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WEIGHTED SHARING AND UNIQUENESS OF DIFFERENTIAL POLYNOMIALS

By

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Abstract. In the paper we study the uniqueness of meromorphic functions concerning differential polynomials and improve a result of Lahiri [8].

1. Introduction, Definitions and Results

In the paper by meromorphic function we always mean a function which is meromorphic in the open complex plane C. We use the standard notations of the value distribution theory without any explanation because these are available in [7] and [12]. Let f be a nonconstant meromorphic function. We denote by $N_{1)}(r,a;f)$ and $\overline{N}_{(2}(r,a;f)$ the counting functions of only simple and only distinct multiple zeros of $f-a$ respectively for $a\in C\cup\{\infty\}$. Also we denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\rightarrow\infty$ and $r\notin E$, where E is a set of finite linear measure and may not be the same in every occurrence.

Let f and g be two nonconstant meromorphic functions and $a\in C\cup\{\infty\}$. We say that f and g share the value a CM (counting multiplicities) if and only if $f-a$ and $g-a$ have the same set of zeros with the same multiplicities. For a meromorphic function f we denote by $f^{(k)}$ the k^{th} derivative of f.

In [8] the following result is proved.

THEOREM 1. *Let f and g be two nonconstant meromorphic functions and $a_1, a_2, \ldots, a_n$ $(a_n\neq 0)$ be finite complex numbers. If*

(i) *f and g share ∞ CM;*
(ii) *F and G share $0, 1$ CM, where $F=\sum_{i=1}^{n}a_if^{(i)}$ and $G=\sum_{i=1}^{n}a_ig^{(i)}$;*
(iii) $\frac{\sum_{a\neq\infty}\delta(a;f)}{1+n(1-\Theta(\infty;f))}-\frac{3(1-\Theta(\infty;f))}{2\sum_{a\neq\infty}\delta(a;f)}>\frac{1}{2}$ *where* $\sum_{a\neq\infty}\delta(a;f)>0$;

then either (a) $F\equiv G$ *or* (b) $F\cdot G\equiv 1$. *If, further, f has at least one pole or F has at least one zero, the case* (b) *does not arise.*

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Now one may as the following question: ***Is it possible in any way to relax the nature of sharing the values and to weaken the condition on deficiences in Theorem 1?***

In the paper we give an affirmative answer to this question and prove two results, one of which improves *Theorem 1.*

To this end we explain the notion of weighted sharing as introduced in [9, 10].

DEFINITION 1 ([9, 10]). Let k be a nonnegative integer or infinity. For $a \in C \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_o is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_o is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Following two are the main results of the paper, the first of which improves *Theorem 1.*

THEOREM 2. *Let f and g be two nonconstant meromorphic functions and $a_1, a_2, \ldots, a_n$ $(a_n \neq 0)$ be finite complex numbers. If*

(i) *f and g share (∞, ∞);*
(ii) *F and G are nonconstant and share $(0, 1), (1, \infty)$, where $F = \sum_{i=1}^{n} \alpha_i f^{(i)}$ and $G = \sum_{i=1}^{n} \alpha_i f^{(i)}$;*
(iii) $\sum_{a \neq \infty} \delta(a; f) > \frac{1}{2}$;

then either (a) $F \equiv G$ *or* (b) $F \cdot G \equiv 1$. *If, further, F has at least one zero or f has at least one pole, the case* (b) *does not arise.*

THEOREM 3. *Let f and g be two nonconstant meromorphic functions and $a_0, a_1, \ldots, a_n$ $(a_n \neq 0)$ be finite complex numbers, where $n \geq 1$. If*

(i) *f and g share (∞, ∞);*
(ii) *F and G are nonconstant and share $(0, 1), (1, \infty)$, where $F = \sum_{i=0}^{n} \alpha_i f^{(i)}$ and $G = \sum_{i=0}^{n} \alpha_i f^{(i)}$;*
(iii) $\delta(0; f) > \frac{1}{2}$;

then either (a) $F \equiv G$ *or* (b) $F \cdot G \equiv 1$. *If, further,* f *has at least one pole or* F *has at least one zero, the case* (b) *does not arise.*

Following example shows that the condition (iii) in *Theorem 2 and 3* is best possible.

