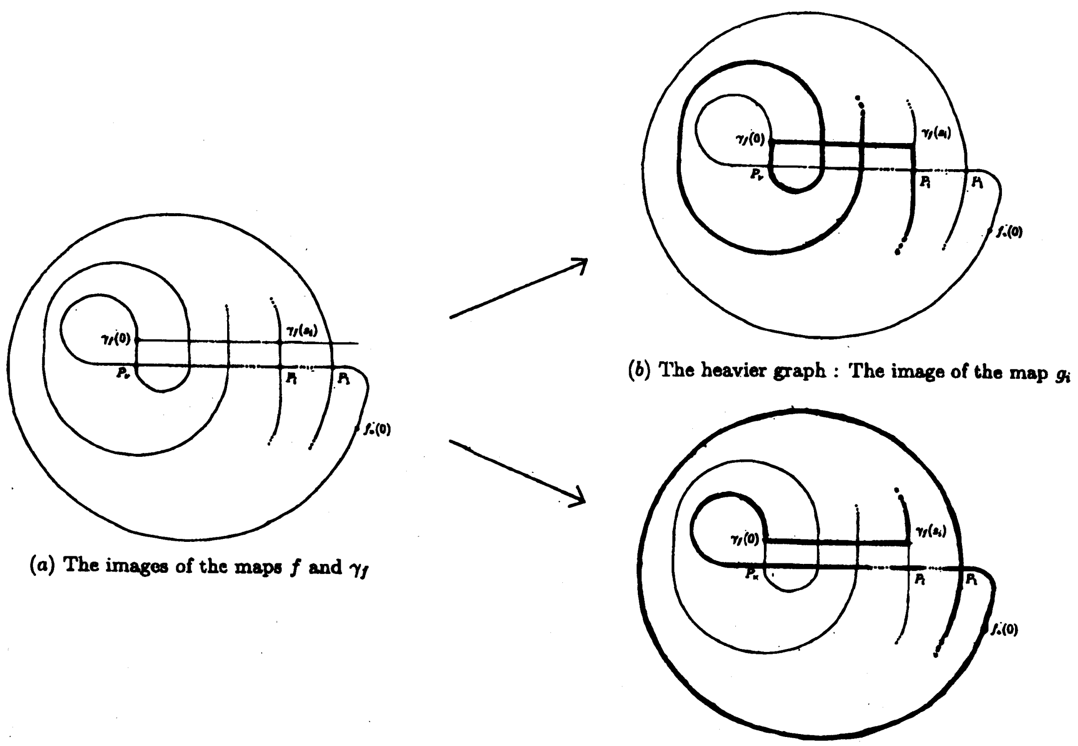
(E3) $g_i(H_i) = h_i(G_i) = \gamma_f([0, s_i])$, and
(E4) $g_i(\delta(0))$ is situated very close to $f_*(t'_i + \varepsilon)$.

We say that the maps g_i and h_i are obtained by *splitting the map f with respect to the double point P_i and the associated principal arc γ_f*. This is a Titus' cut of Type II (see Figure 5).

Note that the number of the double points of g_i and h_i are strictly less than that of the double points of f. Since the map f is properly nested, so are the maps g_i and h_i.

Now our main theorem is the following.

THEOREM 3. *Let f be a properly nested normal immersion which satisfies Titus Condition. Let $\{P_1, P_2, \cdots, P_\nu, \gamma_f\}$ be a principal set of f. Then there exists exactly one integer $0 < i < \nu$ such that the two maps g_i and h_i, obtained by splitting the map f with respect to P_i and γ_f, are properly nested normal immersions and satisfy Titus Condition and TWN Condition.*



(a) The images of the maps f and γ_f

(b) The heavier graph : The image of the map g_i

(c) The heavier graph : The image of the map h_i

Figure 5

2. Proof of propositions

We use all the notations in §1.

