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POSITIVELY RECURRENT MARKOV CHAINS AND THE STEPPING STONE MODEL AS A FLEMING-VIOT PROCESS

By

AKINOBU SHIMIZU AND TAKAHIRO SOSHI

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Abstract. The stepping stone model with infinitely many alleles has been studied in the framework of Fleming-Viot processes by Handa [5]. In this paper, it is investigated the strong-migration limit of the average number of distinct elements in a sample of finite particles in the stationary state of the model, where the results in [5] are applied. To obtain our result, we investigate a problem on positively recurrent Markov chains, and make use of the results in Shiga, Shimizu and Soshi [12].

1. Introduction

We will discuss the stepping stone model as a Fleming-Viot process, and the related Markov chains determined by the migration rates. Fleming-Viot processes have been investigated by many authors. See Ethier and Griffiths [1], Ethier and Kurtz [2] and [3]. One of important applications of Fleming-Viot processes is to formulate mathematically the infinite allele model in population genetics which was given by Kimura and Crow [7]. The Fleming-Viot process describing the stepping stone model with infinitely many alleles has been studied by Handa [5]

First, we explain the Fleming-Viot process describing the stepping stone model, and our problem on the measure-valued diffusion. Let E be the 1-dimensional interval $[0,1]$, and S be a countable (or finite) set. The set $P(E)$ of probability measures on E is endowed with the weak topology. Define $\mathcal{P}$

$$\mathcal{P} = P(E)^S = \{\widetilde{\mu} = \{\mu_k\}_{k\in S}; \mu_k \in P(E), k \in S\},$$

with the product topology. In the following, for a topological space X, $B(X)$ stands for the space of bounded Borel measurable functions on X. For each positive integer k, X^k denotes the k-fold product of X.

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Let $\boldsymbol{N}$ be the set of positive integers, and θ be a positive number. For each $n \in \boldsymbol{N}$ and any $f \in B(E^n)$, define $\boldsymbol{L}^{(n)}f$ by

$$(\boldsymbol{L}^{(n)}f)(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}(\boldsymbol{L}_i f)(x_1, x_2, \ldots, x_n),$$

where

$$(\boldsymbol{L}_i f)(x_1, x_2, \ldots, x_n) = \frac{\theta}{2}\left(\int_0^1 f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n)du - f(x_1, \ldots, x_n)\right).$$

For $n \in \boldsymbol{N}$, $f \in B(E^n)$ and $1 \leq i < j \leq n$, $\boldsymbol{\Phi}_{ij}^{(n)}f$ is defined by

$$(\boldsymbol{\Phi}_{ij}^{(n)}f)(x_1, x_2, \ldots, x_{n-1}) = f(x_1, \ldots, x_{j-1}, x_i, x_j, x_{j+1}, \ldots, x_{n-1}).$$

Here, in the right-hand side of the above equality, the j-th variable x_j is replace by x_i, and the variables $x_{j+1}, x_{j+2}, \ldots, x_n$ are replaced by $x_j, x_{j+1}, \ldots, x_{n-1}$ respectively.

Let us introduce a matrix $\{m_{k'k}\}_{k',k\in S}$, which describes migration rates. Suppose

$$m_{k'k} \geq 0 \quad \text{if} \quad k \neq k' \quad \text{and} \quad m_{kk} = -\sum_{k':k'\neq k} m_{k'k}.$$

Furthermore, we assume that $\sup_{k\in S}|m_{kk}| < +\infty$.
For $\boldsymbol{k} = (k_1, k_2, \ldots, k_n) \in S^n$, $\beta_j\boldsymbol{k}$ and $\gamma_i(k')\boldsymbol{k}$ are defined by

$$\beta_j\boldsymbol{k} = (k_1, k_2, \ldots, k_{j-1}, k_{j+1}, \ldots, d_n) \in S^{n-1},$$

and

$$\gamma_i(k')\boldsymbol{k} = (k_1, \ldots, k_{i-1}, k', k_{i+1}, \ldots, k_n) \in S^n.$$

Namely, β_j eliminates the j-th coordinate of $\boldsymbol{k}$, and $\gamma_i(k')$ replaces the i-th element of $\boldsymbol{k}$ by k'. For $\boldsymbol{k} = (k_1, k_2, \ldots, k_n) \in S^n$, $\overline{\mu}_{\boldsymbol{k}}$ denotes $\mu_{k_1} \times \mu_{k_2} \times \cdots \times \mu_{k_n}$ (*direct product*), and $\phi_{f,\boldsymbol{k}}(\overline{\mu})$ is defined by $\phi_{f,\boldsymbol{k}}(\overline{\mu}) = \langle f, \overline{\mu}_{\boldsymbol{k}}\rangle$ for $f \in B(E^n)$.

