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REAL HYPERSURFACES IN COMPLEX PROJECTIVE SPACE SATISFYING A CERTAIN CONDITION ON THE SECOND FUNDAMENTAL FORM

By

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Abstract. The purpose of the present paper is to characterize A_1, A_2 or a ruled real hypersurface of CP^n under a certain condition on the second fundamental form.

1. Introduction

Let $CP^n, n \geq 2$, be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4, and let M be a real hypersurface of CP^n. Let ν be a unit normal vector field on M and $\xi = -J\nu$, where J denotes the complex structure of CP^n. M has an almost contact metric structure (ϕ, ξ, η, g) induced from J. Many differential geometers have studied M (cf. [1]–[7]) by using the structure (ϕ, ξ, η, g).

Typical examples of real hypersurfaces in CP^n are homogeneous ones. TAKAGI [7] showed that all homogeneous real hypersurfaces in CP^n are realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or rank 2. Namely, he showed the following : Let M be a homogeneous real hypersurface of CP^n. Then M is a tube of radius r over one of the following Kaehler submanifolds:

(A_1) hyperplane CP^{n-1}, where $0 < r < \frac{\pi}{2}$,
(A_2) totally geodesic CP^k $(1 \leq k \leq n-2)$,
(B) complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,
(C) $CP^1 \times CP^{\frac{n-1}{2}}$, where $0 < r < \frac{\pi}{4}$ and $n(\geq 5)$ is odd,
(D) complex Grassmann $CG_{2,5}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,
(E) Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.

Due to his classification, we find that the number of distinct constant principal

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curvatures of a homogeneous real hypersurface is 2, 3 or 5. Here note that the vector ξ of any homogeneous real hypersurface M (which is a tube of radius r) is a principal curvature vector with principal curvature $\alpha = 2\cot 2r$ with multiplicity 1 (See [1]) and that in the case of type A_1 M has two distinct principal curvatures and in the case of type A_2 M has three distinct principal curvatures $t, -\frac{1}{t}$ and $\alpha = t - \frac{1}{t}$.

OKUMURA [5] proved the following remarkable result: a real hypersurface M of CP^n satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to A_1 or A_2.

The purpose of the present paper is to study more weaker condition either

$$(A\phi - \phi A)X = 0 \tag{1}$$

for any $\xi^\perp$ (See [2]) or

$$g((A\phi - \phi A)X, Y) = 0 \tag{2}$$

for any $X, Y \in \xi^\perp$, where g and $\xi^\perp$ denotes the induced metric of M by the metric of CP^n and the orthogonal complemnt of ξ in TM, respectively. Now, we prepare the notion of a ruled real hypersurface (See [3], [4]) which means that there is a foliation of M by complex hypersurfaces CP^{n-1} and that M is a ruled real hypersurface of CP^n if and only if the shape operator A satisfies

$$A\xi = \alpha\xi + \beta U, AU = \beta\xi + \lambda U \quad \text{and} \quad AX = 0 \tag{3}$$

for $X \in \xi^\perp$. Specifically, we shall prove the following :

PROPOSITION. *Let M be a real hypersurface of CP^n. Then M satisfies* (1) *if and only if M satisfies*

$$A\phi - \phi A = 0$$

on M.

THEOREM. *Let M be a real hypersurface of $CP^n, n \geq 3$. Then M satisfies* (2) *if and only if M is locally congruent to* A_1, A_2 *or a ruled real hypersurface.*

REMARK 1. We don't know whether or not the case of $n = 2$ of Theorem is true.

REMARK 2. We note that SUH [6] showed that an η-recurrent real hypersurface M (i.e., $g((\nabla_X A)Y, Z) = \lambda(X)g(AY, Z)$ for some functions $\lambda(X)$ and X, Y and $Z \in \xi^\perp$) satifies (2) if and only if M is locally congruent to A_1, A_2 or a ruled real hypersurface.

2. Preliminaries

Let X be a tangent vector field on M. We write $JX = \phi X + \eta(X)\nu$, where ϕX is the tangent component of JX and $\eta(X) = g(X, \xi)$. As $J^2 = -Id$, where Id denotes the identity endomorphism on TCP^n, we get

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\phi X) = 0, \quad \phi\xi = 0 \tag{4}$$

for any X tangent to M. It is also easy to see that for any X and Y tangent to M

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{5}$$

$$(\nabla_X \xi) = \phi AX, \tag{6}$$

where ∇ denotes the convarinat defferentiation on M. Finally, from the expression of the curvature tensor of CP^n, we see that the curvature tensor R and Codazzi equation of M are given by

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi. \tag{8}$$

3. Proof of Proposition and key Lemma

Let M be a real hypersurface of CP^n. Then we mention again:

Proposition. *Let M be a real hypersurface of CP^n. Then M satisfies*

$$(A\phi - \phi A)X = 0$$

for any X in $\xi^\perp$ if and only if $A\phi - \phi A = 0$ on M (See Introduction).