Proof of Proposition 1. Let D_i be the disk bounded by the simple closed curve C_i. Since the map f is properly nested, we have that $f_*(I_i) \cap f_*([0, 2\pi] - I_i) = \emptyset$. Namely

$$C_i \cap f_*([0, 2\pi] - I_i) = \emptyset$$

The condition $f_*(0) = f_*(2\pi)$ implies that $f_*([0, 2\pi] - I_i)$ is connected and contains the point $f_*(0)$. On the other hand, $f_*(0)$ must be outside of the disk D_i by Condition $(A2)$. Thus the set $f_*([0, 2\pi] - I_i)$ belongs to the outside of the disk D_i. Therefore the orientation of the simple closed curve C_i is positive if $\mu(P_i, f) = +1$, and the orientation of the simple closed curve C_i is negative if $\mu(P_i, f) = -1$ (see Figure 6). ■

Proof of Proposition 2. We use contradiction. Suppose $\mu(P_i, f) = -1$ for a double point P_i on the simple closed curve C_0. Let $P_{i_1}, P_{i_2}, \cdots, P_{i_k}$ be the other

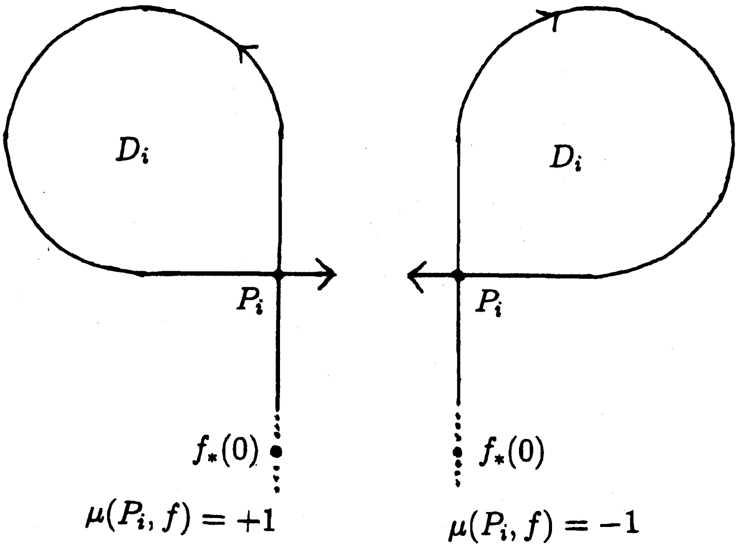


Figure 6

double points on C_i. Now

$$\begin{aligned}\lambda(P_i, f) &= \sum_{I_j \subset I_i} \mu(P_j, f) \\ &= \mu(P_i, f) + \sum_{I_j \subsetneq I_i} \mu(P_j, f) \\ &= \mu(P_i, f) + \sum_{t=1}^{k} \sum_{I_j \subset I_{i_t}} \mu(P_j, f) \\ &= \mu(P_i, f) + \sum_{t=1}^{k} \lambda(P_{i_t}, f)\end{aligned}$$

Since the point P_i is maximal, we have $\lambda(P_i, f) = 0$ by Condition (TC1). Hence we have

$$\sum_{t=1}^{k} \lambda(P_{i_t}, f) = -\mu(P_i, f) = 1 > 0$$

Therefore at least one of $\lambda(P_{i_1}, f), \lambda(P_{i_2}, f), \cdots, \lambda(P_{i_k}, f)$ must be positive. This contradicts to Conditon (TC2). ■

Proof of Proposition 3. Recall that $C_1, C_2, \cdots, C_n$ are simple closed curves corresponding to the double points of the map f. Let $P_{i_1}, P_{i_2}, \cdots, P_{i_k}$ be the double points of f on the simple closed curve C_0. Then

$$\begin{aligned}
\text{the tangent winding number of } f &= \sum_{j=0}^{n} \tau(C_j) \\
&= \tau(C_0) + \sum_{j=1}^{n} \tau(C_j) \\
&= \tau(C_0) + \sum_{j=1}^{n} \mu(P_j, f) \\
&= \tau(C_0) + \sum_{t=1}^{k} \sum_{I_j \subset I_{i_t}} \mu(P_j, f) \\
&= \tau(C_0) + \sum_{t=1}^{k} \lambda(P_{i_t}, f)
\end{aligned}$$

Since each P_{i_t} is maximal, we have $\lambda(P_{i_t}, f) = 0$. Therefore the tangent winding number of $f = \tau(C_0)$. ■

3. Proof of Theorem 3

We use all notations defined in §1.

