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KILLING VECTOR FIELDS OF A 4-SPACE ON $R^4{}_+$

By

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Abstract. We studied the Killing fields of a spacetime with pseudo Metric:

$$ds^2 = \frac{1}{x_4x_4}\left\{\sum_{b,c=1}^{3}\left(\delta_{bc} - \frac{ax_bx_c}{1+ar^2}\right)dx_bdx_c - \frac{1}{1+ax_4x_4}dx_4dx_4\right\},$$

on $R^4{}_+ = R^3 \times R_+$, where $r^2 = \sum_{b=1}^3 x_a x_b$, $a =$ constant > 0 in [13]. In this paper, we shall investigate the analogous problems for the case $a < 0$, for which the above metric has singularity where $1 + ar^2 = 0$ or $1 + ax_4x_4 = 0$.

1. Preliminaries

Let us put $a = -1/r_0{}^2$, $r_0 > 0$. Then the above Ot-metric becomes

$$(1.1)\qquad ds^2 = \frac{1}{x_4x_4}\left\{\sum_{b,c=1}^{3}\left(\delta_{bc} + \frac{x_bx_c}{r_0{}^2 - r^2}\right)dx_bdx_c - \frac{r_0{}^2}{r_0{}^2 - x_4x_4}dx_4dx_4\right\},$$

which is singular at $r = r_0$ or $x_4 = r_0$. Let D_i be the domains of $R^3 \times R_+$ defined by

$$D_1 : 0 < r < r_0,\ 0 < x_4 < r_0; D_2 : r_0 < r,\ 0 < x_4 < r_0;$$
$$D_3 : r_0 < r, \qquad r_0 < x_4; \qquad D_4 : 0 < r < r_0,\ r_0 < x_4$$

and ϵ' and ϵ'' be the auxiliary functions which are constant on each D_i as

$$(1.2)\qquad \begin{array}{lcl} \epsilon' = 1 \text{ on } D_1 \text{ and } D_4 & \text{and} & \epsilon' = -1 \text{ on } D_2 \text{ and } D_3, \\ \epsilon'' = 1 \text{ on } D_1 \text{ and } D_2 & \text{and} & \epsilon'' = -1 \text{ on } D_3 \text{ and } D_4. \end{array}$$

Then we have easily the equalities

$$(1.3)\qquad |r_0{}^2 - r^2| = \epsilon'(r_0{}^2 - r^2) \text{ and } |r_0{}^2 - x_4x_4| = \epsilon''(r_0{}^2 - x_4x_4).$$

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Denoting the metric (1,1) as

$$ds^2 = \sum_{i.j=1}^{4} g_{ij} dx_i dx_j, \quad g_{ij} = g_{ji},$$

$$(1.4) \qquad g_{bc} = \frac{1}{x_4 x_4}\left(\delta_{bc} + \frac{x_b x_c}{{r_0}^2 - r^2}\right), \; g_{b4} = 0, \; g_{44} = -\frac{{r_0}^2}{x_4 x_4({r_0}^2 - x_4 x_4)},$$

$b, c = 1, 2, 3$, from which $(g^{ij}) = (g_{ij})^{-1}$ is given by

$$(1.4') \qquad g^{bc} = x_4 x_4 \left(\delta_{bc} - \frac{x_b x_c}{{r_0}^2}\right), \; g^{b4} = 0, \; g^{44} = -x_4 x_4 \frac{{r_0}^2 - x_4 x_4}{{r_0}^2}.$$

The Christoffel symbols computed from (1.4) are given as

$$(1.5) \qquad \begin{aligned} &\{{}_b{}^e{}_c\} = \frac{x_e}{{r_0}^2}\left(\delta_{bc} + \frac{x_b x_c}{{r_0}^2 - r^2}\right), \qquad \{{}_b{}^4{}_c\} = -\frac{{r_0}^2 - x_4 x_4}{{r_0}^2 x_4}\left(\delta_{bc} + \frac{x_b x_c}{{r_0}^2 - r^2}\right), \\ &\{{}_b{}^e{}_4\} = -\frac{1}{x_4}\delta_b{}^e, \qquad \{{}_b{}^4{}_4\} = 0, \\ &\{{}_4{}^e{}_4\} = 0, \qquad \{{}_4{}^4{}_4\} = -\frac{{r_0}^2 - 2x_4 x_4}{x_4({r_0}^2 - x_4 x_4)} \end{aligned}$$

by (1.3) in [13].

Lemma 1. *The signatures of the metric* (1.1) *on each* D_i *is given as*

$$(+++-) \text{ on } D_1, \; (++--) \text{ on } D_2, \; (++-+) \text{ on } D_3, \; (++++) \text{ on } D_4.$$

Proof. For any X_i we have

$$x_4 x_4 \sum_{i,j} g_{ij} X_i X_j = \sum_b X_b X_b + \frac{1}{{r_0}^2 - r^2}\left(\sum_b x_b X_b\right)^2 - \frac{{r_0}^2}{{r_0}^2 - x_4 x_4} X_4 X_4$$

$$= \left\{\sum_b X_b X_b - \left(\sum_b \frac{x_b}{r} X_b\right)^2\right\} + \frac{{r_0}^2}{{r_0}^2 - r^2}\left(\sum_b \frac{x_b}{r} X_b\right)^2 - \frac{{r_0}^2}{{r_0}^2 - x_4 x_4} X_4 X_4,$$

which implies easily the above claim.

As shown in [13], the components $R_j{}^i{}_{hk}$ of the curvature tensor :

$$R_j{}^i{}_{hk} := \frac{\partial\{{}_j{}^i{}_k\}}{\partial x_h} - \frac{\partial\{{}_j{}^i{}_h\}}{\partial x_k} + \sum_l \{{}_l{}^i{}_h\}\{{}_j{}^l{}_k\} - \sum_l \{{}_l{}^i{}_k\}\{{}_j{}^l{}_h\}$$

satisfy

$$R_j{}^i{}_{hk} = \delta^i_h g_{jk} - \delta^i_k g_{jh}$$

and the Ricci curvature

$$R_{jk} = \sum_l R_j{}^h{}_{hk}$$

and the scalar curvature $R = \sum_{j,k} g^{jk} R_{jk}$ becomes as

$$R_{ij} = 3g_{jk} \qquad \text{and} \qquad R = 12,$$

and hence the metric (1.1) satisfies the Einstein condition

$$R_{ij} = \frac{R}{4} g_{ij}.$$

Let $V = \sum_i v^i \partial/\partial x_i$ be a Killing vector field. It satisfies the condition :

$$v_{i,j} + v_{j,i} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - 2\sum_k \{{}_i{}^k{}_j\} v_k = 0, \; v_i = \sum_j g_{ij} v^j$$

which can be written by (1.4) and (1.5) as

$$\text{(1.6)} \qquad \frac{\partial v_b}{\partial x_c} + \frac{\partial v_c}{\partial x_b} - \frac{2}{r_0{}^2}\left(\delta_{bc} + \frac{x_b x_c}{r_0{}^2 - r^2}\right)\sum_e x_e v_e + \frac{2(r_0{}^2 - x_4 x_4)}{r_0{}^2 x_4}\left(\delta_{bc} + \frac{x_b x_c}{r_0{}^2 - r^2}\right) v_4 = 0,$$

$$\text{(1.7)} \qquad \frac{\partial v_b}{\partial x_4} + \frac{\partial v_4}{\partial x_b} + \frac{2}{x_4} v_b = 0,$$

$$\text{(1.8)} \qquad \frac{\partial v_4}{\partial x_4} + \frac{r_0{}^2 - 2x_4 x_4}{x_4(r_0{}^2 - x_4 x_4)} v_4 = 0.$$

Since the argument deriving the solution of (1.6) ,(1.7) and (1.8) is the same as the case $a > 0$ treated in [13] on D_1, we obtain a general solution:

$$\text{(1.9)} \qquad \begin{aligned} v_b &= \frac{1}{x_4 x_4}\Bigg\{ r_0\sqrt{r_0{}^2 - x_4 x_4}\,\lambda_b + \frac{1}{r_0}\sqrt{r_0{}^2 - r^2}\,p_b - (\mu \times \tilde{x})_b \\ &\quad - \frac{1}{\sqrt{r_0{}^2 - r^2}}\left(p_0\sqrt{r_0{}^2 - x_4 x_4} - \frac{1}{r_0}(p \cdot \tilde{x})\right) x_b \Bigg\}, \\ v_4 &= \frac{1}{x_4\sqrt{r_0{}^2 - x_4 x_4}}\left(r_0(\lambda \cdot \tilde{x}) + p_0\sqrt{r_0{}^2 - r^2}\right), \end{aligned}$$

where p_0 is a constant and

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

are constant vectors in R^3, and $(\lambda \cdot \tilde{x})$ and $(\mu \times \tilde{x})$ denote the inner and outer products of λ and μ with $\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, respectively, (see Theorem 1 in [13]).

2. Solutions of (1.6), (1.7) and (1.8)

Looking over the solution (1.9) of the partial differential equations (1.6), (1.7) and (1.8) on the domain D_1 , we give the following theorem.

THEOREM 1. *The following vector field $V = \sum_i v^i \partial/\partial x_i$ depending on 10 real constants $p_0, \lambda_i, p_i, \mu_i, i = 1.2.3$ given by*

$$
\begin{aligned}
v_b &= \frac{1}{x_4 x_4}\left\{ r_0 \epsilon'' \sqrt{|r_0{}^2 - x_4 x_4|}\lambda_b + \frac{1}{r_0}\epsilon' \sqrt{|r_0{}^2 - r^2|} p_b - (\mu \times \tilde{x})_b \right. \\
&\quad \left. - \frac{1}{\sqrt{|r_0{}^2 - r^2|}}\left(p_0 \epsilon'' \sqrt{|r_0{}^2 - x_4 x_4|} - \frac{1}{r_0}(p \cdot \tilde{x})\right) x_b \right\}, \qquad (2.1)\\
v_4 &= \frac{1}{x_4 \sqrt{|r_0{}^2 - x_4 x_4|}}\left(r_0 (\lambda \cdot \tilde{x}) + p_0 \epsilon' \sqrt{|r_0{}^2 - r^2|}\right), \quad v_i = \sum_j g_{ij} v^j
\end{aligned}
$$

is a Killing vector field on D_1, D_2, D_3 and D_4 for the metric (1.1).

Proof. Using (1.3), first we have

$$
\begin{aligned}
\frac{\partial v_4}{\partial x_4} &= \left(-\frac{1}{x_4{}^2 \sqrt{|r_0{}^2 - x_4 x_4|}} + \frac{x_4 \epsilon''}{x_4 |r_0{}^2 - x_4 x_4|^{3/2}} \right)\left(r_0 (\lambda \cdot \tilde{x}) + p_0 \epsilon' \sqrt{|r_0{}^2 - r^2|}\right) \\
&= \left(-\frac{1}{x_4} + \frac{x_4 \epsilon''}{|r_0{}^2 - x_4 x_4|}\right) v_4 = \left(-\frac{1}{x_4} + \frac{x_4}{r_0{}^2 - x_4 x_4}\right) v_4 = -\frac{r_0{}^2 - 2x_4 x_4}{x_4 (r_0{}^2 - x_4 x_4)} v_4
\end{aligned}
$$

hence (1.8) holds.

Second, we have

$$
\begin{aligned}
\frac{\partial v_b}{\partial x_4} &= -\frac{2}{x_4} v_b + \frac{1}{x_4 x_4}\left\{ -\frac{r_0 x_4}{\sqrt{|r_0{}^2 - x_4 x_4|}}\lambda_b + \frac{p_0 x_4}{\sqrt{|r_0{}^2 - r^2|}\sqrt{|r_0{}^2 - x_4 x_4|}} x_b \right\} \\
\frac{\partial v_4}{\partial x_b} &= \frac{1}{x_4 \sqrt{|r_0{}^2 - x_4 x_4|}}\left\{ r_0 \lambda_b - \frac{p_0}{\sqrt{|r_0{}^2 - r^2|}} x_b \right\}
\end{aligned}
$$

from which we obtain

$$
\frac{\partial v_b}{\partial x_4} + \frac{\partial v_4}{\partial x_b} + \frac{2}{x_4} v_b = 0
$$

and so (1.7) holds. Finally, since we have

$$\begin{aligned}
&\frac{\partial v_b}{\partial x_c} + \frac{\partial v_c}{\partial x_b} \\
&= \frac{1}{x_4x_4}\left\{-\frac{x_c}{r_0\sqrt{|r_0{}^2 - r^2|}}p_b - \frac{x_c\epsilon'}{|r_0{}^2 - r^2|^{3/2}}\left(p_0\epsilon''\sqrt{|r_0{}^2 - x_4x_4|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)x_b\right. \\
&\quad \left. - \frac{\delta_{bc}}{\sqrt{|r_0{}^2 - r^2|}}\left(p_0\epsilon''\sqrt{|r_0{}^2 - x_4x_4|} - \frac{1}{r_0}(p\cdot\tilde{x})\right) + \frac{p_cx_b}{r_0\sqrt{|r_0{}^2 - x_4x_4|}}\right\} \\
&\quad + \frac{1}{x_4x_4}\left\{-\frac{x_b}{r_0\sqrt{|r_0{}^2 - r^2|}}p_c - \frac{x_b\epsilon'}{|r_0{}^2 - r^2|^{3/2}}\left(p_0\epsilon''\sqrt{|r_0{}^2 - x_4x_4|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)x_c\right. \\
&\quad \left. - \frac{\delta_{cb}}{\sqrt{|r_0{}^2 - r^2|}}\left(p_0\epsilon''\sqrt{|r_0{}^2 - x_4x_4|} - \frac{1}{r_0}(p\cdot\tilde{x})\right) + \frac{p_bx_c}{r_0\sqrt{|r_0{}^2 - r^2|}}\right\} \\
&= -\frac{2}{x_4x_4}\left(\frac{x_bx_c\epsilon'}{|r_0{}^2 - r^2|^{3/2}} + \frac{\delta_{bc}}{\sqrt{|r_0{}^2 - r^2|}}\right)\left(p_0\epsilon''\sqrt{|r_0{}^2 - x_4x_4|} - \frac{1}{r_0}(p\cdot\tilde{x})\right) \\
&= -\frac{2}{x_4x_4\sqrt{|r_0{}^2 - r^2|}}\left(\delta_{bc} + \frac{x_bx_c}{r_0{}^2 - r^2}\right)\left(p_0\epsilon''\sqrt{|r_0{}^2 - x_4x_4|} - \frac{1}{r_0}(p\cdot\tilde{x})\right),
\end{aligned}$$