Proof. Assume that $(A\phi - \phi A)X = 0$ for any X in $\xi^\perp$. Then we have for $X \in \xi^\perp$

$$g(\phi A\xi, X) = -g(\xi, A\phi X) = -g(\xi, \phi AX) = 0.$$

Since $g(\phi A\xi, \xi) = 0$, we get $\phi A\xi = 0$, and $A\phi\xi - \phi A\xi = 0$. Therefore $A\phi - \phi A = 0$ on M.

Lemma. *Let M be a real hypersurface of CP^n. Then if $g((A\phi - \phi A)X, Y) = 0, X, Y \in \xi^\perp$ and ξ is principal, then for $X \in \xi^\perp$*

$$(A\phi - \phi A)X = 0.$$

Proof. Assume that $g((A\phi - \phi A)X, Y) = 0, X, Y \in \xi^{\perp}$. Then we have on M

$$A\phi X - \phi AX = g(A\phi X, \xi)\xi - g(X, \xi)\phi A\xi. \tag{9}$$

Since ξ is principal, we have for $X \in \xi^{\perp}$

$$(A\phi - \phi A)X = 0.$$

4. Proof of Theorem

Let M be a real hypersurface of $CP^n, n \geq 3$. By Lemma we have only to prove the case that ξ is not principal, that is, $A\xi = \alpha\xi + \beta U$ for some functions α, β and a vector U which satisfies $g(U, \xi) = 0$ and $g(U, U) = 1$. Let $X, Y \in TM$. From Codazzi equation we have

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi. \tag{10}$$

Taking the inner product (10) by ξ, we get

$$g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) = -2g(\phi X, Y).$$

Since $(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX + (X\beta)U + \beta\nabla_X U - A\phi AX$, we obtain

$$\begin{aligned} 2g(A\phi AX, Y) - \alpha g(A\phi X, Y) - \alpha g(\phi AX, Y) - 2g(\phi X, Y) \\ + (Y\alpha)g(X, \xi) - (X\alpha)g(Y, \xi) \\ + (Y\beta)g(U, X) - (X\beta)g(U, Y) \\ + \beta g(\nabla_Y U, X) - \beta g(\nabla_X U, Y) = 0. \end{aligned} \tag{11}$$

By the assumption of theorem, i.e., (9) the equation (11) yields

$$\begin{aligned} 2g(A\phi AX, Y) - 2\alpha g(A\phi X, Y) - 2g(\phi X, Y) \\ + \alpha g(A\phi X, \xi)g(\xi, Y) - \alpha\beta g(X, \xi)g(\phi U, Y) \\ + (Y\alpha)g(X, \xi) - (X\alpha)g(Y, \xi) \\ + (Y\beta)g(U, X) - (X\beta)g(U, Y) \\ + \beta g(\nabla_Y U, X) - \beta g(\nabla_X U, Y) = 0. \end{aligned} \tag{12}$$

Putting $X = \xi$ in (12), we have

$$\begin{aligned} Y\alpha = -3\beta g(A\phi U, Y) + \alpha\beta g(\phi U, Y) + (\xi\alpha)g(Y, \xi) \\ + (\xi\beta)g(U, Y) + \beta g(\nabla_\xi U, Y). \end{aligned} \tag{13}$$

Now, noting that

$$g(A\phi U, U) = g(\phi AU, U) = -g(AU, \phi U),$$

we can put

$$AU = \beta\xi + \lambda U + Z \tag{14}$$

for some function λ and a vector Z which satisfies $g(Z, \xi) = 0$ and $g(Z, U) = 0$. Also, since $A\phi$ is skew symmetric on $\{X \in \xi^{\perp} | g(X, U) = g(X, \phi U) = 0\} = T$, we can consider

$$A\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & & & & 0 \\ 0 & 0 & \lambda & * & & & & * \\ -\beta & -\lambda & 0 & * & & & & * \\ 0 & 0 & 0 & 0 & \mu_2 & & & \\ 0 & 0 & 0 & -\mu_2 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & \mu_{n-1} \\ & & & & & & -\mu_{n-1} & 0 \end{pmatrix}$$

at each point by an orthonormal basis. Hence (14) yields

$$AU = \beta\xi + \lambda U. \tag{15}$$

Differentiating (15) by $Y \in TM$, we get

$$(\nabla_Y A)U + A\nabla_Y U = (Y\beta)\xi + \beta\phi AY + (Y\lambda)U + \lambda\nabla_Y U. \tag{16}$$