Now, we are in a position to define the operator $\mathcal{L}$ by

$$\begin{aligned}(\mathcal{L}\phi_{f,\boldsymbol{k}})(\overline{\mu}) = &\sum_{1\leq i<j\leq n, k_i=k_j} (\langle \Phi_{ij}^{(n)}f, \overline{\mu}_{\beta_j\boldsymbol{k}}\rangle - \langle f, \overline{\mu}_{\boldsymbol{k}}\rangle) \\ &+ \langle \boldsymbol{L}^{(n)}f, \overline{\mu}_{\boldsymbol{k}}\rangle + \sum_{i=1}^{n}\sum_{k'\in S} m_{k'k_i}\langle f, \overline{\mu}_{\gamma_i(k')\boldsymbol{k}}\rangle. \qquad (1.1)\end{aligned}$$

It is shown in Handa [5] that the martingale problem for the operator $\mathcal{L}$ is well posed, and that the $\mathcal{P}$-valued diffusion $\{\mu(t)\}$ determined by $\mathcal{L}$ has a unique stationary distribution, which is denoted by $\widetilde{Q}(d\overline{\mu})$.

Define

$$I_n = \{\alpha = \{\alpha_k\}_{k\in S}; \alpha_k \in Z_+, \sum_{k\in S} \alpha_k = n\}, \text{ and } \quad |\alpha| = \sum_{k\in S} \alpha_k.$$

where Z_+ denotes the set of non-negative integers, and set

$$I = \bigcup_{n=1}^{+\infty} I_n.$$

Consider the mapping $\Psi : \bigcup_{n=1}^{+\infty} S^n \to I$, defined by

$$\Psi(\boldsymbol{k}) = \{\alpha_k\}_{k\in S}, \boldsymbol{k} \in S^n,$$

where α_k = the cardinality of the set $\{i; k_i = k\}$. Let $\widetilde{\mu}_\alpha = \prod_{k\in S, \alpha_k>0} \mu_k^{\alpha_k}$, where $\mu_k^{\alpha_k}$ denotes the α_k-fold product of μ_k. If a function $f \in B(E^n)$ is symmetric, and $\Psi(\boldsymbol{k}) = \Psi(\boldsymbol{k}') = \alpha \in I_n$, then $\langle f, \widetilde{\mu}_\alpha \rangle$ is well-defined and we have

$$\langle f, \widetilde{\mu}_{\boldsymbol{k}} \rangle = \langle f, \widetilde{\mu}_{\boldsymbol{k}'} \rangle = \langle f, \widetilde{\mu}_\alpha \rangle.$$

Note that for a symmetric function $f \in B(E^n)$, $\Phi_{ij}^{(n)} f$ and $L^{(n)} f$ are symmetric in $B(E^{n-1})$ and $B(E^n)$ respectively. In the following, we assume that the migration rates $m_{k'k}$, k', $k \in S$ are written in the form

$$m_{k'k} = mr_{k'k}, \quad k', k \in S,$$

with a positive constant m. The stationary distribution $\widetilde{Q}(d\widetilde{\mu})$ is denoted by $\widetilde{Q}_m(d\widetilde{\mu})$, since it depends on m. Define $f_n \in B(E^n)$ by $f_n(x_1, x_2, \dots, x_n)$ = the number of distinct elements of the sequence $(x_1, x_2, \dots, x_n)$. Note that the function f_n is symmetric.

Our problem on the Fleming-Viot process is to investigate the asymptotic behavior of the integral

$$\mathcal{A}_m(\alpha) := \int \mathcal{F}_\alpha(\widetilde{\mu}) \widetilde{Q}_m(d\widetilde{\mu}), \tag{1.2}$$

as $m \to \infty$, where

$$\mathcal{F}_\alpha(\widetilde{\mu}) = \langle f_n, \widetilde{\mu}_\alpha \rangle, \alpha \in I_n.$$

Here, we explain what the quantity $\mathcal{A}_m(\alpha)$ means. Choose n genes from the colonies according to $\alpha \in I_n$. Namely, α_k genes are chosen from colony k, for each $k \in S$. $\mathcal{F}_\alpha(\widetilde{\mu})$ means the expectation of the number of distinct elements in a sample of n genes with respect to the measure $\widetilde{\mu} \in \mathcal{P}$. Since $\mathcal{A}_m(\alpha)$ is the average of $\mathcal{F}_\alpha(\overline{\mu})$ with respect to $\widetilde{Q}_m(d\overline{\mu})$, $\mathcal{A}_m(\alpha)$, $\alpha \in I_n$, should be called the average expectation of the number of alleles in a sample of n genes.

The asymptotic behavior of a quantity for stepping stone models, as $m \to \infty$, is called that of the strong-migration limit. The strong-migration limit in geographically structured population was investigated in Nagylaki [8], Notohara [9] and [10]. Notohara [10] discussed the same problem as ours. In [10], it is assumed that the migration process is a random walk on the d-dimensional torus S with lattice points. Under this assumption, he got fruitful results on the strong-migration limit of the stepping stone model. In this paper, the set S is assumed to be a countable set, so that it will be assumed the finiteness of the moments of the first returning time for the migration process to investigate the asymptotic behavior of $\mathcal{A}_m(\alpha)$ in the strong-migration limit.