$$\begin{aligned}
&-\frac{2}{r_0{}^2}\left(\delta_{bc} + \frac{x_bx_c}{r_0{}^2 - r^2}\right)\sum_e x_ev_e \\
&= -\frac{2}{r_0{}^2}\left(\delta_{bc} + \frac{x_bx_c}{r_0{}^2 - r^2}\right)\frac{1}{x_4x_4}\left\{r_0\epsilon''\sqrt{|r_0{}^2 - x_4x_4|}(\lambda\cdot\tilde{x})\right. \\
&\quad \left. + \frac{\epsilon'}{r_0}\sqrt{|r_0{}^2 - r^2|}(p\cdot\tilde{x}) - \frac{r^2}{\sqrt{|r_0{}^2 - r^2|}}\left(p_0\epsilon''\sqrt{|r_0{}^2 - x_4x_4|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)\right\} \\
&= -\frac{2}{x_4x_4}\left(\delta_{bc} + \frac{x_bx_c}{r_0{}^2 - r^2}\right)\left\{\frac{\epsilon''}{r_0}\sqrt{|r_0{}^2 - x_4x_4|}(\lambda\cdot\tilde{x}) + \frac{1}{r_0\sqrt{|r_0{}^2 - r^2|}}(p\cdot\tilde{x})\right. \\
&\quad \left. - \frac{p_0r^2\epsilon''}{r_0{}^2\sqrt{|r_0{}^2 - r^2|}}\sqrt{|r_0{}^2 - x_4x_4|}\right\} \\
&= -\frac{2}{x_4x_4\sqrt{|r_0{}^2 - r^2|}}\left(\delta_{bc} + \frac{x_bx_c}{r_0{}^2 - r^2}\right)\left\{\frac{\epsilon''}{r_0}\sqrt{|r_0{}^2 - r^2|}\sqrt{|r_0{}^2 - x_4x_4|}(\lambda\cdot\tilde{x})\right. \\
&\quad \left. + \frac{1}{r_0}(p\cdot\tilde{x}) - \frac{p_0r^2\epsilon''}{r_0{}^2}\sqrt{|r_0{}^2 - x_4x_4|}\right\}
\end{aligned}$$

and

$$\frac{2(r_0{}^2 - x_4x_4)}{r_0{}^2x_4}\left(\delta_{bc} + \frac{x_bx_c}{r_0{}^2 - r^2}\right)v_4$$

$$= \frac{2\epsilon''\sqrt{|{r_0}^2 - x_4x_4|}}{{r_0}^2 x_4x_4}\left(\delta_{bc} + \frac{x_bx_c}{{r_0}^2 - r^2}\right) \times \left(r_0(\lambda\cdot\tilde{x}) + p_0\epsilon'\sqrt{|{r_0}^2 - r^2|}\right)$$
$$= \frac{2}{x_4x_4\sqrt{|{r_0}^2 - r^2|}}\left(\delta_{bc} + \frac{x_bx_c}{{r_0}^2 - r^2}\right)\left\{\frac{\epsilon''\sqrt{|{r_0}^2 - r^2|}\sqrt{|{r_0}^2 - x_4x_4|}}{r_0}(\lambda\cdot\tilde{x})\right.$$
$$\left. + \frac{p_0\epsilon''\sqrt{|{r_0}^2 - x_4x_4|}({r_0}^2 - r^2)}{{r_0}^2}\right\}$$

we obtain easily

$$\frac{\partial v_b}{\partial x_c} + \frac{\partial v_c}{\partial x_b} - \frac{2}{{r_0}^2}\left(\delta_{bc} + \frac{x_bx_c}{{r_0}^2 - r^2}\right)\sum_e x_ev_e + \frac{2({r_0}^2 - x_4x_4)}{{r_0}^2 x_4}\left(\delta_{bc} + \frac{x_bx_c}{{r_0}^2 - r^2}\right)v_4 = 0,$$

and so (1.6) holds. Hence the vector field V given by (2.1) satisfies (1.6), (1.7) and (1.8) and so a Killing field for the metric (1.1). Q.E.D.

In the following we compute the norm $N(v)$ of the Killing field $v = V$ given by (2.1) :

$$N(v) = \sum_{i,j} g_{ij}v^iv^j = \sum_{i,j} g^{ij}v_iv_j \tag{2.2}$$

Using the notations $L = {r_0}^2 - r^2$ and $M = {r_0}^2 - x_4x_4$ and the identity

$$(\lambda\cdot(\mu\times\tilde{x})) = ((\lambda\times\mu)\cdot\tilde{x}),$$

we have by (1.4) and (2.1)

$$x_4x_4N(v) = \sum_{b,c}\left(\delta^{bc} - \frac{x_bx_c}{{r_0}^2}\right)$$
$$\times\left\{r_0\epsilon''\sqrt{|M|}\lambda_b + \frac{1}{r_0}\epsilon'\sqrt{|L|}p_b - (\mu\times\tilde{x})_b - \frac{1}{\sqrt{|L|}}\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)x_b\right\}$$
$$\times\left\{r_0\epsilon''\sqrt{|M|}\lambda_c + \frac{1}{r_0}\epsilon'\sqrt{|L|}p_c - (\mu\times\tilde{x})_c - \frac{1}{\sqrt{|L|}}\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)x_c\right\}$$
$$- \frac{x_4x_4}{{r_0}^2}\frac{M}{|M|}\left(r_0(\lambda\cdot\tilde{x}) + p_0\epsilon'\sqrt{|L|}\right)^2$$
$$= {r_0}^2|M|(\lambda\cdot\lambda) + 2\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}(\lambda\cdot p) - 2r_0\epsilon''\sqrt{|M|}((\lambda\times\mu)\cdot\tilde{x})$$
$$- 2r_0\epsilon''\frac{\sqrt{|M|}}{\sqrt{|L|}}\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)(\lambda\cdot\tilde{x}) + \frac{1}{{r_0}^2}|L|(p\cdot p)$$
$$- \frac{2}{r_0}\epsilon'\sqrt{|L|}((p\times\mu)\cdot\tilde{x}) - \frac{2}{r_0}\epsilon'\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)(p\cdot\tilde{x})$$

$$
\begin{aligned}
&+ ((\mu \times \tilde{x}) \cdot (\mu \times \tilde{x})) + \frac{r^2}{|L|}\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p \cdot \tilde{x})\right)^2 \\
&- \frac{1}{r_0{}^2}\left\{r_0\epsilon''\sqrt{|M|}(\lambda \cdot \tilde{x}) + \frac{1}{r_0}\epsilon'\sqrt{|L|}(p \cdot \tilde{x}) - \frac{r^2}{|L|}\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p \cdot \tilde{x})\right)\right\}^2 \\
&- \frac{x_4x_4}{r_0{}^2}\epsilon''\left(r_0(\lambda \cdot \tilde{x}) + p_0\epsilon'\sqrt{|L|}\right)^2 \\
&= r_0{}^2|M|(\lambda \cdot \lambda) + 2\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}(\lambda \cdot p) - 2r_0\epsilon''\sqrt{|M|}((\lambda \times \mu) \cdot \tilde{x}) \\
&- 2p_0r_0\frac{|M|}{\sqrt{|L|}}(\lambda \cdot \tilde{x}) + 2\epsilon''\frac{\sqrt{|M|}}{\sqrt{|L|}}(\lambda \cdot \tilde{x})(p \cdot \tilde{x}) + \frac{1}{r_0{}^2}|L|(p \cdot p) \\
&- \frac{2}{r_0}\epsilon'\sqrt{|L|}((p \times \mu) \cdot \tilde{x}) - 2\frac{p_0}{r_0}\epsilon'\epsilon''\sqrt{|M|}(p \cdot \tilde{x}) + \frac{2}{r_0{}^2}\epsilon'(p \cdot \tilde{x})^2 + (\mu \cdot \mu)r^2 \\
&- (\mu \cdot \tilde{x})^2 + \frac{p_0{}^2r^2}{|L|}|M| - \frac{2p_0r^2}{r_0|L|}\epsilon''\sqrt{|M|}(p \cdot \tilde{x}) + \frac{r^2}{r_0{}^2|L|}(p \cdot \tilde{x})^2 - |M|(\lambda \cdot \tilde{x})^2 \\
&- \frac{|L|}{r_0{}^4}(p \cdot \tilde{x})^2 - \frac{r^4}{r_0{}^2|L|}\left(p_0{}^2|M| - \frac{2p_0}{r_0}\epsilon''\sqrt{|M|}(p \cdot \tilde{x}) + \frac{1}{r_0{}^2}(p \cdot \tilde{x})^2\right) \\
&- \frac{2}{r_0{}^2}\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}(\lambda \cdot \tilde{x})(p \cdot \tilde{x}) + \frac{2p_0r^2|M|}{r_0\sqrt{|L|}}(\lambda \cdot \tilde{x}) - \frac{2r^2}{r_0{}^2|L|}\epsilon''\sqrt{|M|} \times \\
&\quad (\lambda \cdot \tilde{x})(p \cdot \tilde{x}) + \frac{2p_0r^2}{r_0{}^3}\epsilon'\epsilon''\sqrt{|M|}(p \cdot \tilde{x}) - \frac{2r^2}{r_0{}^4}\epsilon'(p \cdot \tilde{x})^2 - x_4x_4\epsilon''(\lambda \cdot \tilde{x})^2 \\
&- \frac{2p_0x_4x_4}{r_0}\epsilon'\epsilon''\sqrt{|L|}(\lambda \cdot \tilde{x}) - \frac{p_0{}^2x_4x_4}{r_0{}^2}\epsilon''|L|,
\end{aligned}
$$

which is arranged as follows:

$$
\begin{aligned}
x_4x_4N(v) &= r_0{}^2|r_0{}^2 - x_4x_4|(\lambda \cdot \lambda) + \frac{1}{r_0{}^2}|r_0{}^2 - r^2|(p \cdot p) - r_0{}^2\epsilon''(\lambda \cdot \tilde{x})^2 \\
&+ \frac{1}{r_0{}^2}\epsilon'(p \cdot \tilde{x})^2 + (\mu \cdot \mu)r^2 - (\mu \cdot \tilde{x})^2 + 2\epsilon'\epsilon''\sqrt{|r_0{}^2 - r^2|}\sqrt{|r_0{}^2 - x_4x_4|}(\lambda \cdot p) \\
&- 2r_0\epsilon''\sqrt{|r_0{}^2 - x_4x_4|}((\lambda \times \mu) \cdot \tilde{x}) - \frac{2}{r_0}\epsilon'\sqrt{|r_0{}^2 - r^2|}((p \times \mu) \cdot \tilde{x}) \\
&- 2p_0r_0\epsilon'\epsilon''\sqrt{|r_0{}^2 - r^2|}(\lambda \cdot \tilde{x}) - \frac{2p_0}{r_0}\epsilon'\epsilon''\sqrt{|r_0{}^2 - x_4x_4|}(p \cdot \tilde{x}) \\
&+ p_0{}^2\epsilon'\epsilon''(r_0{}^2 - x_4x_4).
\end{aligned} \tag{2.3}
$$

Now, we denote the special Killing field with

$$p_0 = -1, \quad \lambda = p = \mu = 0$$

in (2.1) by $\xi = \sum_i \xi_i dx_i$. We have

$$\xi_b = \frac{\epsilon''\sqrt{|r_0{}^2 - x_4x_4|}}{x_4x_4\sqrt{|r_0{}^2 - r^2|}}x_b, \quad \xi_4 = -\frac{\epsilon'\sqrt{|r_0{}^2 - r^2|}}{x_4\sqrt{|r_0{}^2 - x_4x_4|}} \tag{2.4}$$

and hence obtain

$$\xi^i = \frac{1}{{r_0}^2}\epsilon'\epsilon''\sqrt{|{r_0}^2 - r^2|}\sqrt{|{r_0}^2 - x_4x_4|}x_i. \tag{2.4'}$$

Lemma 2. *For the Killing Field ξ , we have*

(i) *The sign of $N(\xi)$ is shown in Fig.1;*

(ii) *The Pfaff equation $\xi = \sum_i \xi_i dx_i = 0$ is complete and its solution is ${r_0}^2 - x_4x_4 = c({r_0}^2 - r^2)$, where c is an integral constant.*

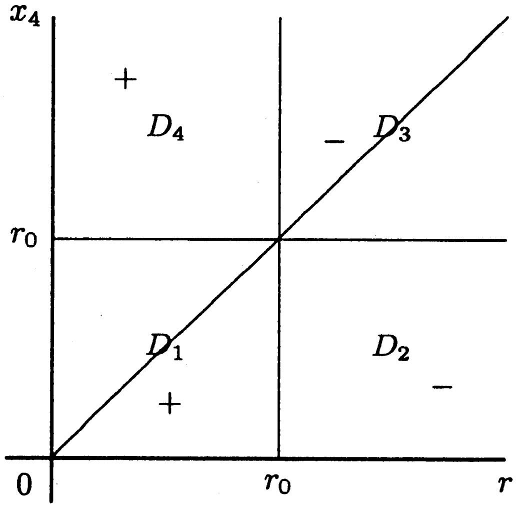

Fig. 1

Proof. By (2.3) we have

$$x_4x_4N(\xi) = \epsilon'\epsilon''({r_0}^2 - x_4x_4),$$

from which we obtain easily (i) by means of (1.2).