Also, applying Codazzi equation to (16), we obtain

$$\begin{aligned}(\nabla_U A)Y + \eta(Y)\phi U - 2g(\phi Y, U)\xi + A\nabla_Y U \\ = (Y\beta)\xi + \beta\phi AY + (Y\lambda)U + \lambda\nabla_Y U.\end{aligned} \tag{17}$$

Taking the inner product (17) by ξ we have

$$Y\beta = g((\nabla_U A)\xi, Y) + 2g(\phi U, Y) - \lambda g(A\phi U, Y) + \alpha g(A\phi U, Y). \tag{18}$$

Also, taking the inner product (17) by U, we get

$$Y\lambda = g((\nabla_U A)U, Y) + 2\beta g(A\phi U, Y). \tag{19}$$

Noting that $A\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & & & & & 0 \\ 0 & 0 & \lambda & 0 & & & & & 0 \\ -\beta & -\lambda & 0 & 0 & & & & & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & & & & \\ 0 & 0 & 0 & -\mu_2 & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & & 0 & \mu_{n-1} \\ & & & & & & & -\mu_{n-1} & 0 \end{pmatrix}$

$$= \begin{pmatrix} \alpha & \beta & & & & & & \\ \beta & \lambda & & & & & & \\ & & \lambda & & & & & \\ & & & \mu_2 & & & & \\ & & & & \mu_2 & & & \\ & & & & & \ddots & & \\ & & & & & & \mu_{n-1} & \\ & & & & & & & \mu_{n-1} \end{pmatrix} \begin{pmatrix} 0 & & & & & & & \\ & 0 & 1 & & & & & \\ & -1 & 0 & & & & & \\ & & & 0 & 1 & & & \\ & & & -1 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix}$$

on each point and $\phi^2 = -1$, we can put $AX = \mu X$ for each $X \in \xi^{\perp}$ orthogonal to U and ϕU. Assume that $\mu = \lambda$. Taking the inner product (17) by X. We obtain

$$(\nabla_U A)X = -\beta A\phi X.$$

Noting that $A\phi X = \mu\phi X$, since $A\phi X = \phi AX$, we have $\mu = 0$. Therefore suppose $\mu \neq \lambda$. Then from (17) we get

$$(\mu - \lambda)g(\nabla_Y U, X) = -g((\nabla_U A)X, Y) - \beta g(Y, A\phi X). \tag{20}$$

Substituting (20) into (12) we obtain

$$\begin{aligned} 2(\mu - \lambda)\mu^2\phi X - 2\alpha(\mu - \lambda)\mu\phi X - 2(\mu - \lambda)\phi X \\ - (\mu - \lambda)(X\alpha)\xi - (\mu - \lambda)(X\beta)U \\ + \beta(-(\nabla_U A)X - \beta\mu\phi X) - \beta(\mu - \lambda)\nabla_X U = 0. \end{aligned} \tag{21}$$

Applying ϕ to (21), we have

$$\begin{aligned} -2(\mu - \lambda)\mu^2 X + 2\alpha(\mu - \lambda)\mu X + 2(\mu - \lambda)X \\ - (\mu - \lambda)(X\beta)\phi U \\ + \beta(-\phi(\nabla_U A)X + \beta\mu X) - \beta(\mu - \lambda)\phi\nabla_X U = 0. \end{aligned} \tag{22}$$

From Codazzi equation we obtain

$$(\nabla_U A)X = (\nabla_X A)U,$$

i.e.,

$$(U\mu)X + (\mu I - A)\nabla_U X = (X\beta)\xi + \beta\mu\phi X + (X\lambda)U + (\lambda I - A)\nabla_X U. \tag{23}$$

Taking the inner product (23) by ϕX, we have

$$\beta\mu + (\lambda - \mu)g(\nabla_X U, \phi X) = 0. \tag{24}$$

Combining (21) with (24), we get

$$(\mu - \lambda)\mu^2 - \alpha(\mu - \lambda)\mu - (\mu - \lambda) - \beta^2\mu = 0. \tag{25}$$

Also, fom Codazzi equation we have

$$(\nabla_X A)\phi U = (\nabla_{\phi U} A)X.$$

and get

$$(\phi U)\mu - (\mu - \lambda)g(\nabla_X U, \phi X) = 0. \tag{26}$$

By (24) and (26) we obtain

$$(\phi U)\mu = \beta\mu. \tag{27}$$

On the other hand, from (13), (18) and (19) we get

$$(\phi U)\alpha = -3\beta\lambda + \alpha\beta + \beta g(\nabla_\xi U, \phi U), \tag{28}$$

$$(\phi U)\beta = -\lambda^2 + \alpha\lambda + \beta^2 + 1, \tag{29}$$

$$(\phi U)\lambda = 3\beta\lambda. \tag{30}$$

We put $g(\nabla_\xi U, \phi U) = \Phi$. Defferentiating the equation (25) by ϕU, we have

$$\begin{aligned}\beta(3\mu^3 - 2\lambda\mu^2 - 3\alpha\mu^2 - \Phi\mu^2 + 3\alpha\mu\lambda \\ -\mu\lambda^2 + \Phi\mu\lambda - 3\mu + 3\lambda - 3\beta^2\mu) = 0.\end{aligned} \tag{31}$$