Let us introduce three kinds of Markov chains.
Define $q_{kk'}$, $k, k' \in S$ by

$$q_{kk'} = r_{k'k}.$$

Let $q_k = -q_{kk}$. The minimal Markov chain generated by $\{q_{kk'}\}_{k,k' \in S}$ is denoted by $\{x(t), P_k\}$. It is conservative since $\{q_k\}_{k \in S}$ is bounded by the assumption on the migration rates. In the following, the Markov chain $\{x(t), P_k\}$ is sometimes called the 1-particle system. We consider the n-fold direct product of the 1-particle system $\{x(t), P_k\}$, which may be called the n-particle system. The n-particle system, whose state space is S^n, is denoted by $\{\boldsymbol{x}(t)\}_{t \geq 0}$. Define the matrix $\widetilde{q}_{\alpha\beta}$ on I_n, by

$$\begin{aligned} \widetilde{q}_{\alpha\beta} &= \alpha_k r_{k'k} && \text{if} \quad \beta = \alpha - \epsilon_k + \epsilon_{k'}, \\ &= \sum_k \alpha_k r_{kk} && \text{if} \quad \beta = \alpha, \\ &= 0 && \text{otherwise,} \end{aligned} \tag{1.3}$$

where $\epsilon_k = \{\delta_{k,l}\}_{l \in S} \in I_1$. The Markov chain with state space I_n, generated by $\{\widetilde{q}_{\alpha\beta}\}_{\alpha,\beta \in I_n}$, is denoted by $\{\widetilde{\alpha}(t)\}_{t \geq 0}$. We see that

$$\left\{\Psi(\boldsymbol{x}(t))\right\}_{t \geq 0} \overset{\text{law}}{=} \left\{\widetilde{\alpha}(t)\right\}_{t \geq 0}.$$

In this paper, we assume the following.

ASSUMPTION 1. The Markov chain $\{x(t), P_k\}$, the 1-particle system, is irreducible and positively recurrent, whose stationary distribution is denoted by ν.

Under this assumption, the n-particle system $\{\boldsymbol{x}(t)\}_{t \geq 0}$ and the Markov chain $\{\alpha(t)\}_{t \geq 0}$, mapped by Ψ from the n-particle system are both positively recurrent, whose stationary distributions are $\nu_n(\boldsymbol{k}) = \prod_{i=1}^n \nu(k_i)$, for $\boldsymbol{k} = (k_1, k_2, \ldots, k_n) \in S^n$, and $\widetilde{\nu}_n = \frac{n!}{\prod_k \alpha_k!}\nu(k)^{\alpha_k}$ for $\alpha = (\alpha_k)_{k\in S}$ respectively.

Second, we explain our problem on the 1-particle system $\{x(t), P_k\}$. Let us consider the equation on a function u_m

$$m\sum_{k'} q_{kk'}u_m(k') - V(k)u_m(k) = -f(k), \qquad k \in S, \tag{1.4}$$

with a positive constant m and $f \in B(S)$, where $V(k)$ satisfies $c_1 \le V(k) \le c_2$ for some positive constants c_1, c_2. Recall that the equation (1.4) has a unique bounded solution u_m if and only if the Markov chain $\{x(t), P_k\}$ is conservative (Feller [4], Shiga [11]), and it is easy to see that

$$\lim_{m\to\infty} u_m(k) = \langle f\rangle/\langle V\rangle,$$

where $\langle g\rangle = \sum_k g(k)\nu(k)$ for $g \in B(S)$. Our problem is the following. Under what condition does the sequence $m(u_m(k) - \langle f\rangle/\langle V\rangle)$ have a finite limit as m tends to infinity? Our result on the problem mentioned above of the Markov chain $\{x(t), P_k\}$ is formulated as the next theorem. The expectation with respect to P_k is denoted by $E_k[\cdot]$. Define $\mathcal{V}$, a class of bounded functions on S, by

$$\mathcal{V} = \{V \in B(S);\ \text{there exist positive constants } c_1 \text{ and } c_2 \\ \text{such that } c_1 \le V(k) \le c_2,\ \text{for any } k \in S\}.$$

Let T be the first returning time, namely

$$T = \inf\{t > 0; x(t) = x(0) \text{ and } x(s) \ne x(0) \text{ for some } s < t\}.$$

THEOREM 1. *There exists a constant c depending on k, V, and f for any $k \in S$, $V \in \mathcal{V}$ and $f \in B(S)$ such that*

$$\lim_{m\to\infty} m(u_m(k) - \langle f\rangle/\langle V\rangle) = c,$$

if and only if

$$E_{k_0}[T^2] < \infty \quad \textit{for some} \quad k_0 \in S. \tag{1.5}$$

Remark. Note that the condition $E_{k_0}[T^2] < \infty$ for some $k_0 \in S$ implies $E_k[T^2] < \infty$ for all $k \in S$. Since the matrix $\{q_{kk'}\}$ is defined at the beginning by the migration rates $\{m_{k'k}\}$ of the stepping stone model, it is assumed in Theorem 1 that q_k is bounded in $k \in S$. However, it suffices to assume on $\{q_{kk'}\}$ that the minimal Markov chain generated by $\{q_{kk'}\}$ is conservative, in order to prove Theorem 1 disregarding the stepping stone model.