Next we have

$$\xi = \sum_i \xi_i dx_i = \frac{\epsilon''\sqrt{|{r_0}^2 - x_4x_4|}}{x_4x_4\sqrt{|{r_0}^2 - r^2|}}\sum_b x_b dx_b - \frac{\epsilon'\sqrt{|{r_0}^2 - r^2|}}{x_4\sqrt{|{r_0}^2 - x_4x_4|}}dx_4$$

and hence the Pfaff equation $\xi = 0$ is equivalent to

$$\epsilon''|{r_0}^2 - x_4x_4|\sum_b x_b dx_b - \epsilon'|{r_0}^2 - r^2|x_4 dx_4 = 0,$$

which becomes

$$({r_0}^2 - x_4x_4)rdr - ({r_0}^2 - r^2)x_4dx_4 = 0.$$

Therefore, its solution is given by

$$r_0{}^2 - x_4x_4 = c(r_0{}^2 - r^2) \tag{2.5}$$

with integral constant c. Q.E.D.

We denote this symmetric in R^3 and quadratic hypersurface in $R^4_+ = R^3 \times R_+$ by $\sum_c$ in the following.

3. Special Killing forms

We shall investigate some special Killing field V given by (2.1) and putting

$$\theta := \sum_b v_b dx_b + v_4 dx_4,$$

such that the system of Pfaff equations :

$$\xi = 0 \qquad \text{and} \qquad \theta = 0 \tag{3.1}$$

is complete, that is, it admits locally a surface satisfying both equations on D_i. As it is well known, it is necessary and sufficient that the following equalities hold

$$\xi \wedge \theta \wedge d\xi = 0 \qquad \text{and} \qquad \xi \wedge \theta \wedge d\theta = 0. \tag{3.2}$$

By Lemma 2, the first equality holds. We shall compute the left hand of the second equality. In computation we shall use the following notations and equalities :

$$L = r_0{}^2 - r^2, \ M = r_0{}^2 - x_4x_4, \ d_2\tilde{x} = \begin{pmatrix} dx_2 \wedge dx_3 \\ dx_3 \wedge dx_1 \\ dx_1 \wedge dx_2 \end{pmatrix}, \ d_3\tilde{x} = dx_1 \wedge dx_2 \wedge dx_3,$$

and

$$\begin{cases} (\lambda \cdot d\tilde{x}) \wedge (\mu \cdot d\tilde{x}) = ((\lambda \times \mu) \cdot d_2\tilde{x}), \\ (\lambda \cdot d\tilde{x}) \wedge (\mu \cdot d_2\tilde{x}) = (\lambda \cdot \mu) d_3\tilde{x}, \ (p \cdot d\tilde{x}) \wedge (\mu \cdot d_2\tilde{x}) = (p \cdot \mu) d_3\tilde{x}, \\ rdr \wedge (\mu \cdot d_2\tilde{x}) = (\mu \cdot \tilde{x}) d_3\tilde{x}, \ ((\mu \times \tilde{x}) \cdot d\tilde{x}) \wedge (\mu \cdot d_2\tilde{x}) = 0. \end{cases} \tag{3.3}$$

First, we have from (2.1)

$$\theta = \frac{1}{x_4x_4}\Big\{ r_0\epsilon''\sqrt{|M|}(\lambda \cdot d\tilde{x}) + \frac{1}{r_0}\epsilon'\sqrt{|L|}(p \cdot d\tilde{x}) - ((\mu \times \tilde{x}) \cdot d\tilde{x}) \tag{3.4}$$

$$-\frac{1}{\sqrt{|L|}}\left(p_0\epsilon''\sqrt{|M|}-\frac{1}{r_0}(p\cdot\tilde{x})\right)rdr+\frac{1}{\sqrt{|M|}}\left(p_0\epsilon'\sqrt{|L|}+r_0(\lambda\cdot\tilde{x})\right)x_4dx_4\Big\}$$

and

$$\begin{aligned} d\theta = &-\frac{2}{x_4}dx_4\wedge\theta \\ &+\frac{1}{x_4x_4}\Big\{-\frac{r_0}{\sqrt{|M|}}x_4dx_4\wedge(\lambda\cdot d\tilde{x})-\frac{1}{r_0\sqrt{|L|}}rdr\wedge(p\cdot\tilde{x})-2(\mu\cdot d_2\tilde{x}) \\ &+\frac{1}{\sqrt{|L|}}\left(\frac{p_0}{\sqrt{|M|}}x_4dx_4+\frac{1}{r_0}(p\cdot d\tilde{x})\right)\wedge rdr \\ &+\frac{1}{\sqrt{|M|}}\left(-\frac{p_0}{\sqrt{|L|}}rdr+r_0(\lambda\cdot d\tilde{x})\right)\wedge x_4dx_4\Big\}, \end{aligned}$$

from which we obtain

$$\begin{aligned} (x_4x_4)^2\theta\wedge d\theta = &\Big\{r_0\epsilon''\sqrt{|M|}(\lambda\cdot d\tilde{x})+\frac{1}{r_0}\epsilon'\sqrt{|L|}(p\cdot d\tilde{x})-((\mu\times\tilde{x})\cdot d\tilde{x}) \\ &-\frac{1}{\sqrt{|L|}}\left(p_0\epsilon''\sqrt{|M|}-\frac{1}{r_0}(p\cdot\tilde{x})\right)rdr+\frac{1}{\sqrt{|M|}}\left(p_0\epsilon'\sqrt{|L|}+r_0(\lambda\cdot\tilde{x})\right)x_4dx_4\Big\} \\ &\wedge 2\Big\{\frac{r_0}{\sqrt{|M|}}(\lambda\cdot d\tilde{x})\wedge x_4dx_4-\frac{1}{r_0\sqrt{|L|}}rdr\wedge(p\cdot\tilde{x}) \\ &-\frac{p_0}{\sqrt{|L|}\sqrt{|M|}}rdr\wedge x_4dx_4-(\mu\cdot d_2\tilde{x})\Big\} \\ = 2\Big[&\epsilon'\frac{\sqrt{|L|}}{\sqrt{|M|}}(p\cdot d\tilde{x})\wedge(\lambda\cdot d\tilde{x})\wedge x_4dx_4-\frac{r_0}{\sqrt{|M|}}((\mu\times\tilde{x})\cdot d\tilde{x})\wedge(\lambda\cdot d\tilde{x})\wedge x_4dx_4 \\ &-\frac{r_0}{\sqrt{|L|}\sqrt{|M|}}\left(p_0\epsilon''\sqrt{|M|}-\frac{1}{r_0}(p\cdot\tilde{x})\right)rdr\wedge(\lambda\cdot d\tilde{x})\wedge x_4dx_4 \\ &-\epsilon''\frac{\sqrt{|M|}}{\sqrt{|L|}}(\lambda\cdot d\tilde{x})\wedge rdr\wedge(p\cdot d\tilde{x}) \\ &+\frac{1}{r_0\sqrt{|L|}}((\mu\times\tilde{x})\cdot d\tilde{x})\wedge rdr\wedge(p\cdot d\tilde{x})-\frac{1}{r_0\sqrt{|L|}\sqrt{|M|}}(p_0\epsilon'\sqrt{|L|}+r_0(\lambda\cdot\tilde{x})) \\ &x_4dx_4\wedge rdr\wedge(p\cdot d\tilde{x}) \\ &-\frac{r_0p_0}{\sqrt{|L|}}\epsilon''(\lambda\cdot d\tilde{x})\wedge rdr\wedge x_4dx_4-\frac{p_0}{r_0\sqrt{|M|}}\epsilon'(p\cdot d\tilde{x})\wedge rdr\wedge x_4dx_4 \\ &+\frac{p_0}{\sqrt{|L|}\sqrt{|M|}}((\mu\times\tilde{x})\cdot d\tilde{x})\wedge rdr\wedge x_4dx_4 \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\sqrt{|M|}}\Big(p_0\epsilon'\sqrt{|L|} + r_0(\lambda\cdot\tilde{x})\Big)(\mu\cdot d_2\tilde{x})\wedge x_4dx_4 + \Big\{ - r_0\epsilon''\sqrt{|M|}(\lambda\cdot\mu) \\
&- \frac{1}{r_0}\epsilon'\sqrt{|L|}(p\cdot\mu) + \frac{1}{\sqrt{|L|}}\Big(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\Big)(\mu\cdot\tilde{x})\Big\}d_3\tilde{x}\Big] \\
&= 2\Big[- \epsilon'\frac{\sqrt{|L|}}{\sqrt{|M|}}((\lambda\times p)\cdot d_2\tilde{x})\wedge x_4dx_4 - \frac{r_0}{\sqrt{|M|}}(((\mu\times\tilde{x})\times\lambda)\cdot d_2\tilde{x})\wedge x_4dx_4 \\
&+ \frac{r_0}{\sqrt{|L|}\sqrt{|M|}}\Big(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\Big)((\lambda\times\tilde{x})\cdot d_2\tilde{x})\wedge x_4dx_4 \\
&+ \epsilon''\frac{\sqrt{|M|}}{\sqrt{|L|}}((\lambda\times p)\cdot d_2\tilde{x})\wedge rdr - \frac{1}{r_0\sqrt{|L|}}(((\mu\times\tilde{x})\times p)\cdot d_2\tilde{x})\wedge rdr \\
&+ \frac{1}{r_0\sqrt{|L|}\sqrt{|M|}}\Big(p_0\epsilon'\sqrt{|L|} + r_0(\lambda\times\tilde{x})\Big)((p\times\tilde{x})\cdot d_2\tilde{x})\wedge x_4dx_4 \\
&- \Big\{\frac{r_0p_0}{\sqrt{|L|}}\epsilon''((\lambda\times\tilde{x})\cdot d_2\tilde{x}) + \frac{p_0}{r_0\sqrt{|M|}}\epsilon'((p\times\tilde{x})\cdot d_2\tilde{x}) \\
&- \frac{p_0}{\sqrt{|L|}\sqrt{|M|}}(((\mu\times\tilde{x})\times\tilde{x})\cdot d_2\tilde{x})\Big\}\wedge x_4dx_4 \\
&+ \Big\{-r_0\epsilon''\sqrt{|M|}(\lambda\cdot\mu) - \frac{1}{r_0}\epsilon'\sqrt{|L|}(p\cdot\mu) \\
&+ \frac{1}{\sqrt{|L|}}\Big(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\Big)(\mu\cdot\tilde{x})\Big\}d_3\tilde{x} \\
&- \frac{1}{\sqrt{|M|}}\Big(p_0\epsilon'\sqrt{|L|} + r_0(\lambda\cdot\tilde{x})\Big)(\mu\cdot d_2\tilde{x})\wedge x_4dx_4\Big].
\end{aligned}
$$

Since we have the identities :

$$
\begin{aligned}
&(\mu\times\tilde{x})\times\lambda = (\lambda\cdot\mu)\tilde{x} - (\lambda\cdot\tilde{x})\mu, \qquad (\mu\times\tilde{x})\times p = (\mu\cdot p)\tilde{x} - (p\cdot\tilde{x})\mu, \\
&(\mu\times\tilde{x})\times\tilde{x} = (\mu\cdot\tilde{x})\tilde{x} - r^2\mu,
\end{aligned}
$$

the above expression is arranged as

$$
\begin{aligned}
\frac{1}{2}(x_4x_4)^2\theta\wedge d\theta = \Big\{ &- \epsilon'\frac{\sqrt{|L|}}{\sqrt{|M|}}((\lambda\times p)\cdot d_2\tilde{x}) - \frac{r_0}{\sqrt{|M|}}(\lambda\cdot\mu)(\tilde{x}\cdot d_2\tilde{x}) \\
&+ \frac{r_0}{\sqrt{|M|}}(\lambda\cdot\tilde{x})(\mu\cdot d_2\tilde{x}) + \frac{r_0}{\sqrt{|L|}\sqrt{|M|}}\Big(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\Big)((\lambda\times\tilde{x})\cdot d_2\tilde{x}) \\
&+ \frac{1}{r_0\sqrt{|L|}\sqrt{|M|}}\Big(p_0\epsilon'\sqrt{|L|} + r_0(\lambda\cdot\tilde{x})\Big)((p\times\tilde{x})\cdot d_2\tilde{x}) - \frac{r_0p_0}{\sqrt{|L|}}\epsilon''((\lambda\times\tilde{x})\cdot d_2\tilde{x})
\end{aligned}
$$

$$- \frac{p_0}{r_0\sqrt{|M|}}\epsilon'((p \times \tilde{x}) \cdot d_2\tilde{x}) + \frac{p_0}{\sqrt{|L|}\sqrt{|M|}}(\mu \cdot \tilde{x})(\tilde{x} \cdot d_2\tilde{x})$$

$$\left. - \frac{p_0 r^2}{\sqrt{|L|}\sqrt{|M|}}(\mu \cdot d_2\tilde{x}) - \frac{1}{\sqrt{|M|}}\left(p_0\epsilon'\sqrt{|L|} + r_0(\lambda \cdot \tilde{x})\right)(\mu \cdot d_2\tilde{x})\right\} \wedge x_4 dx_4$$

$$+ \left\{ \frac{\sqrt{|M|}}{\sqrt{|L|}}\epsilon''((\lambda \times p) \cdot \tilde{x}) - \frac{r^2}{r_0\sqrt{|L|}}(\mu \cdot p) + \frac{1}{r_0\sqrt{|L|}}(p \cdot \tilde{x})(\mu \cdot \tilde{x}) \right.$$

$$- r_0\epsilon''\sqrt{|M|}(\lambda \cdot \mu) - \frac{1}{r_0}\epsilon'\sqrt{|L|}(p \cdot \mu)$$

$$\left. + \frac{1}{\sqrt{|L|}}\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p \cdot \tilde{x})\right)(\mu \cdot \tilde{x}) \right\} d_3\tilde{x}$$

$$= \left\{ \frac{\sqrt{|M|}}{\sqrt{|L|}}\epsilon''((\lambda \times p) \cdot \tilde{x}) - r_0\epsilon''\sqrt{|M|}(\lambda \cdot \mu) - \frac{r_0}{\sqrt{|L|}}(p \cdot \mu) \right.$$

$$\left. + \frac{p_0\epsilon''\sqrt{|M|}}{\sqrt{|L|}}(\mu \cdot \tilde{x}) \right\} d_3\tilde{x} + \frac{1}{\sqrt{|L|}\sqrt{|M|}}\left(\left[- (p \cdot \tilde{x})(\lambda \times \tilde{x}) - L(\lambda \times p) \right.\right.$$

$$\left.\left. + (\lambda \cdot \tilde{x})(p \times \tilde{x}) - p_0 {r_0}^2 \mu + \left(p_0(\mu \cdot \tilde{x}) - r_0\sqrt{|L|}(\lambda \cdot \mu)\right)\tilde{x}\right] \cdot d_2\tilde{x}\right) \wedge x_4 dx_4.$$