Combining (25) with (31), we obtain

$$\mu(\mu - \lambda)(\lambda - \Phi) = 0,$$

since $\beta \neq 0$. If $\mu = 0$, then from (25) we have $\lambda = 0$, which is a contradiction to the assumption of $\mu \neq \lambda$. Thus we obtain

$$\lambda - \Phi = 0. \tag{32}$$

Now, the equation (25), i.e.,

$$x^3 - (\lambda + \alpha)x^2 + (\alpha\lambda - 1 - \beta^2)x + \lambda = 0. \tag{33}$$

has at most three distinct roots. Suppose that μ, ν and τ are distinct roots of (33). Without the loss of generality we may assume that μ is the largest root of (33). Replacing x by $x + \mu$ and using (25), we have

$$x^3 - (-3\mu + \lambda + \alpha)x^2 + (3\mu^2 - 2\mu(\lambda + \alpha) + (\alpha\lambda - 1 - \beta^2))x = 0. \tag{34}$$

Then two roots of (34) which are not zero must have the same signs. Hence

$$3\mu^2 - 2\mu(\lambda + \alpha) + (\alpha\lambda - 1 - \beta^2) \tag{35}$$

is absolutely positive. But the discriminant of (35) is absolutely positive, which is contradiction. Thus (33) has at most two distinct roots. If μ and ν are distinct roots of (33), then we have

$$(\nu - \lambda)\nu^2 - \alpha(\nu - \lambda)\nu - (\nu - \lambda) - \beta^2\nu = 0. \tag{36}$$

Subtracting (36) from (25), we get

$$\mu^2 + \mu\nu + \nu^2 - \lambda\mu - \lambda\nu - \alpha\nu + \alpha\lambda - 1 - \beta^2 = 0, \tag{37}$$

since $\mu \neq \nu$. Multiplying (37) by μ, we have

$$\mu^3 + \mu^2\nu + \mu\nu^2 - \lambda\mu^2 - \lambda\mu\nu - \alpha\mu^2 - \alpha\mu\nu + \alpha\lambda\mu - \mu - \beta^2\mu = 0. \tag{38}$$

Subtracting (38) from (25), we get

$$\mu^2\nu + \mu\nu^2 - \lambda\mu\nu - \alpha\mu\nu - \lambda = 0. \tag{39}$$

From the roots and coefficients of the equation (39) we have

$$\mu + \nu = -\nu + \lambda + \alpha = -\mu + \lambda + \alpha.$$

Therefore (33) has only a root. Thus we get

$$3\mu = \lambda + \alpha, \tag{40}$$

$$3\mu^2 = \alpha\lambda - 1 - \beta^2, \tag{41}$$

$$-\mu^3 = \lambda.$$

By the way, the equation (25) yields

$$(\mu - \lambda)^3 - (\alpha - 2\lambda)(\mu - \lambda)^2 + (\lambda^2 - \alpha\lambda - 1 - \beta^2)(\mu - \lambda) - \beta^2\lambda = 0. \tag{42}$$

Then from (40) and (41) we have

$$(43)\qquad \alpha - 2\lambda = 3(\mu - \lambda),$$

$$(44)\qquad (\lambda^2 - \alpha\lambda - 1 - \beta^2) = 3(\mu - \lambda)^2.$$

Combining (42) with (43) and (44), we obtain

$$(45)\qquad (\mu - \lambda)^3 - \beta^2\lambda = 0.$$

Differentiating (45) by ϕU we have

$$\begin{aligned} &3(\mu - \lambda)^2(\mu - \lambda - 2\lambda) - 3\beta^2\lambda \\ &- 2\lambda(-\lambda^2 + \alpha\lambda + \beta^2 + 1) = 0, \end{aligned}$$

since $\beta \neq 0$. Using (40) and (45), we obtain

$$\begin{aligned} &- 6\lambda(\frac{\lambda + \alpha}{3} - \lambda)^2 \\ &- 2\lambda(-\lambda^2 + \alpha\lambda + \beta^2 + 1), \\ &= \frac{\lambda}{3}(-2\lambda^2 + 2\alpha\lambda - 2\alpha^2 - 6 - 6\beta^2) = 0, \end{aligned}$$

i.e., $\lambda = 0$. By (45) we get $\mu = 0$, which contradicts to the assumption of $\mu \neq \lambda$. Thus M is a ruled real hypersurfaces. Conversely, assume that A_1, A_2 or a ruled hypersurface. Then we can easily confirm that they satisfies (2).

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