Here, we apply this result to the problem of the Fleming-Viot process describing a stepping stone model. It is easy to see by using (3.3) and (3.4) below that, under Assumption 1, the equality

$$\lim_{m\to+\infty} \mathcal{A}_m(\alpha) = 1+\theta\sum_{l=2}^{n}\frac{l}{\widetilde{\nu}_l+(\theta-1)l} = 1+\theta\sum_{l=2}^{n}\frac{1}{(l-1)\sum_k \nu(k)^2+\theta}$$

holds for $\alpha \in I_n$, $n \geq 2$, where $\widetilde{\nu}_l = \sum_k \sum_{\alpha\in I_l} \alpha_k^2 \widetilde{\nu}_l(\alpha) = (l^2-l)\sum_k \nu(k)^2 + l$. The right-hand side of the above equality is denoted by $\mathcal{A}_\infty(\alpha)$. Recall that the average number of alleles in randomly chosen n genes for the ordinary infinite allele model is known to be equal to

$$1+\theta\sum_{l=2}^{n}\frac{1}{l-1+\theta}.$$

See Hartl and Clark [6]. By Shiga, Shimizu and Soshi [12], we see that the finiteness of the expectation of T^2 for the 1-particle system implies that of the second moment of the first returning time for the n-particle system. Therefore, under the assumption that $E_k[T^2] < \infty$ for some $k \in S$, the first returning time for the Markov chain $\widetilde{\alpha}(t)$, mapped by Ψ from the n-particle system has the finite second moment. Making use of Theorem 1 and the results of Shiga, Shimizu and Soshi [12], we can obtain the next theorem.

THEOREM 2. *Assume for the* 1*-particle system* $\{x(t), P_k\}$ *that* $E_k[T^2] < \infty$ *for some* k. *Then, there exists a constant* c' *depending on* α *for each* $\alpha \in I_2$ *such that*

$$\lim_{m\to+\infty} m\{\mathcal{A}_m(\alpha) - \mathcal{A}_\infty(\alpha)\} = c'.$$

2. Proof of Theorem 1

By the Feynman-Kac formula, we see that the solution $u_m(k)$ of (1.4) can be written as follows (Shiga [11]):

$$u_m(k) = \frac{1}{m}\int_0^{+\infty} E_k\left[\exp\left\{-\frac{1}{m}\int_0^s V(x(u))du\right\} f(x(s))\right] ds \tag{2.1}$$

$$= \int_0^{+\infty} E_k\left[\exp\left\{-\frac{1}{m}\int_0^{mt} V(x(u))du\right\} f(x(mt))\right] dt. \tag{2.2}$$

The ergodic property of the 1-particle system and (2.2) imply that

$$\lim_{m\to\infty} u_m(k) = \langle f\rangle/\langle V\rangle.$$

Making use of the strong Markov property of the 1-particle system and (2.1), we obtain

$$u_m(k) = \frac{E_k\left[\int_0^T \exp\{-\frac{1}{m}\int_0^t V(x(u))du\} f(x(t))dt\right]}{m\left(1 - E_k\left[\exp\{-\frac{1}{m}\int_0^T V(x(u))du\}\right]\right)}. \tag{2.3}$$

Define $\phi(z)$, $\psi(z)$ and $v(z)$ by

$$\phi(z) = E_k\left[\int_0^T \exp\left\{-z\int_0^t V(x(u))du\right\} f(x(t))dt\right],$$

$$\psi(z) = \frac{1}{z}\left\{1 - E_k\left[\exp\left\{-z\int_0^T V(x(u))du\right\}\right]\right\} \equiv \frac{1}{z}\{1-\eta(z)\},$$

and

$$v(z) = \frac{\phi(z)}{\psi(z)}$$

for a positive number z, respectively. Note that $\phi(0+) = E_k[T]\langle f\rangle$, $\psi(0+) = E_k[T]\langle V\rangle$, and that $v(0) = \langle f\rangle/\langle V\rangle$. First, we assume that $E_{k_0}[T^2] < \infty$ for some $k_0 \in S$. In order to show that the condition (1.5) is sufficient, it suffices to prove that $v(z)$ is right-differentiable at $z = 0$. Since $E_k[T^2] < +\infty$ for any $k \in S$, as mentioned in Remark of Section 1, we see that

$$\lim_{z\downarrow 0} \phi'(z) = -E_k\left[\int_0^T\int_0^t V(x(u))du f(x(t))dt\right].$$

By a simple calculation, we see

$$\psi'(z) = \frac{\eta(z) - 1 - z\eta'(z)}{z^2}.$$

Noting that $E_k[T^2] < +\infty$ implies the existence of $\eta''(0+)$, we get

$$\lim_{z\downarrow 0} \psi'(z) = -\frac{1}{2}\eta''(0+) = -\frac{1}{2}E_k\left[\left(\int_0^T V(x(u))du\right)^2\right].$$

Hence, we conclude that $v'(0+)$ exists, namely,

$$v'(0+) = \frac{\phi'(0+)\psi(0+) - \phi(0+)\psi'(0+)}{\psi(0+)^2}.$$

To complete the proof of Theorem 1, it is sufficient to show the next lemma.