Thus we obtain an important formula for the Killing form $\theta = \sum_i v_i dx_i$ of the Killing vector field $V = \sum_i v^i \partial/\partial x_i$ given by (2.1) as follows :

$$\frac{1}{2}(x_4x_4)^2\theta \wedge d\theta = \left\{ \frac{\epsilon''\sqrt{|{r_0}^2 - x_4x_4|}}{\sqrt{|{r_0}^2 - r^2|}}((\lambda \times p) \cdot \tilde{x}) + p_0(\mu \cdot \tilde{x})) \right.$$

$$\left. - r_0\epsilon''\sqrt{|{r_0}^2 - x_4x_4|}(\lambda \cdot \mu) - \frac{r_0}{\sqrt{|{r_0}^2 - r^2|}}(p \cdot \mu) \right\} d_3\tilde{x} \tag{3.5}$$

$$+ \frac{1}{\sqrt{|{r_0}^2 - r^2|}\sqrt{|{r_0}^2 - x_4x_4|}}\left(\left[- (p \cdot \tilde{x})(\lambda \times \tilde{x}) + (\lambda \cdot \tilde{x})(p \times \tilde{x}) \right.\right.$$

$$- ({r_0}^2 - r^2)(\lambda \times p) - p_0{r_0}^2\mu + (p_0(\mu \cdot \tilde{x})$$

$$\left.\left. - r_0\sqrt{|{r_0}^2 - r^2|}(\lambda \cdot \mu))\tilde{x}\right] \cdot d_2\tilde{x}\right) \wedge x_4 dx_4,$$

from which and the Killing form :

$$\xi = \sum_i \xi_i dx_i = \frac{\epsilon''\sqrt{|{r_0}^2 - x_4x_4|}}{x_4x_4\sqrt{|{r_0}^2 - r^2|}}\sum_b x_b dx_b - \frac{\epsilon'\sqrt{|{r_0}^2 - r^2|}}{x_4\sqrt{|{r_0}^2 - x_4x_4|}}dx_4$$

we obtain

$$\frac{1}{2}(x_4x_4)^3\xi \wedge \theta \wedge d\theta$$

$$
\begin{aligned}
&= \frac{\epsilon''}{\sqrt{|{r_0}^2 - r^2|}} \Bigg(\bigg[-(p\cdot\tilde{x})(\lambda\times\tilde{x}) + (\lambda\cdot\tilde{x})(p\times\tilde{x}) - ({r_0}^2 - r^2)(\lambda\times p) \\
&\quad - p_0{r_0}^2\mu + (p_0(\mu\cdot\tilde{x}) - r_0\sqrt{|{r_0}^2 - r^2|}(\lambda\cdot\mu))\tilde{x}\bigg]\cdot\tilde{x}\Bigg) d_3\tilde{x}\wedge x_4 dx_4 \\
&\quad + \frac{\epsilon'\sqrt{|{r_0}^2 - r^2|}}{\sqrt{|{r_0}^2 - x_4x_4|}} \times \Bigg\{ \frac{\epsilon''\sqrt{|{r_0}^2 - x_4x_4|}}{\sqrt{|{r_0}^2 - r^2|}}(((\lambda\times p)\cdot\tilde{x}) + p_0(\mu\cdot\tilde{x})) \\
&\quad - r_0\epsilon''\sqrt{|{r_0}^2 - x_4x_4|}(\lambda\cdot\mu) - \frac{r_0}{\sqrt{|{r_0}^2 - r^2|}}(p\cdot\mu)\Bigg\} d_3\tilde{x}\wedge x_4 dx_4 \\
&= \Bigg\{ \frac{\epsilon''}{\sqrt{|{r_0}^2 - r^2|}}(-({r_0}^2 - r^2)(((\lambda\times p)\cdot\tilde{x}) - p_0{r_0}^2(\mu\cdot\tilde{x}) + p_0(\mu\cdot\tilde{x}) \\
&\quad - r_0\sqrt{|{r_0}^2 - r^2|}(\lambda\cdot\mu))r^2) + \epsilon'\epsilon''(((\lambda\times p)\cdot\tilde{x}) + p_0(\mu\cdot\tilde{x})) \\
&\quad - r_0\epsilon'\epsilon''\sqrt{|{r_0}^2 - r^2|}(\lambda\cdot\mu) - \frac{r_0\epsilon'}{\sqrt{|{r_0}^2 - x_4x_4|}}(p\cdot\mu)\Bigg\} d_3\tilde{x}\wedge x_4 dx_4 \\
&= -x_4\left\{ \frac{{r_0}^3\epsilon''}{\sqrt{|{r_0}^2 - r^2|}}(\lambda\cdot\mu) + \frac{r_0\epsilon'}{\sqrt{|{r_0}^2 - x_4x_4|}}(p\cdot\mu)\right\} d_3\tilde{x}\wedge dx_4
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{1}{2}(x_4x_4)^3\xi\wedge\theta\wedge d\theta \\
&\quad = -x_4\left\{ \frac{{r_0}^3\epsilon''}{\sqrt{|{r_0}^2 - r^2|}}(\lambda\cdot\mu) + \frac{r_0\epsilon'}{\sqrt{|{r_0}^2 - x_4x_4|}}(p\cdot\mu)\right\} dx_1\wedge\cdots\wedge dx_4.
\end{aligned}
\tag{3.6}
$$

From this relation we obtain immediately the following theorem.

THEOREM 2. *For the Killing form θ given by* (2.1), *Pfaffian equation : $\theta = 0$ forms complete system with $\xi = 0$, if and only if its constants λ, p, μ and p_0 satisfy the following conditions :*

(i) $\mu = 0$, *or* (ii) $\mu \neq 0$ *and* $(\lambda\cdot\mu) = (p\cdot\mu) = 0$,

different from $\lambda = \mu = p = 0$ which gives $\theta = -p_0\xi$.

Last, in the following, we shall search for the solution for the pair of Pfaffian equations : $\xi = 0$ and $\theta = 0$ with $(\lambda\cdot\mu) = (p\cdot\mu) = 0$. First, for $\xi = 0$ we have from (2.5)

$$\sum_c : \quad {r_0}^2 - x_4x_4 = c({r_0}^2 - r^2) \text{ where } c \text{ is an integral constant.}$$

We see easily that

Case 1 : $c > 0$. $\sum_c$ must lie in $D_1 \cup D_3$, where $\epsilon' = \epsilon'' = 1$ on D_1 and -1 on D_3, respectively ;

Case 2 : $c < 0$. $\sum_c$ must lie in $D_2 \cup D_4$, where $\epsilon' = -\epsilon'' = -1$ on D_2 and 1 on D_4, respectively.

Since $\theta = 0$ is equivalent to

$$\begin{aligned}
&r_0\epsilon''\sqrt{|M|}(\lambda\cdot d\tilde{x}) + \frac{1}{r_0}\epsilon'\sqrt{|L|}(p\cdot d\tilde{x}) - ((\mu\times\tilde{x})\cdot d\tilde{x})\\
&-\frac{1}{\sqrt{|L|}}\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)rdr + \frac{1}{\sqrt{|M|}}\left(p_0\epsilon'\sqrt{|L|} + r_0(\lambda\cdot\tilde{x})\right)x_4dx_4\\
&= 0,
\end{aligned}$$

which is reduced to

$$\begin{aligned}
&r_0\epsilon''\sqrt{|c|}\sqrt{|L|}(\lambda\cdot d\tilde{x}) + \frac{1}{r_0}\epsilon'\sqrt{|L|}(p\cdot d\tilde{x}) - ((\mu\times\tilde{x})\cdot d\tilde{x})\\
&+\left\{-\frac{1}{\sqrt{|L|}}\left(p_0\sqrt{|c|}\epsilon''\sqrt{|L|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)\right.\\
&\left.+\frac{c}{\sqrt{|c|}\sqrt{|L|}}\left(p_0\epsilon'\sqrt{|L|} + r_0(\lambda\cdot\tilde{x})\right)\right\}rdr = 0
\end{aligned} \tag{3.7}$$

on $\sum_c$ by means of $M = cL$ and $x_4dx_4 = crdr$.

For Case 1, (3.7) becomes

$$\begin{aligned}
&\epsilon'\sqrt{|L|}\left(\sqrt{c}r_0(\lambda\cdot d\tilde{x}) + \frac{1}{r_0}(p\cdot d\tilde{x})\right)\\
&-((\mu\times\tilde{x})\cdot d\tilde{x}) + \frac{1}{\sqrt{|L|}}\left(\sqrt{c}r_0(\lambda\cdot\tilde{x}) + \frac{1}{r_0}(p\cdot\tilde{x})\right)rdr = 0.
\end{aligned} \tag{3.7$'$}$$

Since we have

$$d\sqrt{|L|} = -\frac{1}{\sqrt{|L|}}\epsilon' rdr,$$

the above expression is equivalent to

$$\begin{aligned}
&\sqrt{|L|}\left(\sqrt{c}r_0(\lambda\cdot d\tilde{x}) + \frac{1}{r_0}(p\cdot d\tilde{x})\right)\\
&-\left(\sqrt{c}r_0(\lambda\cdot\tilde{x}) + \frac{1}{r_0}(p\cdot\tilde{x})\right)d\sqrt{|L|} - \epsilon'((\mu\times\tilde{x})\cdot d\tilde{x}) = 0.
\end{aligned}$$

For simplicity, we use the constant vector in R^3 :

$$q = \sqrt{c}r_0\lambda + \frac{1}{r_0}p,$$

then (3.7') becomes

(3. 8) $$\sqrt{|L|}(q\cdot d\tilde{x})-(q\cdot\tilde{x})d\sqrt{|L|}-\epsilon'((\mu\times\tilde{x})\cdot d\tilde{x})=0.$$

When $\mu=0$, we have

$$d\log(q\cdot\tilde{x})-d\log\sqrt{|L|}=0,$$

from which we obtain by integration

$$(q\cdot\tilde{x})^2={c_1}^2|L| \qquad \text{with integral constant } c_1>0,$$

that is

(3. 9) $$\Gamma_{c_1}:\left(\left(\sqrt{c}r_0\lambda+\frac{1}{r_0}p\right)\cdot\tilde{x}\right)^2={c_1}^2|L|={c_1}^2\epsilon'({r_0}^2-r^2).$$

When $\mu\neq 0$, we may put $\mu_1=\mu_2=0, \mu_3\neq 0$. Then, we have $\lambda_3=p_3=0$ by $(\lambda\cdot\mu)=(p\cdot\mu)=0$ and (3.8) turns into

(3. 8') $$z(q_1dx_1+q_2dx_2)-(q_1x_1+q_2x_2)dz-\epsilon'\mu_3(x_1dx_2-x_2dx_1)=0,$$

where $z=\sqrt{|{r_0}^2-{x_1}^2-{x_2}^2-{x_3}^2|}$. Let $\Phi(x_1,x_2,z)$ be an integral multiplier. Denoting the above expression as

$$(q_1z+\epsilon'\mu_3x_2)dx_1+(q_2z-\epsilon'\mu_3x_1)dx_2-(q_1x_1+q_2x_2)dz=0,$$

we have

$$\begin{aligned}
\frac{\partial}{\partial x_2}((q_1z+\epsilon'\mu_3x_2)\Phi)&=\frac{\partial}{\partial x_1}((q_2z-\epsilon'\mu_3x_1)\Phi),\\
\frac{\partial}{\partial z}((q_1z+\epsilon'\mu_3x_2)\Phi)&=-\frac{\partial}{\partial x_1}((q_1x_1+q_2x_2)\Phi),\\
\frac{\partial}{\partial z}((q_2z-\epsilon'\mu_3x_1)\Phi)&=-\frac{\partial}{\partial x_2}((q_1x_1+q_2x_2)\Phi),
\end{aligned}$$

which become

$$\begin{aligned}
&(q_1z+\epsilon'\mu_3x_2)\frac{\partial\Phi}{\partial x_2}-(q_2z-\epsilon'\mu_3x_1)\frac{\partial\Phi}{\partial x_1}+2\epsilon'\mu_3\Phi=0,\\
&(q_1z+\epsilon'\mu_3x_2)\frac{\partial\Phi}{\partial z}+(q_1x_1+q_2x_2)\frac{\partial\Phi}{\partial x_1}+2q_1\Phi=0,\\
&(q_2z-\epsilon'\mu_3x_1)\frac{\partial\Phi}{\partial z}+(q_1x_1+q_2x_2)\frac{\partial\Phi}{\partial x_2}+2q_2\Phi=0,
\end{aligned}$$

respectively. From the second and third equalities, we obtain

$$(q_1x_1+q_2x_2)\left(z\frac{\partial\Phi}{\partial z}+x_1\frac{\partial\Phi}{\partial x_1}+x_2\frac{\partial\Phi}{\partial x_2}+2\Phi\right)=0,$$

and so

$$x_1\frac{\partial\Phi}{\partial x_1}+x_2\frac{\partial\Phi}{\partial x_2}+z\frac{\partial\Phi}{\partial z}+2\Phi=0,\ \text{supposing } (q_1,q_2)\neq(0,0). \tag{3.10}$$