Example ([8]). Let $f = \exp(z) - (\frac{1}{2})^n \exp(2z)$ and $g = (-1)^n \exp(-z) - (\frac{-1}{2})^n \exp(-2z)$. Then $F = f^{(n)} = \exp(z) - \exp(2z)$ and $G = g^{(n)} = \exp(-z) - \exp(-2z)$. Also it is easy to see that (i) f and g share (∞, ∞), (ii) F and G share $(0, \infty), (1, \infty)$, (iii) $\delta(0; f) = \sum_{a \neq \infty} \delta(a; f) = \frac{1}{2}$ but neither $F \equiv G$ nor $F \cdot G \equiv 1$.

Now we require the following definitions and lemmas.

DEFINITION 2 ([5]). For a meromorphic function f we define

$$T_o(r, f) = \int_1^r \frac{T(t, f)}{t} dt,$$
$$N_o(r, a; f) = \int_1^r \frac{N(t, a; f)}{t} dt,$$
$$m_o(r, a; f) = \int_1^r \frac{m(t, a; f)}{t} dt,$$
$$\overline{N}_o(r, a; f) = \int_1^r \frac{\overline{N}(t, a; f)}{t} dt,$$
$$S_o(r, f) = \int_1^r \frac{S(t, f)}{t} dt$$

etc. where $a \in C \cup \{\infty\}$.

DEFINITION 3 ([5]). For a meromorphic function f we put

$$\delta_o(a; f) = 1 - \limsup_{r \to \infty} \frac{N_o(r, a; f)}{T_o(r, f)} = \liminf_{r \to \infty} \frac{m_o(r, a; f)}{T_o(r, f)}.$$

LEMMA 1 ([5]). *For a meromorphic function* f

$$\lim_{r \to \infty} \frac{S_o(r, f)}{T_o(r, f)} = 0$$

through all values of r.

LEMMA 2 ([11]). *If* f *is a meromorphic function and* $a \in C \cup \{\infty\}$ *then* $\delta(a; f) \leq \delta_o(a; f)$.

LEMMA 3 ([13]). *If* $\beta_i(\not\equiv 0,\ i=1,2)$ *are meromorphic functions such that* $T(r,\beta_i)=S(r,f)+S(r,g)$ *for* $i=1,2$ *and* $\beta_1 f+\beta_2 g\equiv 1$ *then*

$$T(r,f)\le \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,f)+S(r,f)+S(r,g).$$

LEMMA 4. *Let* f, g *be two nonconstant meromorphic functions sharing* $(0,1)$, $(1,\infty)$ *and* (∞,∞). *If* $f\not\equiv g$ *then*

$$\overline{N}_{(2}(r,0;f)=\overline{N}_{(2}(r,0;g)=S(r,f)=S(r,g)$$

and

$$\overline{N}_{(2}(r,\infty;f)=\overline{N}_{(2}(r,\infty;g)=S(r,f)=S(r,g).$$

Proof. Since f, g share $(0,1),(1,\infty)$ and (∞,∞), by the second fundamental theorem we get

$$T(r,f)\le 3T(r,g)+S(r,f) \text{ and } T(r,g)\le 3T(r,f)+S(r,g)$$

and so $S(r,f)=S(r,g)$. If $\overline{N}(r,0;f)=S(r,f)$ and $\overline{N}(r,\infty;f)=S(r,f)$, the lemma is obvious. So we suppose that $\overline{N}(r,0;f)\ne S(r,f)$ and $\overline{N}(r,\infty;f)\ne S(r,f)$ Let