LEMMA 2.1. *Let $f(k) \equiv 1$, $V(k) = I_{\{k_0\}}(k) + 1$. Then, the statement that*

$$u_m(k_0) - \frac{1}{1+\nu(k_0)} = O(1/m), \quad m \to +\infty,$$

holds if and only if $E_{k_0}[T^2] < +\infty$.

Proof. Noting that $f(k) \equiv 1$, we have for the initial state k_0

$$\phi(z) = E_{k_0}\left[\int_0^T \exp\left\{-z\int_0^t V(x(u))du\right\}dt\right].$$

Let ζ be the first jumping time, and T_0 be the first passage time to the state k_0, then

$$\begin{aligned}
\phi(z) &= E_{k_0}\left[\int_0^\zeta \exp\left\{-z\int_0^t V(x(u))du\right\}dt\right] + E_{k_0}\left[\int_\zeta^T \exp\left\{-z\int_0^t V(x(u))du\right\}dt\right] \\
&= E_{k_0}\left[\int_0^\zeta \exp\{-2zt\}dt\right] + E_{k_0}\left[\int_\zeta^T \exp\{-z(2\zeta + t - \zeta)\}dt\right] \\
&= E_{k_0}\left[\frac{1}{2z}(1 - e^{-2z\zeta})\right] + E_{k_0}\left[e^{-2z\zeta}E_{x_\zeta}\left[\int_0^{T_0} \exp(-zu)du\right]\right] \\
&= \frac{1}{q_{k_0} + 2z} + \frac{q_{k_0}}{z(q_{k_0} + 2z)}(1 - E_{k_0}[E_{x_\zeta}[e^{-zT_0}]]) \\
&= \frac{1}{z(q_{k_0} + 2z)}(z + q_{k_0} - q_{k_0}E_{k_0}[E_{x_\zeta}[e^{-zT_0}]]).
\end{aligned}$$

Similarly, we see for the initial state k_0 that

$$\psi(z) = \frac{1}{z(q_{k_0} + 2z)}(2z + q_{k_0} - q_{k_0}E_{k_0}[E_{x_\zeta}[e^{-zT_0}]]).$$

Hence, we obtain

$$v(z) = 1 - \frac{1}{q_{k_0} + 2z}\frac{1}{\psi(z)},$$

so that $v(z)$ is right-differentiable at $z = 0$ if and only if $\frac{\psi(z)-\psi(0+)}{z}$ has a finite limit as $z \downarrow 0$. Since $\psi(0+) = E_{k_0}\left[\int_0^T V(x(u))du\right]$, we see that

$$\begin{aligned}
\frac{\psi(z) - \psi(0+)}{z} &= \frac{1}{z^2}\left(1 - E_{k_0}\left[z\int_0^T V(x(u))du\right] - E_{k_0}\left[\exp\left\{-z\int_0^T V(x(u))du\right\}\right]\right) \\
&= (-1)E_{k_0}\left[\int_0^{\int_0^T V(x(u))du} dt \int_0^t e^{-zs}ds\right].
\end{aligned}$$

Consequently,

$$\lim_{z\downarrow 0}\frac{\psi(z) - \psi(0+)}{z} = (-1)\frac{1}{2}E_{k_0}\left[\left(\int_0^T V(x(u))du\right)^2\right].$$

Noting that

$$T^2 \leq \left(\int_0^T V(x(u))du\right)^2 \leq 4T^2,$$

we can conclude that there exists a finite limit $\lim_{z\downarrow 0}\frac{\psi(z)-\psi(0+)}{z}$ if and only if $E_{k_0}[T^2] < +\infty$. Thus, the proof of Lemma 2.1 is complete.

3. Proof of Theorem 2

First, let us introduce another infinitesimal matrix $q^m_{\alpha\beta}$ on I defined by

$$
\begin{aligned}
q^m_{\alpha\beta} &= \binom{\alpha_k}{2} + \frac{\theta}{2}\alpha_k && \text{if} \quad \beta = \alpha - \epsilon_k \quad \text{and} \quad |\alpha| \geq 2 \\
&= \alpha_k m r_{k'k} && \text{if} \quad \beta = \alpha - \epsilon_k + \epsilon_{k'}, \ k \neq k' \\
&= -\sum_{k \in S}\left(\binom{\alpha_k}{2} + \frac{\theta}{2}\alpha_k\right) I_{\{|\alpha| \geq 2\}}(\alpha) + \sum_{k \in S} \alpha_k m r_{kk} && \text{if} \quad \beta = \alpha \\
&= 0 && \text{otherwise,}
\end{aligned}
$$

where $\alpha + \beta$ and $\alpha - \beta$ are defined componentwisely for $\alpha, \beta \in I$. Here, $\binom{n}{k}$ means 0 if $n < k$.