Since the first equality is written as

$$z\left(q_1\frac{\partial\Phi}{\partial x_2}-q_2\frac{\partial\Phi}{\partial x_1}\right)+\epsilon'\mu_3\left(x_1\frac{\partial\Phi}{\partial x_1}+x_2\frac{\partial\Phi}{\partial x_2}+2\Phi\right)=0,$$

into which substituting (3.10) we obtain

$$z\left(q_1\frac{\partial\Phi}{\partial x_2}-q_2\frac{\partial\Phi}{\partial x_1}-\epsilon'\mu_3\frac{\partial\Phi}{\partial z}\right)=0.$$

Next, the second one is written as

$$q_1\left(z\frac{\partial\Phi}{\partial z}+x_1\frac{\partial\Phi}{\partial x_1}+2\Phi\right)+\epsilon'\mu_3x_2\frac{\partial\Phi}{\partial z}+q_2x_2\frac{\partial\Phi}{\partial x_1}=0,$$

into which substituting (3.10) we obtain

$$x_2\left(-q_1\frac{\partial\Phi}{\partial x_2}+\epsilon'\mu_3\frac{\partial\Phi}{\partial z}+q_2\frac{\partial\Phi}{\partial x_1}\right)=0.$$

Analogously, from the third one we obtain

$$x_1\left(q_2\frac{\partial\Phi}{\partial x_1}+\epsilon'\mu_3\frac{\partial\Phi}{\partial z}-q_1\frac{\partial\Phi}{\partial x_2}\right)=0.$$

From these equalities it must be

$$q_2\frac{\partial\Phi}{\partial x_1}-q_1\frac{\partial\Phi}{\partial x_2}+\epsilon'\mu_3\frac{\partial\Phi}{\partial z}=0. \tag{3.11}$$

As an general solution of (3.10), we have

$$\Phi=\frac{a_1}{{x_1}^2}+\frac{a_2}{{x_2}^2}+\frac{a_3}{z^2}+\frac{a_4}{x_1x_2}+\frac{a_5}{x_1z}+\frac{a_6}{x_2z},$$

where $a_1,\cdots,a_6$ are integral constants and for which in order to satisfy (3.11) it must be $a_1=a_2=\cdots=a_6$. Hence the above argument is only hold the case

$$\sqrt{c}r_0\lambda+\frac{1}{r_0}p=0, \tag{3.12}$$

and the solution is a hyperplane parallel to x_4 axis and containing a straight line through the origin of R^3 in $D_1 \cup D_3$.

For Case 2, (3.7) becomes

$$-r_0\epsilon'\sqrt{-c}\sqrt{|L|}(\lambda\cdot d\tilde{x})+\frac{1}{r_0}\epsilon'\sqrt{|L|}(p\cdot d\tilde{x})-((\mu\times\tilde{x})\cdot d\tilde{x})$$
$$+\left\{\frac{1}{\sqrt{|L|}}\left(p_0\sqrt{-c}\epsilon'\sqrt{|L|}+\frac{1}{r_0}(p\cdot\tilde{x})\right)+\frac{c}{\sqrt{-c}\sqrt{|L|}}\left(p_0\epsilon'\sqrt{|L|}+r_0(\lambda\cdot\tilde{x})\right)\right\}rdr$$
$$=0$$

that is

$$\epsilon'\sqrt{|L|}\left(-\sqrt{-c}r_0(\lambda\cdot d\tilde{x})+\frac{1}{r_0}(p\cdot d\tilde{x})\right)$$
$$-((\mu\times\tilde{x})\cdot d\tilde{x})+\frac{1}{\sqrt{|L|}}\left(-\sqrt{-c}r_0(\lambda\cdot\tilde{x})+\frac{1}{r_0}(p\cdot\tilde{x})\right)rdr=0. \tag{3.7''}$$

For simplicity , we use the constant vector in R^3 :

$$q'=-\sqrt{-c}r_0\lambda+\frac{1}{r_0}p,$$

then (3.7″) becomes

$$\sqrt{|L|}(q'\cdot d\tilde{x})-(q'\cdot\tilde{x})d\sqrt{|L|}-\epsilon'((\mu\times\tilde{x})\cdot d\tilde{x})=0, \tag{3.8''}$$

which is the same equality as (3.8) replaced q by q'. Hence, we can obtain the following results.

When $\mu=0$, its solution is

$$\Gamma'_{c_1}\qquad\left(\left(-\sqrt{-c}r_0\lambda+\frac{1}{r_0}p\right)\cdot\tilde{x}\right)^2={c_1}^2\epsilon'({r_0}^2-r^2). \tag{3.9''}$$

When $\mu\neq0$ and

$$-\sqrt{-c}r_0\lambda+\frac{1}{r_0}p=0, \tag{3.12'}$$

its solution is a hyperplane parallel to x_4 axis and containing a straight line through the origin of R^3 in $D_2\cup D_4$, otherwise no solutions.

As a conclusion of these arguments, we obtain the following claim.

COROLLARY. *The solution for $\xi=0$ and $\theta=0$ in Theorem 2 with $\mu\neq 0, (\lambda\cdot\mu)=(p\cdot\mu)=0$ is given by the intersection of quadratic hypersurface*

$$\Sigma_c: {r_0}^2-x_4x_4=c({r_0}^2-r^2),\quad c=constant\neq0$$

and a hyperplane parallel to x_4 axis and containing a plane through the origin of R^3, and then $p = -\sqrt{c}{r_0}^2\lambda$ or $p = \sqrt{-c}{r_0}^2\lambda$ and Σ_c in $D_1 \cup D_3$ or Σ_c in $D_2 \cup D_4$ according to $c > 0$ or $c < 0$. When $\mu = 0$, the solution is given by the intersection of Σ_c and an another quadratic hypersurface

$$\Gamma_c : \left(\left(\sqrt{c}r_0\lambda + \frac{1}{r_0}p\right)\cdot\tilde{x}\right)^2 = {c_1}^2\epsilon'({r_0}^2 - r^2)$$

or

$$\Gamma_c{}' : \left(\left(-\sqrt{-c}r_0\lambda + \frac{1}{r_0}p\right)\cdot\tilde{x}\right)^2 = {c_1}^2\epsilon'({r_0}^2 - r^2),$$

where c_1 is an integral constant.

4. Special Killing vector fields

In this section, we shall investigate Killing vector fields that the pair of contravariant vector fields $X = \sum_i X^i\partial/\partial x_i$ and $Y = \sum_i Y^i\partial/\partial x_i$ by (2.4′) and (2.1) respectively as

$$X^i = \xi^i = \frac{1}{r_0^2}\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}x_i, \tag{4.1}$$

and

$$\begin{aligned}
Y^b& \\
&= \theta^b = \sum_c g^{bc}\frac{1}{x_4x_4}\left\{r_0\epsilon''\sqrt{|M|}\lambda_c + \frac{1}{r_0}\epsilon'\sqrt{|L|}p_c - (\mu\times\tilde{x})_c\right.\\
&\quad\left. - \frac{1}{\sqrt{|L|}}\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)x_c\right\}\\
&= r_0\epsilon''\sqrt{|M|}\lambda_b + \frac{1}{r_0}\epsilon'\sqrt{|L|}p_b - (\mu\times\tilde{x})_b - \frac{1}{\sqrt{|L|}}\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)x_b\\
&\quad - \frac{1}{{r_0}^2}\left\{r_0\epsilon''\sqrt{|M|}(\lambda\cdot\tilde{x}) + \frac{1}{r_0}\epsilon'\sqrt{|L|}(p\cdot\tilde{x}) - \frac{r^2}{\sqrt{|L|}}\left(p_0\epsilon''\sqrt{|M|} - \frac{1}{r_0}(p\cdot\tilde{x})\right)\right\}x_b\\
&= r_0\epsilon''\sqrt{|M|}\lambda_b + \frac{1}{r_0}\epsilon'\sqrt{|L|}p_b - (\mu\times\tilde{x})_b - \frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}\left(r_0(\lambda\cdot\tilde{x}) + p_0\epsilon'\sqrt{|L|}\right)x_b,\\
Y^4& \\
&= \theta^4 = g^{44}\frac{1}{x_4\sqrt{|M|}}\left(r_0(\lambda\cdot\tilde{x}) + p_0\epsilon'\sqrt{|L|}\right)\\
&= -\frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}\left(r_0(\lambda\cdot\tilde{x}) + p_0\epsilon'\sqrt{|L|}\right)x_4
\end{aligned}$$

that is

$$
(4.2)\qquad
\begin{aligned}
Y^b &= r_0\epsilon''\sqrt{|M|}\lambda_b + \frac{1}{r_0}\epsilon'\sqrt{|L|}p_b - (\mu\times\tilde{x})_b \\
&\quad - \frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}\left(r_0(\lambda\cdot\tilde{x}) + p_0\epsilon'\sqrt{|L|}\right)x_b, \\
Y^4 &= -\frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}\left(r_0(\lambda\cdot\tilde{x}) + p_0\epsilon'\sqrt{|L|}\right)x_4,
\end{aligned}
$$

forms a Lie algebra of order 2.

For the bracket $[X, Y]$ of X and Y, we have

$$
[X,Y]^i = \sum_j \left(X^j\frac{\partial Y^i}{\partial x_j} - Y^j\frac{\partial X^i}{\partial x_j} \right)
$$

Putting $S = r_0(\lambda\cdot\tilde{x}) + p_0\epsilon'\sqrt{|L|})$ for simplicity, from (4.1) and (4.2) we obtain

$$
\begin{aligned}
&[X,Y]^b \\
&= \sum_j X^j\frac{\partial Y^b}{\partial x_j} - \sum_c Y^c\frac{\partial X^b}{\partial x_c} - Y^4\frac{\partial X^b}{\partial x_4} \\
&= \frac{1}{{r_0}^2}\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}\sum_j x^j\frac{\partial Y^b}{\partial x_j} \\
&\quad -\sum_c\left\{r_0\epsilon''\sqrt{|M|}\lambda_c + \frac{1}{r_0}\epsilon'\sqrt{|L|}p_c - (\mu\times\tilde{x})_c - \frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}Sx_c\right\} \\
&\quad \times\frac{1}{{r_0}^2}\left\{\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}\delta_{bc} - \epsilon''\frac{\sqrt{|M|}}{\sqrt{|L|}}x_bx_c\right\} \\
&\quad +\frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}Sx_4\left(-\frac{1}{{r_0}^2\sqrt{|M|}}\epsilon'\sqrt{|L|}x_4\right)x_b \\
&= \frac{1}{{r_0}^2}\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}\left\{ -\frac{r_0x_4x_4}{\sqrt{|M|}}\lambda_b - \frac{r^2}{{r_0}^2\sqrt{|L|}}p_b - (\mu\times\tilde{x})_b + \frac{x_4x_4}{{r_0}^2\sqrt{|M|}}Sx_b \right. \\
&\quad \left. -\frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}\left(r_0(\lambda\cdot\tilde{x}) - p_0\frac{r^2}{\sqrt{|L|}}\right)x_b - \frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}Sx_b\right\} \\
&\quad -\frac{1}{{r_0}^2}\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}\left\{r_0\epsilon''\sqrt{|M|}\lambda_b + \frac{1}{r_0}\epsilon'\sqrt{|L|}p_b - (\mu\times\tilde{x})_b - \frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}Sx_b\right\} \\
&\quad +\frac{1}{{r_0}^2\sqrt{|L|}}\epsilon''\sqrt{|M|}\left\{r_0\epsilon''\sqrt{|M|}(\lambda\cdot\tilde{x}) + \frac{1}{r_0}\epsilon'\sqrt{|L|}(p\cdot\tilde{x}) - \frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}r^2S\right\}x_b \\
&\quad -\frac{1}{{r_0}^4}\epsilon'\epsilon''\sqrt{|L|}x_4x_4Sx_b
\end{aligned}
$$

$$= -\epsilon'\epsilon'' r_0\sqrt{|L|}\lambda_b - \frac{1}{r_0}\epsilon'\epsilon''\sqrt{|M|}p_b - \frac{1}{{r_0}^4\sqrt{|L|}}\epsilon''({r_0}^2 - x_4x_4)r^2Sx_b$$
$$+\left\{\frac{1}{{r_0}^3\sqrt{|L|}}\epsilon''({r_0}^2 - x_4x_4)r^2(\lambda\cdot\tilde{x})\right.$$
$$\left.+\frac{1}{{r_0}^3}\epsilon'\epsilon''\sqrt{|M|}(p\cdot\tilde{x}) + \frac{1}{{r_0}^4}\epsilon'\epsilon''({r_0}^2 - x_4x_4)p_0r^2\right\}x_b$$
$$= -\epsilon'\epsilon''\left\{r_0\sqrt{|L|}\lambda_b + \frac{1}{r_0}\sqrt{|M|}p_b - \frac{1}{{r_0}^3}\sqrt{|M|}(p\cdot\tilde{x})x_b\right\}$$