$$h=\frac{f'}{f-1}-\frac{g'}{g-1}.$$

We see that $h\not\equiv 0$ because $f\not\equiv g$. Since f, g share $(0,1)$ it follows that a multiple zero of f is a multiple zero of g and conversely. Hence a multiple zero of f (and so of g) is a zero of h. So by the first fundamental theorem and Milloux theorem {p.55 [7]} we get

$$\overline{N}_{(2}(r,0;f)\le N(r,0;h)\le N(r,h)+S(r,f)+S(r,g)=N(r,h)+S(r,f).$$

Since the possible poles of h occur at the poles and 1-points of f, g and f, g share $(1,\infty),(\infty,\infty)$, it follows that h has no pole at all. Therefore

$$\overline{N}_{(2}(r,0;f)=S(r,f).$$

Let $f_1=1/f$ and $g_1=1/g$. Then f_1, g_1 share $(0,\infty),(1,\infty)$ and $(\infty,1)$. Let

$$\phi=\frac{f_1'}{f_1-1}-\frac{g_1'}{g_1-1}.$$

Then $\phi\not\equiv 0$ because $f_1\not\equiv g_1$. Since f_1, g_1 share $(0,\infty)$, it follows by the first fundamental theorem and Milloux theorem {p.55 [7]} that

$$\overline{N}_{(2}(r,0;f_1)\le N(r,0;\phi)\le N(r,\phi)+S(r,f)+S(r,g).$$

Since f_1, g_1 share $(1,\infty)$ and $(\infty,1)$, it follows that poles of ϕ occur only at those poles of f_1 whose multiplicities are different from the multiplicities of the corresponding poles of g_1 and we denote by $\overline{N}_*(r,\infty;f_1,g_1)$ the reduced counting function of such poles.

Since f_1, g_1 share $(\infty;1)$ and ϕ has only simple poles, we get from above

$$\begin{aligned}\overline{N}_{(2}(r,0;f_1) &\leq \overline{N}_*(r,\infty;f_1,g_1)+S(r,f)\\ &\leq \overline{N}_{(2}(r,\infty;f_1)+S(r,f)\end{aligned}$$

i.e.

$$\overline{N}_{(2}(r,\infty;f)\leq \overline{N}_{(2}(r,0;f)+S(r,f)=S(r,f).$$

Since $\overline{N}_{(2}(r,0;f)=\overline{N}_{(2}(r,0;g)$ and $\overline{N}_{(2}(r,\infty;f)=\overline{N}_{(2}(r,\infty,g)$, the lemma is proved. □

2. Proof of Theorem 2 and Theorem 3

Since the proofs of *Theorem 2* and *Theorem 3* are similar, we only prove *Theorem 2*.

Let $F\not\equiv G$. We shall prove that $F\cdot G\equiv 1$.

From conditions (i) and (ii) of *Theorem 2*, it follows that F and G share $(0,1)$, $(1,\infty)$ and (∞,∞). So by the second fundamental theorem we get

$$\begin{aligned}&T(r,F)\leq 3T(r,G)+S(r,F),\\ &T(r,G)\leq 3T(r,F)+S(r,G)\\ \text{and}\quad &S(r,F)=S(r,G).\end{aligned}\tag{1}$$

Since F, G has only multiple poles, it follows from *Lemma 4* that $\overline{N}(r,F)=\overline{N}_{(2}(r,\infty;F)=S(r,F)$ and $\overline{N}(r,G)=\overline{N}_{(2}(r,\infty;G)=S(r,G)$. Let $b_1,b_2,\ldots,b_p$ be finite deficient values of f. Since $\sum_{n=1}^{p}m(r,b_n;f)\leq m(r,0;F)+S(r,f)$, integrating we get $\sum_{n=1}^{p}m_o(r,b_n;f)\leq m_o(r,0;F)+S_o(r,f)$ and so

$$\sum_{n=1}^{p}\frac{m_o(r,b_n;f)}{T_o(r,f)}\leq\frac{m_o(r,0;F)}{T_o(r,F)}\frac{T_o(r,F)}{T_o(r,f)}+\frac{S_o(r,f)}{T_o(r,f)}.$$