Note that the equalities

$$
\begin{aligned}
&(L_i f_n)(x_1, x_2, \ldots, x_n) \\
&\quad = \frac{\theta}{2}(1 + f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - f_n(x_1, x_2, \ldots, x_n)),
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathcal{L}\mathcal{F}_\alpha)(\bar{\mu}) = \sum_{k \in S} \binom{\alpha_k}{2} &(\langle f_{n-1}, \bar{\mu}_{\alpha - \epsilon_k}\rangle - \langle f_n, \bar{\mu}_\alpha\rangle) + \frac{\theta}{2}|\alpha| \\
&+ \sum_{k \in S} \frac{\theta}{2}(\langle f_{n-1}, \bar{\mu}_{\alpha-\epsilon_k}\rangle - \langle f_n, \bar{\mu}_\alpha\rangle) \\
&+ \sum_{k \in S}\sum_{k' \in S} \alpha_k m_{k'k} \langle f_n, \bar{\mu}_{\alpha - \epsilon_k + \epsilon_{k'}}\rangle \qquad \text{for} \quad |\alpha| = n \geq 2.
\end{aligned}
$$

hold. Since

$$\int (\mathcal{L}\mathcal{F}_\alpha)(\bar{\mu})\bar{Q}_m(d\bar{\mu}) = 0,$$

we obtain

$$\sum_\beta q^m_{\alpha\beta}\mathcal{A}_m(\beta) + \frac{\theta}{2}|\alpha| = 0 \quad \text{if} \quad |\alpha| \geq 2, \tag{3. 1}$$

$$\mathcal{A}_m(\alpha) = 1 \quad \text{if} \quad |\alpha| = 1. \tag{3. 2}$$

Let $(\alpha(t), P^m_\alpha)$, $t \geq 0$, $\alpha \in I$, be the Markov chain generated by $\{q^m_{\alpha\beta}\}_{\alpha,\beta \in I}$. E^m_α denotes the expectation with respect to P^m_α. Define $\mathcal{T}$ and $\mathcal{T}_1$ by

$$\mathcal{T} = \inf\{t \geq 0; |\alpha(t)| = 1\},$$

and

$$\mathcal{T}_1 = \inf\{t \geq 0; |\alpha(t)| < |\alpha(0)|\},$$

where $\inf\emptyset = +\infty$. By (3.1) and (3.2), we obtain for $\alpha \in I_n$, $n \geq 2$,

$$\mathcal{A}_m(\alpha) = 1 + E_\alpha^m\left[\int_0^T \frac{\theta}{2}|\alpha(s)|ds\right]$$
$$= \frac{n\theta}{2}E_\alpha^m[\mathcal{T}_1] + E_\alpha^m[\mathcal{A}_m(\alpha(\mathcal{T}_1))]. \tag{3.3}$$

Observe that $\mathcal{A}_m^1(\alpha) := E_\alpha^m[\mathcal{T}_1]$ satisfies

$$\sum_{\beta\in I_n} \widetilde{q}_{\alpha\beta}\mathcal{A}_m^1(\beta) - \frac{1}{m}\frac{1}{2}\Big(\sum_i \alpha_i^2 + (\theta-1)n\Big)\mathcal{A}_m^1(\alpha) = -\frac{1}{m}, \tag{3.4}$$

with the matrix $\{\widetilde{q}_{\alpha\beta}\}_{\alpha,\beta\in I_n}$ defined by (1.3). Since the minimal Markov chain $\{\widetilde{\alpha}(t), P_\alpha\}$ on I_n generated by the matrix $\{\widetilde{q}_{\alpha\beta}\}_{\alpha,\beta\in I_n}$ is conservative and positively recurrent, as explained in Section 1. Therefore, we can apply Theorem 1 to $\{\widetilde{\alpha}(t), P_\alpha\}$. Let $\boldsymbol{T}$ be the first returning time of $\{\widetilde{\alpha}(t), P_\alpha\}$, then by the results of Shiga, Shimizu and Soshi [12], as mentioned in Section 1, we see that $E_\alpha[\boldsymbol{T}^2] < +\infty$, where $E_\alpha[\cdot]$ denotes the expectation with respect to P_α. Thus, we obtain the next lemma.

Lemma 3.1. *For each $\alpha \in I_n$, there exists a finite limit* $\lim_{m\to\infty} m(\mathcal{A}_m^1(\alpha) - \frac{2}{\widetilde{\nu}_n+(\theta-1)n})$.

If $\alpha \in I_2$, the second term of the right-hand side of (3.3) is equal to 1, so that Lemma 3.1 proves Theorem 2.