Since the map f satisfies Titus Condition, the arc $f_*([0, t_\nu])$ is simple by Condition (TC2). Hence all the double points $P_1, P_2, \cdots, P_{\nu-1}$ are principal. For each $j = 1, 2, \cdots, \nu - 1$, let

$$Q_j = f_*(t'_j + \varepsilon). \tag{1}$$

For each $i = 1, 2, \cdots, \nu - 1$, let

$$\begin{aligned}
X_i &= \text{the set of the double points of } g_i \\
Y_i &= \text{the set of the double points of } h_i, \text{ and} \\
Z_i &= \{P_1, P_2, \cdots, P_{i-1}, P_{\nu+1}, P_{\nu+2}, \cdots, P_n\}.
\end{aligned}$$

Then we have the followings (see Figure 5):

$$\begin{aligned}
X_i \cup Y_i &= Z_i \cup \{Q_{i+1}, Q_{i+2}, \cdots, Q_{\nu-1}\}, \\
X_i &\supset \{Q_{i+1}, Q_{i+2}, \cdots, Q_{\nu-1}\}, \\
Y_i &\supset \{P_1, P_2, \cdots, P_{i-1}\}, \text{ and} \\
X_i \cap Y_i &= \emptyset.
\end{aligned}$$

Hence the numbers of double points of g_i and h_i are strictly less than that of the double points of f.

Now we shall find the integer i so that

(1) $\lambda(Q_{i+1}, g_i) = 0$, and $\lambda(Q_j, g_i) \leqq 0$ $(j = i+2, \cdots, \nu-1)$
i.e. Titus Condition for the double points $Q_{i+1}, Q_{i+2}, \cdots, Q_{\nu-1}$ of the map g_i,

(2) $\lambda(P, g_i) = \lambda(P, f) = 0$ for all the maximal double point P of g_i on the arc $f_*([t'_{i+1} + \varepsilon, t'_i))$,
i.e. Titus Condition for the double points of the map g_i on the simple arc $f_*([t'_{i+1} + \varepsilon, t'_i))$, and

(3) $\lambda(P_1, h_i) = 0$, and $\lambda(P_j, h_i) \leqq 0$ $(j = 2, 3, \cdots, i-1)$,
i.e. Titus Condition for the double points $P_1, P_2, \cdots, P_{i-1}$ of the map h_i.

Note that Titus Condition is satisfied for the other double points of the map h_i, because the map f satisfies Titus Condition. Similarly Titus Condition is satisfied for the other double points of the map g_i except the maximal double points of g_i which lie on the arc $f_*([t'_{i+1} + \varepsilon, t'_i))$.

Let

$$u = \sum_{I_q \subsetneqq I_\nu} \mu(P_q, f), \text{ and } v_0 = \sum_{I_q \not\subset I_1} \mu(P_q, f).$$

For each $j = 1, 2, \cdots, \nu-1$, let

$$v_j = \sum_{I_q \subset J_j} \mu(P_q, f), \text{ where } J_j = I_j - I_{j+1}.$$