Since $\overline{N}(r,F)=S(r,F)$ implies $\overline{N}_o(r,F)=S_o(r,F)$, it follows from above

$$\sum_{n=1}^{p}\frac{m_o(r,b_n;f)}{T_o(r,f)}\leq\frac{m_o(r,0;F)}{T_o(r,F)}\frac{T_o(r,f)+S_o(r,f)}{T_o(r,f)}+\frac{S_o(r,f)}{T_o(r,f)}.$$

Thus we obtain in view of *Lemma 1* $\sum_{n=1}^{p}\delta_o(b_n;f)\leq\delta_o(0;F)$ and since p is arbitrary, it follows that $\sum_{b\neq\infty}\delta_o(b;f)\leq\delta_o(0;F)$

Now by *Lemma 2* and condition (ii) we get

$$\delta_o(0;F) > \frac{1}{2} \tag{2}$$

Set

$$\psi = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}. \tag{3}$$

Since F and G share $(1,\infty)$, by a simple computation, we see that if z_o is a simple zero of $F-1$ and $G-1$ then $\psi(z_o)=0$.

Next we shall prove that $\psi \equiv 0$. Suppose, on the contrary, that $\psi \not\equiv 0$. Then

$$\begin{aligned} N_{1)}(r,1;F) = N_{1)}(r,1;G) &\le N(r,0;\psi) \le T(r,\psi) + O(1) \\ &\le N(r,\psi) + S(r,F) + S(r,G). \end{aligned} \tag{4}$$

Let z_o be a zero of $F-1$. Since F, G share $(1,\infty)$ we can deduce by a simple calculation from (3) that $\psi(z_o) \neq \infty$. Similarly we can see that if z_o is a pole of F then $\psi(z_o) \neq \infty$. So the poles of ψ only occur at the zeros of F' and G'. Hence we obtain from (3)-(4) and *Lemma 4* that

$$\begin{aligned} N_{1)}(r,1;F) &\le N_*(r,0;F') + N(r,0;G') \\ &\quad + S(r,F) + S(r,G), \end{aligned} \tag{5}$$

where $N_*(r,0;F')$ counts only those zeros of F which are not zeros of $F(F-1)$.

By the second fundamental theorem we get

$$T(r,F) \le \overline{N}(r,F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) - N_*(r,0;F') + S(r,F) \tag{6}$$

and

$$T(r,G) \le \overline{N}(r,G) + \overline{N}(r,0;G) + \overline{N}(r,1;G) - N_*(r,0;G') + S(r,G). \tag{7}$$

Also we see that

$$\overline{N}(r,1;F) + \overline{N}(r,1;G) = 2\overline{N}(r,1;G) \le N_{1)}(r,1;G) + N(r,1;G). \tag{8}$$

Since $\overline{N}(r,F) = S(r,F)$ and $\overline{N}(r,G) = S(r,G)$, we get from (1) and (5)-(8)

$$\begin{aligned} T(r,F) + T(r,G) &\le \overline{N}(r,0;F) + \overline{N}(r,0;G) + N(r,1;G) + S(r,F) \\ &\le 2N(r,0;F) + T(r,G) + S(r,F) \end{aligned}$$

i.e. $T(r,F) \le 2N(r,0;F) + S(r,F)$, which gives on integration $T_o(r,F) \le 2N_o(r,0;F) + S_o(r,F)$. This implies that $\delta_o(0;F) \le \frac{1}{2}$ which contradicts (2). Therefore $\psi \equiv 0$ and so

$$\frac{F''}{F'} - \frac{2F'}{F-1} \equiv \frac{G''}{G'} - \frac{2G'}{G-1}. \tag{9}$$

Solving (9) we get

$$\frac{1}{F-1} = \frac{a}{G-1} + b,$$

where $a(\neq 0)$ and b are constants. Thus we obtain

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}. \tag{10}$$

From (10) we see that

$$T(r,F) = T(r,G) + O(1). \tag{11}$$

Now we consider three possibilities.