4. An Example

Here, we present a Markov chain for which one can get the critical parameter for the second moment of the first returning time T to be finite. Furthermore, for the example, it is made clear the speed of convergence of the solution u_m for the equation (1.4) with specified functions V and f, when the second moment of T is infinite. Proposition 4.1 and Lemma 4.2, given below, are essentially due to Professor T. Shiga.

Consider a continuous time minimal Markov chain $\{x(t), P_i\}$ in $S = Z_+$ which is governed by the following infinitesimal matrix $Q = \{q_{i,j}\}$.

$$q_{i,j} = \begin{cases} 1 & (\text{if } j = i+1) \\ \alpha_1 & (\text{if } j = 0) \\ -(1+\alpha_i) & (\text{if } j = i > 0), \end{cases}$$

and $$q_{0,j} = \delta_{1,j} \qquad (j \geq 1) \qquad q_{0,0} = -1,$$

$$q_{i,j} = 0 \quad \text{for any other} \quad i, j \in S.$$

Recall that T denotes the first returning time of the continuous time Markov chain $\{x(t), P_i\}$. Then it is easy to show that

$$E_0\big(e^{-\lambda T}\big) = \sum_{i=1}^{\infty} \frac{\alpha_i}{1+\lambda} \prod_{k=1}^{i} \frac{1}{1+\lambda+\alpha_k}, \tag{4.1}$$

from which it follows that $\{x(t), P_i\}$ is recurrent if and only if

$$\sum_{i=1}^{\infty} \alpha_i = \infty. \tag{4.2}$$

Now we specialize $\{\alpha_i\}$ as follows:

$$\alpha_i = \frac{\kappa}{i} \quad (i \geq 1), \tag{4.3}$$

with a constant $\kappa > 0$. Inserting this to (4.1) we see

$$\begin{aligned} E_0\big(e^{-\lambda T}\big) &= \sum_{n=1}^{\infty} \frac{\kappa \Gamma(n) \Gamma\big(\frac{\kappa}{1+\lambda}+1\big)}{(1+\lambda)^{n+1} \Gamma\big(\frac{\kappa}{1+\lambda}+n+1\big)} \\ &= \sum_{n=1}^{\infty} \frac{\kappa}{(1+\lambda)^{n+1}} B\Big(n, \frac{\kappa}{1+\lambda}+1\Big), \end{aligned}$$

where $B(\alpha, \beta)$ denotes the Beta function, so that it holds

$$E_0\big(e^{-\lambda T}\big) = \frac{\kappa}{1+\lambda} \int_0^1 \frac{t^{\frac{\kappa}{1+\lambda}}}{\lambda+t} dt. \tag{4.4}$$

PROPOSITION 4.1.

$$E_0(T^{\alpha}) < \infty \quad \textit{if and only if} \quad \alpha < \kappa.$$

Proof. Using Taylor's expantion of $1/(\lambda+t)$, from (4.4) one gets

$$\begin{aligned} E_0\big(e^{-\lambda T}\big) &= \kappa \sum_{m=0}^{n} \frac{(-\lambda)^m}{\kappa - m(\lambda+1)} + \frac{\kappa(-\lambda)^{n+1}}{\lambda+1} \int_0^1 \frac{t^{\frac{\kappa}{\lambda+1}-n-1}}{\lambda+t} dt \\ &= \kappa \sum_{m=0}^{n} \frac{(-\lambda)^m}{\kappa - m(\lambda+1)} + \frac{\kappa(-1)^{n+1}}{\lambda+1} \lambda^{\kappa/(\lambda+1)} \int_0^{1/\lambda} \frac{t^{\frac{\kappa}{\lambda+1}-n-1}}{1+t} dt. \end{aligned}$$

Let $n < \kappa \leq n+1$ $(n \in Z_+)$. Denoting the last term by $b(\lambda)$, we obtain that if $n < \kappa < n+1$,

$$\lim_{\lambda \searrow 0} (-1)^{n+1} \frac{b(\lambda)}{\lambda^{\kappa}} = \kappa \int_0^{\infty} \frac{t^{\kappa-n-1}}{1+t} dt, \tag{4.5}$$

and if $\kappa = n+1$,

$$b(\lambda) \sim \kappa(-\lambda)^{n+1} \log \frac{1}{\lambda} \quad (\lambda \searrow 0). \tag{4.6}$$

Therefore, by applying the following lemma, the proof of Proposition 4.1 is completed.

LEMMA 4.1. *For a nonnegative random variable Y, let*

$$\varphi(\lambda) = E\big(e^{-\lambda Y}\big) \quad (\lambda > 0).$$

Suppose that $\varphi(\lambda)$ has the form,

$$\varphi(\lambda) = \sum_{m=0}^{n} a_m \lambda^m + b(\lambda),$$

where for a $\kappa \in (n, n+1]$, either

$$b(\lambda) \sim \beta_0 \lambda^{\kappa}, \qquad as \quad \lambda \downarrow 0,$$

with $\beta_0(\neq 0) \in R$ or

$$\lim_{\lambda \searrow 0} (-1)^{n+1} \frac{b(\lambda)}{\lambda^{\kappa}} = \infty$$

and for every $\beta < \kappa$

$$\lim_{\lambda \searrow 0} (-1)^{n+1} \frac{b(\lambda)}{\lambda^{\beta}} = 0.$$

Then it holds that

$$E(Y^{\alpha}) < \infty \quad for \quad \alpha < \kappa, \quad and \quad E(Y^{\kappa}) = \infty.$$

Proof of Lemma 4.2 is standard, so it is omitted.