Note that $u \leqq 0$, $v_0 = 0$, $v_j \leqq 0$ $(j = 1, 2, \cdots, \nu-1)$ by Titus Condition. From Titus Condition for the map f, we have the followings:

$$\begin{array}{llcl}
(F_1) & \lambda(P_{\nu-1}, f) & = & u + v_{\nu-1} \leqq 0, \\
(F_2) & \lambda(P_{\nu-2}, f) & = & u + v_{\nu-1} + v_{\nu-2} + 1 \leqq 0, \\
\vdots & \vdots & & \vdots \\
(F_j) & \lambda(P_{\nu-j}, f) & = & u + v_{\nu-1} + v_{\nu-2} + \cdots + v_{\nu-j} + j - 1 \leqq 0, \\
\vdots & \vdots & & \vdots \\
(F_{\nu-2}) & \lambda(P_2, f) & = & u + v_{\nu-1} + v_{\nu-2} + \cdots + v_2 + \nu - 3 \leqq 0, \\
(F_{\nu-1}) & \lambda(P_1, f) & = & u + v_{\nu-1} + v_{\nu-2} + \cdots + v_1 + \nu - 2 = 0, \\
(F_\nu) & & & v_0 = 0.
\end{array}$$

That is

$$\begin{array}{ll}
(F_j)^* & u + v_{\nu-1} + v_{\nu-2} + \cdots + v_{\nu-j} \leqq -(j-1) \quad (j = 1, 2, \cdots, \nu\text{-}2), \\
(F_{\nu-1})^* & u + v_{\nu-1} + v_{\nu-2} + \cdots + v_1 = -(\nu - 2), \\
(F_\nu)^* & v_0 = 0.
\end{array}$$

Under the condition that the map f satisfies Titus Condition, the map g_i

satisfies Titus Condition if and only if the following $\nu - i$ conditions are satisfied:

$$\begin{array}{lll}
(G^i_1) & \lambda(Q_{\nu-1}, g_i) = & v_{\nu-1} + 1 \leqq 0, \\
\vdots & \vdots & \vdots \\
(G^i_j) & \lambda(Q_{\nu-j}, g_i) = & v_{\nu-1} + v_{\nu-2} + \cdots + v_{\nu-j} + j \leqq 0, \\
\vdots & \vdots & \vdots \\
(G^i_{\nu-i-2}) & \lambda(Q_{i+2}, g_i) = & v_{\nu-1} + v_{\nu-2} + \cdots + v_{i+2} + \nu - i - 2 \leqq 0, \\
(G^i_{\nu-i-1}) & \lambda(Q_{i+1}, g_i) = & v_{\nu-1} + v_{\nu-2} + \cdots + v_{i+1} + \nu - i - 1 = 0, \\
(G^i_{\nu-i}) & & v_i = 0.
\end{array}$$

where the last condition $v_i = 0$ comes from Titus Condition for all the maximal double points on the set $f_*([t'_{i+1} + \varepsilon, t'_i))$ of the map g_i. That is

$$\begin{array}{ll}
(G^i_j)^* & v_{\nu-1} + v_{\nu-2} + \cdots + v_{\nu-j} \leqq -j \quad (j = 1, 2, \cdots, \nu\text{-}i\text{-}2), \\
(G^i_{\nu-i-1})^* & v_{\nu-1} + v_{\nu-2} + \cdots + v_{i+1} = -(\nu - i - 1), \\
(G^i_{\nu-i})^* & v_i = 0.
\end{array}$$

Under the condition that the map f satisfies Titus Condition, the map h_i satisfies Titus Condition if and only if the following i conditions are satisfied:

$$\begin{array}{lll}
(H^i_1) & \lambda(P_{i-1}, h_i) = & u + v_{i-1} + 1 \leqq 0, \\
\vdots & \vdots & \vdots \\
(H^i_j) & \lambda(P_{i-j}, h_i) = & u + v_{i-1} + v_{i-2} + \cdots + v_{i-j} + j \leqq 0, \\
\vdots & \vdots & \vdots \\
(H^i_{i-2}) & \lambda(P_2, h_i) = & u + v_{i-1} + v_{i-2} + \cdots + v_2 + i - 2 \leqq 0, \\
(H^i_{i-1}) & \lambda(P_1, h_i) = & u + v_{i-1} + v_{i-2} + \cdots + v_1 + i - 1 = 0, \\
(H^i_i) & & v_0 = 0.
\end{array}$$

where the last condition is Titus Conditon for maximal double points. That is

$$\begin{array}{ll}
(H^i_j)^* & u + v_{i-1} + v_{i-2} + \cdots + v_{i-j} \leqq -j \quad (j = 1, 2, \cdots, i\text{-}2), \\
(H^i_{i-1})^* & u + v_{i-1} + v_{i-2} + \cdots + v_1 = -(i - 1), \\
(H^i_i)^* & v_0 = 0.
\end{array}$$

Now we need two lemmata.