and

$$[X,Y]^4$$
$$= \sum_j X^j\frac{\partial Y^4}{\partial x_j} - \sum_c Y^c\frac{\partial X^4}{\partial x_c} - Y^4\frac{\partial X^4}{\partial x_4}$$
$$= \frac{1}{{r_0}^2}\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}\sum_j x_j\frac{\partial Y^4}{\partial x_j} - \sum_c\left\{r_0\epsilon''\sqrt{|M|}\lambda_c + \frac{1}{r_0}\epsilon'\sqrt{|L|}p_c - (\mu\times\tilde{x})_c\right.$$
$$\left.-\frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}Sx_c\right\}\left(-\frac{1}{{r_0}^2\sqrt{|L|}}\epsilon''\sqrt{|M|}x_cx_4\right)$$
$$+\frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}Sx_4\frac{1}{{r_0}^2}\left(\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|} - \frac{1}{\sqrt{|M|}}\epsilon'\sqrt{|L|}x_4x_4\right)$$
$$= \frac{1}{{r_0}^2}\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}\left\{\frac{x_4x_4}{{r_0}^2\sqrt{|M|}}Sx_4 - \frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}\left(r_0(\lambda\cdot\tilde{x}) - p_0\frac{r^2}{\sqrt{|L|}}\right)x_4\right.$$
$$\left.-\frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}Sx_4\right\} + \frac{1}{{r_0}^2\sqrt{|L|}}\epsilon''\sqrt{|M|}\left\{r_0\epsilon''\sqrt{|M|}(\lambda\cdot\tilde{x}) + \frac{1}{r_0}\epsilon'\sqrt{|L|}(p\cdot\tilde{x})\right.$$
$$\left.-\frac{1}{{r_0}^2}\epsilon''\sqrt{|M|}r^2S\right\}x_4 + \frac{1}{{r_0}^4}\epsilon''\sqrt{|M|}\left\{\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|} - \frac{1}{\sqrt{|M|}}\epsilon'\sqrt{|L|}x_4x_4\right\}Sx_4$$
$$= -\frac{1}{{r_0}^4\sqrt{|L|}}\epsilon''({r_0}^2 - x_4x_4)r^2Sx_4 + \left\{-\frac{1}{{r_0}^3}\epsilon'\epsilon''\sqrt{|L|}M(\lambda\cdot\tilde{x}) + \frac{1}{{r_0}^4}p_0r^2\epsilon'\epsilon''M\right.$$
$$\left.+\frac{1}{r_0\sqrt{|L|}}\epsilon''M(\lambda\cdot\tilde{x}) + \frac{1}{{r_0}^3}\epsilon'\epsilon''\sqrt{|M|}(p\cdot\tilde{x})\right\}x_4 = \frac{1}{{r_0}^3}\epsilon'\epsilon''\sqrt{|M|}(p\cdot\tilde{x})x_4,$$

that is

$$(4.3)\quad \begin{cases} [X,Y]^b = -\epsilon'\epsilon''\left(r_0\sqrt{|L|}\lambda_b + \dfrac{1}{r_0}\sqrt{|M|}p_b\right) + \dfrac{1}{{r_0}^3}\epsilon'\epsilon''\sqrt{|M|}(p\cdot\tilde{x})x_b, \\ [X,Y]^4 = \dfrac{1}{{r_0}^3}\epsilon'\epsilon''\sqrt{|M|}(p\cdot\tilde{x})x_4. \end{cases}$$

Then, we compute the vector field :

$$Z = [X,Y] - Y - p_0X.$$

From (4.1), (4.2) and (4.3), we obtain the equalities as

$$
\begin{aligned}
&Z^b \\
&= -\left(\epsilon'\epsilon'' r_0\sqrt{|L|} + r_0\epsilon''\sqrt{|M|}\right)\lambda_b - \left(\epsilon'\epsilon''\frac{\sqrt{|M|}}{r_0} + \epsilon'\frac{\sqrt{|L|}}{r_0}\right)p_b + (\mu\times\tilde{x})_b \\
&\quad + \left\{\frac{1}{r_0{}^3}\epsilon'\epsilon''\sqrt{|M|}(p\cdot\tilde{x}) + \frac{1}{r_0{}^2}\epsilon''\sqrt{|M|}\left(r_0(\lambda\cdot\tilde{x}) + p_0\epsilon'\sqrt{|L|}\right)\right. \\
&\quad \left. - \frac{1}{r_0{}^2}\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}p_0\right\}x_b \\
&= -\epsilon'\epsilon'' r_0\sqrt{|L|}(\lambda_b + \frac{1}{r_0{}^2}\epsilon'' p_b) - \epsilon'' r_0\sqrt{|M|}\left(\lambda_b + \frac{1}{r_0{}^2}\epsilon' p_b\right) + (\mu\times\tilde{x})_b \\
&\quad + \frac{1}{r_0}\epsilon''\sqrt{|M|}\left(\left(\lambda + \frac{1}{r_0{}^2}\epsilon' p\right)\cdot\tilde{x}\right)x_b, \\
&Z^4 \\
&= \left\{\frac{1}{r_0{}^3}\epsilon'\epsilon''\sqrt{|M|}(p\cdot\tilde{x}) + \frac{1}{r_0{}^2}\epsilon''\sqrt{|M|}\left(r_0(\lambda\cdot\tilde{x}) + p_0\epsilon'\sqrt{|L|}\right)\right. \\
&\quad \left. - \frac{p_0}{r_0{}^2}\epsilon'\epsilon''\sqrt{|L|}\sqrt{|M|}\right\}x_4 \\
&= \frac{1}{r_0}\epsilon''\sqrt{|M|}\left(\left(\lambda + \frac{1}{r_0{}^2}\epsilon' p\right)\cdot\tilde{x}\right)x_4
\end{aligned}
$$

that is

$$
(4.4) \quad \begin{cases}
Z^b = -\epsilon'\epsilon'' r_0\sqrt{|L|}\left(\lambda_b + \frac{1}{r_0{}^2}\epsilon'' p_b\right) - \epsilon'' r_0\sqrt{|M|}\left(\lambda_b + \frac{1}{r_0{}^2}\epsilon' p_b\right) \\
\qquad + (\mu\times\tilde{x})_b + \frac{1}{r_0}\epsilon''\sqrt{|M|}\left(\left(\lambda + \frac{1}{r_0{}^2}\epsilon' p\right)\cdot\tilde{x}\right)x_b, \\
Z^4 = \frac{1}{r_0}\epsilon''\sqrt{|M|}\left(\left(\lambda + \frac{1}{r_0{}^2}\epsilon' p\right)\cdot\tilde{x}\right)x_4.
\end{cases}
$$

By the definition of (1.2) of ϵ' and ϵ'' , (4.4) becomes exactly as follows.

$$
(4.4/1) \quad \begin{aligned}
Z^b &= -r_0\sqrt{|L|}\left(\lambda_b + \frac{1}{r_0{}^2}p_b\right) - r_0\sqrt{|M|}\left(\lambda_b + \frac{1}{r_0{}^2}p_b\right) \\
&\quad + (\mu\times\tilde{x})_b + \frac{1}{r_0}\sqrt{|M|}\left(\left(\lambda + \frac{1}{r_0{}^2}p\right)\cdot\tilde{x}\right)x_b, \\
Z^4 &= \frac{1}{r_0}\sqrt{|M|}\left(\left(\lambda + \frac{1}{r_0{}^2}p\right)\cdot\tilde{x}\right)x_4, \text{ on } D_1 ;
\end{aligned}
$$

$$
(4.4/2) \quad \begin{aligned}
Z^b &= r_0\sqrt{|L|}\left(\lambda_b + \frac{1}{r_0{}^2}p_b\right) - r_0\sqrt{|M|}\left(\lambda_b - \frac{1}{r_0{}^2}p_b\right) \\
&\quad + (\mu\times\tilde{x})_b + \frac{1}{r_0}\sqrt{|M|}\left(\left(\lambda - \frac{1}{r_0{}^2}p\right)\cdot\tilde{x}\right)x_b,
\end{aligned}
$$

$$Z^4 = \frac{1}{r_0}\sqrt{|M|}\left(\left(\lambda - \frac{1}{{r_0}^2}p\right)\cdot\tilde{x}\right)x_4, \text{ on } D_2\ ;$$

$$(4.4/3)\qquad Z^b = -r_0\sqrt{|L|}\left(\lambda_b - \frac{1}{{r_0}^2}p_b\right) + r_0\sqrt{|M|}\left(\lambda_b - \frac{1}{{r_0}^2}p_b\right) + (\mu\times\tilde{x})_b - \frac{1}{r_0}\sqrt{|M|}\left(\left(\lambda - \frac{1}{{r_0}^2}p\right)\cdot\tilde{x}\right)x_b,$$

$$Z^4 = -\frac{1}{r_0}\sqrt{|M|}\left(\left(\lambda - \frac{1}{{r_0}^2}p\right)\cdot\tilde{x}\right)x_4, \text{ on } D_3\ ;$$

$$(4.4/4)\qquad Z^b = r_0\sqrt{|L|}\left(\lambda_b - \frac{1}{{r_0}^2}p_b\right) + r_0\sqrt{|M|}\left(\lambda_b + \frac{1}{{r_0}^2}p_b\right) + (\mu\times\tilde{x})_b - \frac{1}{r_0}\sqrt{|M|}\left(\left(\lambda + \frac{1}{{r_0}^2}p\right)\cdot\tilde{x}\right)x_b,$$

$$Z^4 = -\frac{1}{r_0}\sqrt{|M|}\left(\left(\lambda + \frac{1}{{r_0}^2}p\right)\cdot\tilde{x}\right)x_4, \text{ on } D_4.$$

Looking over these expressions, we obtain the following theorem.

THEOREM 3. *Killing vector fields X and Y given by (4.1) and (4.2) respectively have the property as follows: Setting $\hat{p} = \begin{pmatrix} p \\ 0 \end{pmatrix}$ and $\hat{\lambda} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$,*

(i) *if $\mu = 0$ and $\lambda + \frac{1}{{r_2}^2}p = 0$, they form Lie algebra of order 2 on D_1 as $[X,Y] = Y + p_0X$ and*

$$[X,Y] - Y - p_0X = \frac{2}{r_0}\sqrt{|M|}\hat{p} - \frac{2}{{r_0}^3}\sqrt{|M|}(p\cdot\tilde{x})x \text{ on } D_2,$$
$$= \frac{2}{r_0}\left(\sqrt{|L|} - \sqrt{|M|}\right)\hat{p} + \frac{2}{{r_0}^3}\sqrt{|M|}(p\cdot\tilde{x})x \text{ on } D_3,$$
$$= -\frac{2}{r_0}\sqrt{|L|}\hat{p} \text{ on } D_4;$$

(ii) *if $\mu = 0$ and $\lambda - \frac{1}{{r_2}^2}p = 0$, they form Lie algebra of order 2 on D_3 of the same type and*

$$[X,Y] - Y - p_0X = -\frac{2}{r_0}\left(\sqrt{|L|} + \sqrt{|M|}\right)\hat{p} + \frac{2}{{r_0}^3}\sqrt{|M|}(p\cdot\tilde{x})x \text{ on } D_1$$
$$= \frac{2}{r_0}\sqrt{|L|}\hat{p} \text{ on } D_2,$$
$$= \frac{2}{r_0}\sqrt{|M|}\hat{p} - \frac{2}{{r_0}^3}\sqrt{|M|}(p\cdot\tilde{x})x \text{ on } D_4.$$

REMARK 1. Related with Theorem 2 in §3 for Killing forms and Theorem 3 above for Killing vector fields, we consider a condition : $\mu = \mu_0(\lambda\times p)$, which implies $(\lambda \cdot \mu) = (p \cdot \mu) = 0$ and

$$(\mu \times \tilde{x}) = \mu_0(\lambda \cdot \tilde{x})p - \mu_0(p \cdot \tilde{x})\lambda.$$

Then , (4.4) is reduced to

$$[X,Y] - Y - p_0 X = -\left\{r_0\epsilon''\left(\epsilon'\sqrt{|L|} + \sqrt{|M|}\right) + \mu_0(p \cdot \tilde{x})\right\}\hat{\lambda}$$
$$-\left\{\frac{1}{r_0}\epsilon'\left(\sqrt{|L|} + \epsilon''\sqrt{|M|}\right) - \mu_0(\lambda \cdot \tilde{x})\right\}\hat{p} + \frac{1}{r_0}\epsilon''\sqrt{|M|}\left(\left(\lambda + \frac{1}{r_0{}^2}\epsilon' p\right) \cdot \tilde{x}\right)x.$$

We have especially on D_1

$$[X,Y] - Y - p_0 X = -\left(\sqrt{|L|} + \sqrt{|M|}\right)\left(r_0\hat{\lambda} + \frac{1}{r_0}\hat{p}\right) - \mu_0\left(\left(p \cdot \tilde{x})\hat{\lambda} - \lambda \cdot \tilde{x}\right)\hat{p}\right)$$
$$+\frac{1}{r_0{}^2}\sqrt{|M|}\left(\left(r_0\lambda + \frac{1}{r_0}p\right) \cdot \tilde{x}\right)x$$

and on D_3

$$[X,Y] - Y - p_0 X = -\left(\sqrt{|L|} - \sqrt{|M|}\right)\left(r_0\hat{\lambda} - \frac{1}{r_0}\hat{p}\right) - \mu_0((p \cdot \tilde{x})\hat{\lambda} - (\lambda \cdot \tilde{x})\hat{p})$$
$$-\frac{1}{r_0{}^2}\sqrt{|M|}\left(\left(r_0\lambda - \frac{1}{r_0}p\right) \cdot \tilde{x}\right)x$$

and they vanish on D_1 if $\lambda = -\frac{1}{r_0{}^2}p$ and on D_3 if $\lambda = \frac{1}{r_0{}^2}p$, respectively.