We see by Proposition 4.1 that $\{x(t), P_i\}$ is positively recurrent if and only if $\kappa > 1$, so that we investigate the case of $\kappa > 1$ in the following. Let us consider (1.4) for this example, with specified functions V and f. Let $V(i) = 1 + I_{\{0\}}(i)$, $f(i) \equiv 1$, where $I_{\{0\}}$ is the indicator function on the set $\{0\}$. Namely, we consider the equation

$$m \sum_j q_{ij} u_m(j) - V(i) u_m(i) = -1, \tag{4.7}$$

with a positive constant m. The unique solution u_m of (4.7) satisfies

$$\lim_{m \to +\infty} u_m(0) = \frac{1}{1 + \nu(0)},$$

with $\nu(0) = 1 - \frac{1}{\kappa}$, and, by the same argument as the proof of Lemma 2.1, it is shown that

$$u_m(0) = 1 - \frac{1}{1+2z}\frac{1}{\psi(z)}, \tag{4.8}$$

where $z = 1/m$, and

$$\psi(z) = \frac{1}{z(1+2z)}\left(2z + 1 - E_1[e^{-zT_0}]\right), \tag{4.9}$$

where T_0 denotes the first passage time to the state 0. Note that

$$E_1[e^{-zT_0}] = \kappa \int_0^1 t^{\frac{\kappa}{1+z}} \frac{1}{z+t} dt. \tag{4.10}$$

Then, we obtain the next statement.

Proposition 4.2. *If $\kappa > 2$,*

$$u_m(0) - \frac{1}{1+\nu(0)} \sim -k_1(\kappa)\frac{1}{m}, \quad m \to +\infty,$$

if $\kappa = 2$,

$$u_m(0) - \frac{1}{1+\nu(0)} \sim -\frac{2}{9}\frac{1}{m}\log m, \quad m \to +\infty,$$

and if $1 < \kappa < 2$,

$$u_m(0) - \frac{1}{1+\nu(0)} \sim -k_2(\kappa)\left(\frac{1}{m}\right)^{\kappa-1}, \quad m \to +\infty,$$

where

$$k_i(\kappa) > 0, \quad (i = 1, 2), \quad \lim_{\kappa \downarrow 2} k_1(\kappa) = +\infty, \quad \lim_{\kappa \uparrow 2} k_2(\kappa) = +\infty,$$

and $\nu(0) = 1 - 1/\kappa$.

Proof. Recall the equality in the proof of Proposition 4.1,

$$E_0[e^{-zT}] = \kappa \sum_{l=0}^{n} \frac{(-z)^l}{\kappa - l(z+1)} + b(z), \tag{4.11}$$

where

$$b(z) = \frac{\kappa(-1)^{n+1}}{z+1} z^{\frac{\kappa}{z+1}} \int_0^{\frac{1}{z}} \frac{t^{\kappa/(1+z)-n-1}}{1+t} dt. \tag{4.12}$$

We prove Proposition 4.2 only for the case of $\kappa = 2$, because one can prove the assertions similarly for $\kappa > 2$ and $1 < \kappa < 2$ making use of (4.8), (4.9), (4.10), (4.11) and (4.12). For the case of $\kappa = 2$, by (4.11) with $n = 1$, we see that

$$a(z) \equiv E_0[e^{-zT}] = 1 - 2z + b(z) + O(z^2), \quad z \downarrow 0,$$

and that

$$b(z) \sim 2z^2 \log \frac{1}{z}, \quad z \downarrow 0.$$

Note that we have by (4.9)

$$\psi(z) = -\frac{1}{1+2z}\frac{a(z)-1}{z} + \frac{2-a(z)}{1+2z},$$

and therefore $\psi(0+) = 3$. Consequently, we have

$$\begin{aligned}\frac{\psi(z)-\psi(0+)}{z} &= -\frac{1}{1+2z}\left(\frac{b(z)}{z^2} + O(1)\right),\\ &\sim -2\log\frac{1}{z}.\end{aligned}$$

Hence, we conclude that

$$\frac{v(z)-v(0+)}{z} \sim -\frac{2}{\psi(0+)^2}\log\frac{1}{z} = -\frac{2}{9}\log\frac{1}{z}, \quad z \downarrow 0.$$

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Akinobu Shimizu
Institute of Natural Sciences
Nagoya City University
Mizuho-cho, Mizuho-ku, Nagoya 467-8501
Japan

Takahiro Soshi
Department of Applied Mathematics
Faculty of Engineering
Yokohama National University
Tokiwadai, Hodogaya-ku, Yokohama 240-0067
Japan