Lemma 1. *Under the condition that the map f satisfies Titus Condition, if the map g_i satisfies Titus Condition, so does the map h_i.*

Proof. From $(G^i_{\nu-i-1})^*$ and $(G^i_{\nu-i})^*$ for the map g_i, we have the following equation:

$$v_{\nu-1} + v_{\nu-2} + \cdots + v_i = -(\nu - i - 1).$$

For $j = 1, 2, \cdots, i-1$, put the equation into $(F_{\nu-i+j})^*$ to get $(H^i_j)^*$ for the map h_i. ■

LEMMA 2. *Under the condition that the map f satisfies Titus Condition, among the maps $g_1, g_2, \cdots, g_{\nu-1}$, at most one map g_i is able to satisfy Titus Condition.*

Proof. Suppose that g_i and g_k $(i \neq k)$ satisfy Titus Condition. Without loss of generality we can assume that $i < k$. Then from Condition $(G^k_{\nu-k-1})^*$ and $(G^k_{\nu-k})^*$ for the map g_k, we have the following equation:

$$v_{\nu-1} + v_{\nu-2} + \cdots + v_{k+1} + v_k = -(\nu - k - 1).$$

But by Condition $(G^i_{\nu-k})^*$ for the map g_i we have

$$v_{\nu-1} + v_{\nu-2} + \cdots + v_{k+1} + v_k \leqq -(\nu - k).$$

Hence $-(\nu - k - 1) \leqq -(\nu - k)$. This is a contradiction. ■

To finish the proof of Theorem 3 we must find a map g_i satisfies Titus Condition. There are two cases.

Case 1. $v_{\nu-1} = 0$. In this case, $g_{\nu-1}$ is the desired map. For, Condition $(G^{\nu-1}_1)^*$ for the map $g_{\nu-1}$ is to be checked. Since Condition $(G^{\nu-1}_1)^*$ for the map $g_{\nu-1}$ is $v_{\nu-1} = 0$. So nothing is to be checked.

Case 2. $v_{\nu-1} \leqq -1$. Let $A_0 = (0, v_{\nu-1})$. For $j = 1, 2, \cdots, \nu-1$, let

$$B_j = (j\ ,\ v_{\nu-1} + v_{\nu-2} + \cdots + v_{\nu-j}),\ \textit{and}$$
$$A_j = (j\ ,\ v_{\nu-1} + v_{\nu-2} + \cdots + v_{\nu-j} + v_{\nu-j-1}).$$

Let L be the broken line connecting the points $A_0, B_1, A_1, B_2, A_2, \cdots, A_{\nu-2}, B_{\nu-1}$ in succesion. Let

$$U = \{(x, y) \mid x + y > 0\},\ \textit{and}$$
$$V = \{(x, y) \mid x + y < 0\}.$$

Then the point A_0 belongs to the region V. Since $u \leqq 0$, Condition $(F_{\nu-1})^*$ implies

$$\nu - 1 + v_{\nu-1} + v_{\nu-2} + \cdots + v_1 = \nu - 1 - u - \nu + 2 = 1 - u > 0.$$