POSSIBILITY 1. Let $b \neq 0, -1$. If $a-b-1 \neq 0$ then by (10) we know $\overline{N}(r, \frac{1+b-a}{1+b}; G) = \overline{N}(r,0;F)$. Now by the second fundamental theorem we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G) + \overline{N}(r,0;G) + \overline{N}(r, \tfrac{1+b-a}{1+b}); G) + S(r,G) \\ &\leq \overline{N}(r,0;G) + \overline{N}(r,0;F) + S(r,G) \\ &\leq 2\overline{N}(r,0;F) + S(r,G) \end{aligned}$$

and so by (11) $T(r,F) \leq 2N(r,0;F) + S(r,F)$. By integration it follows that $T_o(r,F) \leq 2N_o(r,0;F) + S_o(r,F)$ and so $\delta_o(0;F) \leq \frac{1}{2}$ which contradicts (2). Then $a-b-1=0$ and so we get from (10) that

$$F = \frac{aG}{(a-1)G+1}. \tag{12}$$

Clearly $a \neq 0, 1$ because $b \neq 0, -1$ and $a-b-1=0$. Let $H = \frac{G}{F}$. Then we get from (12) that $aH - (a-1)G \equiv 1$. Since F, G share $(0,1)$, $(1,\infty)$ and (∞,∞), it follows from *Lemma 4* that $\overline{N}(r,0;H) \leq \overline{N}_{(2}(r,0;G) = S(r,F)$. Now by *Lemma 3* we get $T(r,G) \leq \overline{N}(r,0;G) + \overline{N}(r,G) + S(r,G) + S(r,H) = \overline{N}(r,0;G) + S(r,G) + S(r,F)$ and so by (11) it follows that $T(r,F) \leq N(r,0;F) + S(r,F)$. Integrating we get $T_o(r,F) \leq N_o(r,0;F) + S_o(r,F)$ and this implies $\delta_o(0;F) = 0$, which contradicts (2). Therefore the possibility 1 does not arise.

POSSIBILITY 2. Let $b = -1$. Then (10) gives $F = \frac{a}{(a+1)-G}$. If $a \neq -1$, we get

$$\frac{G}{a+1} + \frac{a}{(a+1)F} \equiv 1$$

and this implies by *Lemma 3* and (11) that

$$T(r,F) \leq \overline{N}(r,0;G) + \overline{N}(r,F) + \overline{N}(r,G) + S(r,F) + S(r,G)$$

i.e.

$$T(r,F) \le N(r,0;F) + S(r,F).$$

Integrating we get $T_o(r,F) \le N_o(r,0;F) + S_o(r,F)$ and this implies $\delta_o(0;F) = 0$, which contradicts (2). Hence $a = -1$ and so $F \cdot G \equiv 1$.

Possibility 3. Let $b = 0$. Then (10) gives $F = \frac{G+(a-1)}{a}$. If $a \ne 1$, we get $\frac{a}{a-1}F - \frac{1}{a-1}G \equiv 1$ and this implies by *Lemma 3* and (11) that

$$\begin{aligned} T(r,F) &\le \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,F) + S(r,F) + S(r,G) \\ &\le 2N(r,0;F) + S(r,F) \end{aligned}$$

Integrating we get $T_o(r,F) \le 2N_o(r,0;F) + S_o(r,F)$ and this implies $\delta_o(0;F) \le \frac{1}{2}$, which contradicts (2). Hence $a = 1$ and so $F \equiv G$ which is not possible by our supposition.

Further if f has at least one pole, say z_o, then g has a pole at z_o. Hence F and G has poles at z_o which is impossible if $F \cdot G \equiv 1$. Similarly if F has at least one zero then G has a zero at the same point and implies a contradiction when $F \cdot G \equiv 1$. Therefore if f has at least one pole or F has at least one zero the case (b) does not arise. This proves the theorem. □

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