REMARK 2. We give an example of Lie algebra of real 2×2 -matrices which is isomorphic to the Lie algebra generated by X and Y in Theorem 3 on D_1 with $\mu = 0$ and $\lambda + \frac{1}{r_0{}^2}p = 0$. Since we may replace Y with $Y + p_0X$, therefore we take generators X, Y such that $[X,Y] = Y$. Let $A = (a_{\alpha\beta}), B = (b_{\alpha\beta})$ which satisfy the condition : $[A, B] = AB - BA = B$, then we obtain the results as follows:

(i) if $(a_{11} - a_{22})^2 + 4a_{12}a_{21} = (\mathrm{tr}A)^2 - 4\det A \neq 1$, then $B = 0$;
(ii) if $(a_{11} - a_{22})^2 + 4a_{12}a_{21} = 1$, then

$$b_{11} = -b_{22} = \frac{1}{2}\{(a_{12} - a_{21})(a_{11} - a_{22}) - (a_{12} + a_{21})\},$$
$$b_{12} = a_{12}{}^2 + \frac{1}{4}(a_{11} - a_{22} + 1)^2,\ b_{21} = -a_{21}{}^2 - \frac{1}{4}(a_{11} - a_{22} - 1)^2.$$

Especially, putting $a_{11} = a_{22} = a, a_{12} = \rho$, we have

$$A = \begin{pmatrix} a & \rho \\ \dfrac{1}{4\rho} & a \end{pmatrix},\quad B = \begin{pmatrix} -\dfrac{1}{2}\left(\rho + \dfrac{1}{4\rho}\right) & \rho^2 + \dfrac{1}{4} \\ -\dfrac{1}{16\rho^2} - \dfrac{1}{4} & \dfrac{1}{2}\left(\rho + \dfrac{1}{4\rho}\right) \end{pmatrix}.$$

5. Integral curves of Killing vector fields

In this section, we shall investigate the integral curves of certain Killing vector fields

$$X^i = v^i = \sum_j g^{ij} v_j \text{ given by (2.1).}$$

First we take the special one given by (4.1), that is

$$X^i = \frac{1}{{r_0}^2}\epsilon'\epsilon''\sqrt{|{r_0}^2 - r^2|}\sqrt{|{r_0}^2 - x_4x_4|}x_i,$$

and solve the differential equations :

$$\text{(5. 1)} \qquad \frac{dx_i}{dt} = X^i = \frac{1}{{r_0}^2}\epsilon'\epsilon''\sqrt{|{r_0}^2 - r^2|}\sqrt{|{r_0}^2 - x_4x_4|}x_i.$$

We can put

$$x_i = l_i f(t), \qquad l_i = \text{constant and } l_4 = r_0,$$

and

$$\frac{df}{dt} = \frac{1}{r_0}\epsilon'\epsilon''\sqrt{|1 - f^2|}\sqrt{|{r_0}^2 - (\tilde{l}\cdot\tilde{l})f^2|}f,$$

where $l = (l_i), \tilde{l} = (l_b)$, which is replaced by $F = f^2$ and

$$\frac{dF}{dt} = \frac{2}{r_0}\epsilon'\epsilon''\sqrt{|1 - F|}\sqrt{|{r_0}^2 - (\tilde{l}\cdot\tilde{l})F|}F,$$

Since ϵ' and ϵ'' are constants 1 or -1 on each D_i , we have

$$\frac{2\epsilon'\epsilon'' dt}{r_0} = \frac{dF}{F\sqrt{|(1-F)({r_0}^2 - (\tilde{l}\cdot\tilde{l})F)|}}.$$

Supposing $f(0) = 1$, where $(1-F)({r_0}^2 - (\tilde{l}\cdot\tilde{l})F) > 0$, we obtain by integration

$$\text{(5. 2)} \quad 2\epsilon'\epsilon''(t + c) = \log\frac{2{r_0}^2 - ({r_0}^2 + (\tilde{l}\cdot\tilde{l}))F - 2r_0\sqrt{(1-F)({r_0}^2 - (\tilde{l}\cdot\tilde{l})F)}}{F}$$

and $2\epsilon'\epsilon'' c = \log({r_0}^2 - (\tilde{l}\cdot\tilde{l}))$, and where $(1-F)({r_0}^2 - (\tilde{l}\cdot\tilde{l})F) < 0$ we obtain by integration

$$\text{(5. 3)} \qquad 2\epsilon'\epsilon''(t + c_1) = \sin^{-1}\frac{({r_0}^2 + (\tilde{l}\cdot\tilde{l})F) - 2{r_0}^2}{|{r_0}^2 - (\tilde{l}\cdot\tilde{l})|F},$$

and $2\epsilon'\epsilon'' c_1 = \sin^{-1} \frac{(\tilde{l}\cdot\tilde{l})-r_0{}^2}{|r_0{}^2-(\tilde{l}\cdot\tilde{l})|}$. (5.2) or (5.3) gives implicitly $f(t)$.

Next, we take another special Killing vector field X given by (2.1) with

$$p_0 = 0, \qquad \mu = \mu_0(\lambda \times p), \qquad \lambda \times p \neq 0 \tag{5.4}$$

which implies $(\mu \cdot \lambda) = (\mu \cdot p) = 0$ and hence X is in the case of Theorem 2. We notice that for the case :

$$\lambda \times p = 0, \qquad \mu \neq 0, \qquad (\mu \cdot \lambda) = (\mu \cdot p) = 0$$

is not treated by this way. From (5.4) we have

$$\mu \times \tilde{x} = \mu_0((\lambda \times p) \times \tilde{x}) = \mu_0((\lambda \cdot \tilde{x})p - (p \cdot \tilde{x})\lambda),$$

and hence the component X^i can be written as

$$\begin{aligned} X^b &= \left(r_0\epsilon''\sqrt{|M|} + \mu_0(p\cdot\tilde{x})\right)\lambda_b + \left(\frac{1}{r_0}\epsilon'\sqrt{|L|} - \mu_0((\lambda\cdot\tilde{x})\right)p_b \\ &\quad - \frac{1}{r_0}\epsilon''\sqrt{|M|}(\lambda\cdot\tilde{x})x_b, \\ X^4 &= -\frac{1}{r_0}\epsilon''\sqrt{|M|}(\lambda\cdot\tilde{x})x_4 \end{aligned} \tag{5.5}$$

If we use the oblique coordinate system : $(\lambda, p, \lambda \times p)$ for R^3 and we put as

$$\tilde{x} = u\lambda + vp + w(\lambda \times p), \tag{5.6}$$

then the differential equations:

$$\frac{dx_i}{dt} = X^i, \qquad i = 1.2.3.4 \tag{5.7}$$

are equivalent to

$$\begin{cases} \dfrac{du}{dt} = r_0\epsilon''\sqrt{|M|} + \mu_0(p\cdot\tilde{x}) - \dfrac{1}{r_0}\epsilon''\sqrt{|M|}(\lambda\cdot\tilde{x})u, \\ \dfrac{dv}{dt} = \dfrac{1}{r_0}\epsilon'\sqrt{|L|} - \mu_0(\lambda\cdot\tilde{x}) - \dfrac{1}{r_0}\epsilon''\sqrt{|M|}(\lambda\cdot\tilde{x})v, \\ \dfrac{dw}{dt} = -\dfrac{1}{r_0}\epsilon''\sqrt{|M|}(\lambda\cdot\tilde{x})w, \\ \dfrac{dx_4}{dt} = -\dfrac{1}{r_0}\epsilon''\sqrt{|M|}(\lambda\cdot\tilde{x})x_4 \end{cases} \tag{5.8}$$

For simplicity, we set

$$(\lambda\cdot\lambda) = A, \qquad (p\cdot p) = B, \qquad (\lambda\cdot p) = C, \tag{5.9}$$

then we have

$$(5.10)\qquad \begin{aligned} &(\lambda\cdot\tilde{x}) = Au + Cv, \quad (p\cdot\tilde{x}) = Cu + Bv,\\ &r^2 = (\tilde{x}\cdot\tilde{x}) = Au^2 + 2Cuv + Bv^2 + (AB - C^2)w^2. \end{aligned}$$

From (5.8), we can put for the solution $x_i(t)$ of (5.7)

$$(5.11)\qquad x_4 = c_0 w, \qquad c_0 = \text{constant}.$$

Then we have

$$\begin{aligned} M &= {r_0}^2 - x_4x_4 = {r_0}^2 - {c_0}^2w^2,\\ L &= {r_0}^2 - r^2 = {r_0}^2 - Au^2 - 2Cuv - Bv^2 - (AB - C^2)w^2, \end{aligned}$$

and hence (5.8) is equivalent to (5.11) and

$$(5.12)\qquad \begin{cases} \dfrac{du}{dt} = r_0\epsilon''\sqrt{|{r_0}^2 - {c_0}^2w^2|} + \mu_0(Cu + Bv)\\ \qquad - \dfrac{1}{r_0}\epsilon''\sqrt{|{r_0}^2 - {c_0}^2w^2|}(Au + Cv)u,\\ \dfrac{dv}{dt} = \dfrac{1}{r_0}\epsilon'\sqrt{|{r_0}^2 - Au^2 - 2Cuv - Bv^2 - (AB - C^2)w^2|}\\ \qquad - \mu_0(Au + Cv) - \dfrac{1}{r_0}\epsilon''\sqrt{|{r_0}^2 - {c_0}^2w^2|}(Au + Cv)v,\\ \dfrac{dw}{dt} = -\dfrac{1}{r_0}\epsilon''\sqrt{|{r_0}^2 - {c_0}^2w^2|}(Au + Cv)w, \end{cases}$$

which we change certain linear differential equations as follow. First we obtain from (5.12)

$$\begin{aligned} \frac{d}{dt}\left(\frac{u}{w}\right) &= \frac{1}{w}\frac{du}{dt} - u\frac{1}{w^2}\frac{dw}{dt}\\ &= r_0\epsilon''\sqrt{\left|\left(\frac{r_0}{w}\right)^2 - {c_0}^2\right|} + \mu_0\left(C\frac{u}{w} + B\frac{v}{w}\right),\\ \frac{d}{dt}\left(\frac{v}{w}\right) &= \frac{1}{w}\frac{dv}{dt} - v\frac{1}{w^2}\frac{dw}{dt}\\ &= \frac{1}{r_0}\epsilon'\sqrt{\left|\left(\frac{r_0}{w}\right)^2 - A\left(\frac{u}{w}\right)^2 - 2C\frac{u}{w}\frac{v}{w} - B\left(\frac{v}{w}\right)^2 - AB + C^2\right|}\\ &\quad - \mu_0\left(A\frac{u}{w} + C\frac{v}{w}\right), \end{aligned}$$

$$\frac{d}{dt}\sqrt{\left|\left(\frac{r_0}{w}\right)^2 - {c_0}^2\right|} = \frac{\epsilon''}{\sqrt{\left|\left(\frac{r_0}{w}\right)^2 - {c_0}^2\right|}}{r_0}^2\left(-\frac{1}{w^3}\frac{dw}{dt}\right)$$

$$= r_0\left(A\frac{u}{w}+C\frac{v}{w}\right),$$

and

$$\begin{aligned}
&\frac{d}{dt}\sqrt{\left|(\frac{r_0}{w})^2-A(\frac{u}{w})^2-2C\frac{u}{w}\frac{v}{w}-B(\frac{v}{w})^2-AB+C^2\right|}\\
&=\frac{\epsilon'}{\sqrt{\left|\left(\frac{r_0}{w}\right)^2-A\left(\frac{u}{w}\right)^2-2C\frac{u}{w}\frac{v}{w}-B\left(\frac{v}{w}\right)^2-AB+C^2\right|}}\\
&\quad\times\left\{{r_0}^2\frac{-1}{w^3}\frac{dw}{dt}-\left(A\frac{u}{w}+C\frac{v}{w}\right)\frac{d}{dt}\frac{u}{w}-\left(C\frac{u}{w}+B\frac{v}{w}\right)\frac{d}{dt}\frac{v}{w}\right\}\\
&=-\frac{1}{r_0}\left(C\frac{u}{w}+B\frac{v}{w}\right).
\end{aligned}$$

Hence , if we set

$$\text{(5. 13)}\qquad \begin{aligned}
&u_1=\frac{u}{w},\qquad u_2=\frac{v}{w},\qquad u_3=\sqrt{\left|\left(\frac{r_0}{w}\right)^2-{c_0}^2\right|},\\
&u_4=\sqrt{\left|\left(\frac{r_0}{w}\right)^2-A\left(\frac{u}{w}\right)^2-2C\frac{u}{w}\frac{v}{w}-B\left(\frac{v}{w}\right)^2-AB+C^2\right|},
\end{aligned}$$

we obtain linear differential equations with constant coefficients on each D_i as

$$\text{(5. 14)}\qquad \begin{cases}
\dfrac{du_1}{dt}=\mu_0Cu_1+\mu_0Bu_2+r_0\epsilon''u_3,\\
\dfrac{du_2}{dt}=-\mu_0Au_1-\mu_0Cu_2+\dfrac{1}{r_0}\epsilon'u_4,\\
\dfrac{du_3}{dt}=r_0Au_1+r_0Cu_2,\\
\dfrac{du_4}{dt}=-\dfrac{C}{r_0}u_1-\dfrac{B}{r_0}u_2.
\end{cases}$$

Now taking indeterminate constants $\rho_1,\rho_2,\rho_3,\rho_4$, we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\sum_i\rho_iu_i\right)&=\left(\rho_1\mu_0C-\rho_2\mu_0A+\rho_3r_0A-\frac{1}{r_0}\rho_4C\right)u_1\\
&\quad+\left(\rho_1\mu_0B-\rho_2\mu_0C+\rho_3r_0C-\frac{1}{r_0}\rho_4B\right)u_2\\
&\quad+\rho_1r_0\epsilon''u_3+\frac{1}{r_0}\rho_2\epsilon'u_4.
\end{aligned}$$