Thus the point $B_{\nu-1}$ belongs to the region U. Since the broken line L is descending from $A_0 \in V$ to $B_{\nu-1} \in U$ at each step width $= 1$ and each step height $\geqq 0$, the broken line L must penetrate the line $y = -x$ from the region V to the region U. Let $B_{\nu-i-1}$ be the highest penetration point, where $1 \leqq \nu - i - 1 < \nu - 1$ (see Figure 7). Then we have

$$B_{\nu-i-1} = A_{\nu-i-1} = (\nu - i - 1, -(\nu - i - 1)).$$

Since $\nu - (\nu - i - 1) = i + 1$, we have

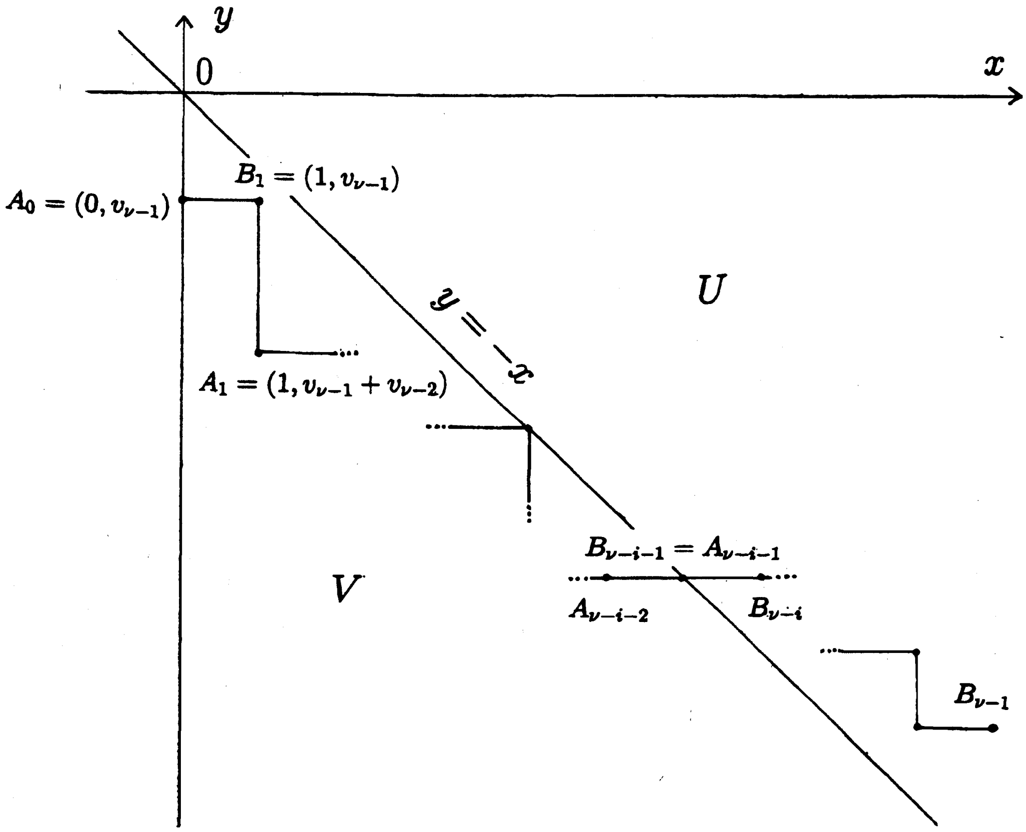


Figure 7

$$B_{\nu-i-1} = (\nu - i - 1, v_{\nu-1} + v_{\nu-2} + \cdots + v_{i+1}), \text{ and}$$
$$A_{\nu-i-1} = (\nu - i - 1, v_{\nu-1} + v_{\nu-2} + \cdots + v_{i+1} + v_i).$$

Thus $B_{\nu-i-1} = A_{\nu-i-1}$ implies $v_i = 0$. Hence the map g_i satisfies Condition $(G^i_{\nu-i})^*$. And $B_{\nu-i-1} = (\nu - i - 1, -(\nu - i - 1))$ implies

$$v_{\nu-1} + v_{\nu-2} + \cdots + v_{i+1} = -(\nu - i - 1).$$

Hence the map g_i satisfies Condition $(G^i_{\nu-i-1})^*$. Since $B_1, B_2, \cdots,$ $B_{\nu-i-1-1}$ belong to the closure of V, we have

$$v_{\nu-1} + v_{\nu-2} + \cdots + v_{\nu-j} \leqq -j \quad (j = 1, 2, \cdots, \nu\text{-}i\text{-}2).$$

Thus the map g_i satisfies Condition $(G^i_1)^* \sim (G^i_{\nu-i-2})^*$. Therefore the map g_i satisfies Titus Condition. Looking at TWN circles of the two maps g and h, it is easy to check that the two maps satisfy TWN Condition by Proposition 3. This completes the proof of Theorem 3. ■

4. Proof of Theorem 1 and Theorem 2

First we shall prove Theorem 1. Let f be a properly nested normal immersion of the unit 1-spehre into the plane. Let D be the unit disk. Let n be the number of double points of the map f. We use the induction argument on the number of double points of f.

Suppose that the properly nested normal immersion f satisfies Titus Condition. If $n = 0$, then the map f easily extends to an immersion of the disk D into the plane. Suppose $n > 0$. Let $\{P_1, P_2, \cdots, P_\nu, \gamma_f\}$ be a principal set of the map f. By Theorem 3 for some integer $0 < i < \nu$, the maps g_i and h_i, obtained by splitting the map f with respect to P_i and γ_f, satisfy Titus Condition and TWN Conditon. Since the numbers of the double points of g_i and h_i are less than that of double points of f, the maps g_i and h_i extend to immersions $\tilde{g}_i$ and $\tilde{h}_i$ of D into the plane by the induction hypothesis. Thus the map f extend to an immersion of D into the plane by using the two maps $\tilde{g}_i$ and $\tilde{h}_i$.

Conversely suppose that the map f extends to an orientation preserving immersion $\tilde{f}$ of D into the plane. Then the map f is regularly homotopic to the inclusion map of D into the plane. Hence the tangent winding number of f is $+1$. If $n = 0$, then the map f clearly satisfies Titus Condition. Suppose $n > 0$. Since the tangent winding number of the map f is $+1$, the map f must have a double point P with $\mu(P) = -1$. Let $\{P_1, P_2, \cdots, P_\nu, \gamma_f\}$ be a principal set of the map f.

Now we use the notations in §1. Let L be the image of γ_f. Then $\tilde{f}^{-1}(L)$ consists of simple arcs in D. One of the arcs connects the point $\delta(t'_\nu - \varepsilon)$ and a point $\delta(t'_i + \varepsilon)$ for some integer $0 < i < \nu$. This arc assures that the maps g_i and h_i, obtained by splitting the map f with respect to P_i and γ_f, extend to orientaion preserving immersions of D into the plane. Hence the two maps satisfy TWN Conditon. Of course, the numbers of double points of g_i and h_i are less than that of double points of f. Thus the maps g_i and h_i satisfy Titus Condition by the induction hypothesis. Since the map h_i satisfies Titus Condition, $f_*([0, t_\nu])$ is a simple arc. Hence all the points $P_1, P_2, \cdots, P_{\nu-1}$ are principal with respect to the map f.

Now we use the notations u and v_i ($i = 0, 1, \cdots, \nu-1$) in §2. Since the maps g_i and h_i satisfy Titus Condition, all the double points of f, except $P_1, P_2, \cdots, P_\nu$, satisfy Titus Condition. Thus we have that $u \leqq 0$, $v_i \leqq 0$ $(0 < i < \nu)$. To prove the map f satisfies Titus Conditon, we have to show that Condition $(F_j)^*$ ($j = 1, 2, \cdots, \nu-1$) in §2 hold. That is

$$\begin{array}{lll} (F_j)^* & u + v_{\nu-1} + v_{\nu-2} + \cdots + v_{\nu-j} \leqq -(j-1) & (j = 1, 2, \cdots, \nu\text{-}2), \\ (F_{\nu-1})^* & u + v_{\nu-1} + v_{\nu-2} + \cdots + v_1 = -(\nu-2), & \\ (F_\nu)^* & v_0 = 0. & \end{array}$$