Looking over the coefficients of u_3 and u_4 , we put

$$\rho_1 r_0 \epsilon'' : \frac{1}{r_0}\rho_2\epsilon' = \rho_1 : \frac{1}{{r_0}^2}\epsilon'\epsilon''\rho_2 = \rho_3 : \rho_4$$

that is

$$\rho_1 = \sigma\rho_3, \qquad \rho_2 = \sigma\epsilon'\epsilon''{r_0}^2\rho_4 \tag{5.15}$$

and hence we have

$$\rho_1 r_0\epsilon'' u_3 + \frac{1}{r_0}\rho_2\epsilon' u_4 = \sigma r_0\epsilon''(\rho_3 u_3 + \rho_4 u_4)$$

Now, we put

$$\begin{aligned}\sigma r_0\epsilon''\rho_1 &= \rho_1\mu_0 C - \rho_2\mu_0 A + \rho_3 r_0 A - \frac{1}{r_0}\rho_4 C,\\ \sigma r_0\epsilon''\rho_2 &= \rho_1\mu_0 B - \rho_2\mu_0 C + \rho_3 r_0 C - \frac{1}{r_0}\rho_4 B,\end{aligned}$$

into which substituting (5.15) we obtain the equalities:

$$\begin{cases}\left(\sigma^2\epsilon'' r_0 - \sigma\mu_0 C - r_0 A\right)\rho_3 + \left(\sigma\epsilon'\epsilon''\mu_0{r_0}^2 A + \dfrac{1}{r_0}C\right)\rho_4 = 0,\\ (\sigma\mu_0 B + r_0 C)\,\rho_3 - \left(\sigma^2\epsilon'{r_0}^3 + \sigma\epsilon'\epsilon''\mu_0{r_0}^2 C + \dfrac{1}{r_0}B\right)\rho_4 = 0.\end{cases} \tag{5.16}$$

For $(\rho_3, \rho_4) \neq (0,0)$ the following equality must hold good

$$\begin{aligned}&\left(\sigma^2\epsilon'' r_0 - \sigma\mu_0 C - r_0 A\right)\left(\sigma^2\epsilon'{r_0}^3 + \sigma\epsilon'\epsilon''\mu_0{r_0}^2 C + \frac{1}{r_0}B\right)\\ &+ \left(\sigma\epsilon'\epsilon''\mu_0{r_0}^2 A + \frac{1}{r_0}C\right)(\sigma\mu_0 B + r_0 C) = 0,\end{aligned}$$

that is

$$\begin{aligned}&\epsilon'\epsilon''{r_0}^4\sigma^4 + \left\{\epsilon'\epsilon''{\mu_0}^2{r_0}^2(AB - C^2) - \epsilon'{r_0}^4 A + \epsilon'' B\right\}\sigma^2\\ &\quad - (AB - C^2) = 0.\end{aligned} \tag{5.17}$$

For σ satisfying (5.17), we can put

$$\rho_3 = \sigma^2\epsilon'{r_0}^3 + \sigma\epsilon'\epsilon''\mu_0{r_0}^2 C + \frac{1}{r_0}B, \qquad \rho_4 = \sigma\mu_0 B + r_0 C, \tag{5.18}$$

and for $\rho_1, \rho_2, \rho_3, \rho_4$ satisfying (5.18) and (5.15) , we obtain

$$\frac{d}{dt}\left(\sum_i \rho_i u_i\right) = \sigma r_0 \epsilon'' \sum_i \rho_i u_i$$

and by integration

(5. 19)
$$\sum_i \rho_i u_i = \exp(\sigma r_0 \epsilon'' t) \times \text{constant.}$$

Now for the 4 solutions $\sigma_{(j)}, j = 1, 2, 3, 4,$ of (5.17) we denote the corresponding ρ_i by $\rho_{(j)i}$, and

$$\sum_i \rho_{(j)i} u_i = c_j \exp(\sigma_{(j)} r_0 \epsilon'' t), \qquad c_j = \text{constant.}$$

If $\det(\rho_{(j)i}) \neq 0$ then we obtain the solution

(5. 20)
$$(u_i) = (\rho_{(j)i})^{-1}(c_j \exp(\sigma_{(j)} r_0 \epsilon'' t)).$$

In the following, we compute $\det(\rho_{(j)i})$. Since

$$\rho_{(j)3} = {\sigma_{(j)}}^2 \epsilon' {r_0}^3 + \sigma_{(j)} \epsilon' \epsilon'' \mu_0 {r_0}^2 C + \frac{1}{r_0} B, \quad \rho_{(j)4} = \sigma_{(j)} \mu_0 B + r_0 C,$$
$$\rho_{(j)1} = \sigma_{(j)} \rho_{(j)3}, \qquad \rho_{(j)2} = \sigma_{(j)} \epsilon' \epsilon'' \mu_0 {r_0}^2 \rho_{(j)4},$$

we obtain

$$\begin{aligned}
\det(\rho_{(j)i}) &= \det\left(\sigma_{(j)}\rho_{(j)3} \;\; \sigma_{(j)}\epsilon'\epsilon''{r_0}^2\rho_{(j)4} \;\; \rho_{(j)3} \;\; \rho_{(j)4}\right) \\
&= - \sum_{j_1<j_2, j_3<j_4} (-1)^{j_1+j_2} \begin{vmatrix} \sigma_{(j_1)}\rho_{(j_1)3} & \sigma_{(j_1)}\epsilon'\epsilon''{r_0}^2\rho_{(j_1)4} \\ \sigma_{(j_2)}\rho_{(j_2)3} & \sigma_{(j_2)}\epsilon'\epsilon''{r_0}^2\rho_{(j_2)4} \end{vmatrix} \begin{vmatrix} \rho_{(j_3)3} & \rho_{(j_3)4} \\ \rho_{(j_4)3} & \rho_{(j_4)4} \end{vmatrix} \\
&= - \sum_{j_1<j_2, j_3<j_4} (-1)^{j_1+j_2} \epsilon'\epsilon''\sigma_{(j_1)}\sigma_{(j_2)}{r_0}^2(\rho_{(j_1)3}\rho_{(j_2)4} - \rho_{(j_1)4}\rho_{(j_2)3}) \times \\
&\quad (\rho_{(j_3)3}\rho_{(j_4)4} - \rho_{(j_3)4}\rho_{(j_4)3})
\end{aligned}$$

which is reduced to

$$\begin{aligned}
\frac{\det(\rho_{(j)i})}{\epsilon'\epsilon''{r_0}^2} &= (\sigma_{(1)}\sigma_{(2)} + \sigma_{(3)}\sigma_{(4)})(\rho_{(1)3}\rho_{(2)4} - \rho_{(1)4}\rho_{(2)3})(\rho_{(3)3}\rho_{(4)4} - \rho_{(3)4}\rho_{(4)3}) \\
&+ (\sigma_{(1)}\sigma_{(3)} + \sigma_{(4)}\sigma_{(2)})(\rho_{(1)3}\rho_{(3)4} - \rho_{(1)4}\rho_{(3)3})(\rho_{(4)3}\rho_{(2)4} - \rho_{(4)4}\rho_{(2)3}) \\
&+ (\sigma_{(1)}\sigma_{(4)} + \sigma_{(2)}\sigma_{(3)})(\rho_{(1)3}\rho_{(4)4} - \rho_{(1)4}\rho_{(4)3})(\rho_{(2)3}\rho_{(3)4} - \rho_{(2)4}\rho_{(3)3})
\end{aligned}$$

Since we have

$$\rho_{(j_1)3}\rho_{(j_2)4} - \rho_{(j_1)4}\rho_{(j_2)3}$$

$$
= \begin{vmatrix} \sigma_{(j_1)}{}^2\epsilon' r_0{}^3 + \sigma_{(j_1)}\epsilon'\epsilon''\mu_0 r_0{}^2 C + B/r_0 & \sigma_{(j_1)}\mu_0 B + r_0 C \\ \sigma_{(j_2)}{}^2\epsilon' r_0{}^3 + \sigma_{(j_2)}\epsilon'\epsilon''\mu_0 r_0{}^2 C + B/r_0 & \sigma_{(j_2)}\mu_0 B + r_0 C \end{vmatrix}
$$
$$
= (\sigma_{(j_1)} - \sigma_{(j_2)})\{\epsilon' r_0{}^3\mu_0 B\sigma_{(j_1)}\sigma_{(j_2)} + \epsilon' r_0{}^4 C(\sigma_{(j_1)} + \sigma_{(j_2)}) - \mu_0 B^2/r_0 + \epsilon'\epsilon'' r_0{}^3 C^2\}
$$

and

$$
\begin{aligned}
&(\rho_{(1)3}\rho_{(2)4} - \rho_{(1)4}\rho_{(2)3})(\rho_{(3)3}\rho_{(4)4} - \rho_{(3)4}\rho_{(4)3}) \\
&= (\sigma_{(1)} - \sigma_{(2)})(\sigma_{(3)} - \sigma_{(4)})\,\{r_0{}^6\mu_0{}^2 B^2\sigma_{(1)}\sigma_{(2)}\sigma_{(3)}\sigma_{(4)} \\
&+ r_0{}^7\mu_0 BC(\sigma_{(1)}\sigma_{(2)}(\sigma_{(3)} + \sigma_{(4)}) + \sigma_{(3)}\sigma_{(4)}(\sigma_{(1)} + \sigma_{(2)})) \\
&+ (-\epsilon' r_0{}^2\mu_0{}^2 B^3 + \epsilon'' r_0{}^6\mu_0{}^2 BC^2)(\sigma_{(1)}\sigma_{(2)} + \sigma_{(3)}\sigma_{(4)}) + r_0{}^8 C^2(\sigma_{(1)} + \sigma_{(2)})(\sigma_{(3)} + \sigma_{(4)}) \\
&+ (-\epsilon' r_0{}^3\mu_0 B^2 C + \epsilon'' r_0{}^7\mu_0 C^3)(\sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)} + \sigma_{(4)}) \\
&+ \mu_0{}^2 B^4/r_0{}^2 - 2\epsilon'\epsilon'' r_0{}^2 B^2 C^2 + \mu_0{}^2 r_0{}^6 C^4\},
\end{aligned}
$$

we obtain finally the expression :

$$
\begin{aligned}
&\det(\rho_{(j)i}) \\
&= \epsilon'\epsilon'' r_0{}^4(\epsilon'\mu_0{}^2 B^3 - \epsilon'' r_0{}^4\mu_0{}^2 BC^2 + r_0{}^6 C^2) \\
&\times \{(\sigma_{(1)}\sigma_{(2)} + \sigma_{(3)}\sigma_{(4)})(\sigma_{(1)}{}^2 - \sigma_{(2)}{}^2)(\sigma_{(3)}{}^2 - \sigma_{(4)}{}^2) \\
&\quad + (\sigma_{(1)}\sigma_{(3)} + \sigma_{(4)}\sigma_{(2)})(\sigma_{(1)}{}^2 - \sigma_{(3)}{}^2)(\sigma_{(4)}{}^2 - \sigma_{(2)}{}^2) \\
&\quad + (\sigma_{(1)}\sigma_{(4)} + \sigma_{(2)}\sigma_{(3)})(\sigma_{(1)}{}^2 - \sigma_{(4)}{}^2)(\sigma_{(2)}{}^2 - \sigma_{(3)}{}^2)\,\} \\
&= \epsilon'\epsilon'' r_0{}^4(\epsilon'\mu_0{}^2 B^3 - \epsilon'' r_0{}^4\mu_0{}^2 BC^2 + r_0{}^6 C^2)(\sigma_{(1)} - \sigma_{(2)})(\sigma_{(1)} - \sigma_{(3)}) \\
&(\sigma_{(1)} - \sigma_{(4)})(\sigma_{(2)} - \sigma_{(3)})(\sigma_{(2)} - \sigma_{(4)})(\sigma_{(3)} - \sigma_{(4)})
\end{aligned}
\tag{5.21}
$$

by working out long computations. Thus we have

PROPOSITION 1. *The integral curve of the Killing vector field*

$$
X^i = \frac{1}{r_0{}^2}\epsilon'\epsilon''\sqrt{|r_0{}^2 - r^2|}\sqrt{|r_0{}^2 - x_4 x_4|}\,x_i
$$

is given as $x_i = l_i f(t), l_i =$ *constant, where* $f(t)$ *is given implicitly by* (5.2) *or* (5.3).

PROPOSITION 2. *The integral curve of the Killing vector field X given by* (2.1) *with*

$$
p_0 = 0,\ \ \mu = \mu_0(\lambda \times p),\ \ \lambda \times p \neq 0
$$

is given by (5.20), if

$$
\epsilon'\mu_0{}^2 B^3 - \epsilon'' r_0{}^4\mu_0{}^2 BC^2 + r_0{}^6 C^2 \neq 0
$$

and

$$\{\mu_0{}^2 r_0{}^2(AB-C^2) - \epsilon'' r_0{}^4 A + \epsilon' B\}^2 + 4\epsilon'\epsilon'' r_0{}^4(AB-C^2) \neq 0$$

where

$$A = (\lambda\cdot\lambda), B = (p\cdot p), C = (\lambda\cdot p).$$

Proof. In order that the quadratic equation (5.17) on σ admits double roots, the expression :

$$\{\epsilon'\epsilon''\mu_0{}^2 r_0{}^2(AB-C^2) - \epsilon' r_0{}^4 A + \epsilon'' B\}^2 + 4\epsilon'\epsilon'' r_0{}^4(AB-C^2)$$

$$= \{\mu_0{}^2 r_0{}^2(AB-C^2) - \epsilon'' r_0{}^4 A + \epsilon' B\}^2 + 4\epsilon'\epsilon'' r_0{}^4(AB-C^2)$$

must vanish. This expression is positive on D_1 and D_3, since $\epsilon'\epsilon'' = 1$ there. We can obtain the claim by (5.21). Q.E.D.

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