Since the maps g_i and h_i satisfy Titus Condition , the same conditions $(G^i_j)^*$ ($j = 1, 2, \cdots, \nu\text{-}i$) and $(H^i_j)^*$ ($j = 1, 2, \cdots, i$) in §2 hold. That is

$$\begin{array}{lll} (G^i_j)^* & v_{\nu-1} + v_{\nu-2} + \cdots + v_{\nu-j} \leqq -j & (j = 1, 2, \cdots, \nu\text{-}i\text{-}2), \\ (G^i_{\nu-i-1})^* & v_{\nu-1} + v_{\nu-2} + \cdots + v_{i+1} = -(\nu - i - 1), & \\ (G^i_{\nu-i})^* & v_i = 0, & \\ (H^i_j)^* & u + v_{i-1} + v_{i-2} + \cdots + v_{i-j} \leqq -j & (j = 1, 2, \cdots, i\text{-}2), \\ (H^i_{i-1})^* & u + v_{i-1} + v_{i-2} + \cdots + v_1 = -(i-1), & \\ (H^i_i)^* & v_0 = 0. & \end{array}$$

For each $j = 1, 2, \cdots, \nu - i - 1$, Condition $(G^i_j)^*$ and $u \leqq 0$ implies $(F_j)^*$. From $(G^i_{\nu-i-1})^*$ and $(G^i_{\nu-i})^*$ we have the following equation:

$$v_{\nu-1} + v_{\nu-2} + \cdots + v_{i+1} + v_i = -(\nu - i - 1).$$

The equation and $u \leqq 0$ implies $(F_{\nu-i})^*$.

For each $j = 1, 2, \cdots, i-1$, the above equation and $(H^i_j)^*$ implies $(F_{\nu-i+j})^*$. Thus all the points $P_1, P_2, \cdots, P_{\nu-1}$ satisfy Titus Condition.

The point P_ν clearly satisfies Titus Condition, because the fact $u \leqq 0$ implies $\lambda(P_\nu, f) = u - 1 < 0$. Therefore the map f satisfies Titus Condition. This completes the proof of Theorem 1.

Now we shall prove Theorem 2. We shall prove only the case that the map f is a properly nested normal immersion of the 1-sphere into the plane which extends to an orientation preserving immersion of the unit disk D into the plane. Again we use the induction argument on the number of double points of the map f. Let n be the number of double points of f. If $n = 0$, then Theorem 2 is trivial. Suppose $n > 0$. Suppose that the map f extends to immersions $\tilde{f}$ and $\hat{f}$ of D into the plane. Then the map f satisfies Titus Condition by Theorem 1. Let $\{P_1, P_2, \cdots, P_\nu, \gamma_f\}$ be a principal set of the map f. By the same way as the one in the proof of Theorem 1, the immersion $\tilde{f}$ determines an integer i $(1 \leqq i < \nu)$ such that the maps g_i and h_i , obtained by splitting the map f with respect to P_i and γ_f, extend to immersions $\tilde{g}_i$ and $\tilde{h}_i$ of D into the plane respectively. Similarly the immersion $\hat{f}$ determines an integer j such that the maps g_j and h_j , obtained by splitting the map f with respect to P_j and γ_f, extend to immersions $\hat{g}_j$ and $\hat{h}_j$ of D into the plane respectively. Then the maps g_i, h_i, g_j, and h_j satisfy Titus Conditon by Theorem 1. Hence by Theorem 3, we have $i = j$. Thus the immersion g_i extends to immersions $\tilde{g}_i$ and $\hat{g}_j$ of D into the plane. Since the map g_i satisfies Titus Condition and the number of the double points of each map is less than n, the maps $\tilde{g}_i$ and $\hat{g}_j$ are topologically equivalent by the induction hypothesis. Similarly maps $\tilde{h}_i$ and $\hat{h}_j$ are topologically equivalent. Therefore maps $\tilde{f}$ and $\hat{f}$ are topologically equivalent. This completes the proof of Theorem 2.

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