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A CRITERION SPHERE FOR EXTENSIONS OF A NICE IMMERSION OF THE 2-SPHERE INTO THE EUCLIDEAN 3-SPACE

By

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Abstract. We give a method to construct all of the extensions of a nice immersion of the 2-sphere into the Euclidean 3-space by using a sphere.

§ 1. Preliminary

Titus [5] gave a necessary and sufficient condition whether a properly nested immersion of a circle into the plane extends to an immersion of a disk into the plane. Smale [4] classified immersed spheres in the Euclidean 3-space by regular homotopies, and Hirsch [1] built an obstruction theory to classify immersions up to regular homotopies. Blank [2] extends the Titus' result to all immersions of a circle into the plane. Pappus [3] gives a method for constructing all of the extensions of an immersion by using Morse theory. We give a method to construct all of the extensions of a nice immersion of the 2-sphere into the Euclidean 3-space by using a sphere.

We work in the piecewise linear category.

Throughout this paper let $f : S \to \mathbb{R}^3$ be a *nice* immersion of a 2-sphere S into the Euclidean 3-space $\mathbb{R}^3$, i.e. the singularities of the map f consist of finitely many *double curves* in which two sheets pierce each other and isolated *triple points* in which three sheets pierce each other. Let Z_f be the unbounded domain of $\mathbb{R}^3 - f(S)$.

Proper surfaces $F_1, F_2, \cdots, F_n$ in a 3-manifold are said to be *admissible* provided that (see Figure 1)

(1) $F_i \cap F_j \neq \emptyset$ implies $\partial F_i \cap \partial F_j \neq \emptyset$,where $\partial(\cdots)$ means the boundary of $(\cdots)$, and

(2) $\bigcup_{i=1}^{n} \partial F_i$ contains finite double points but no triple point.

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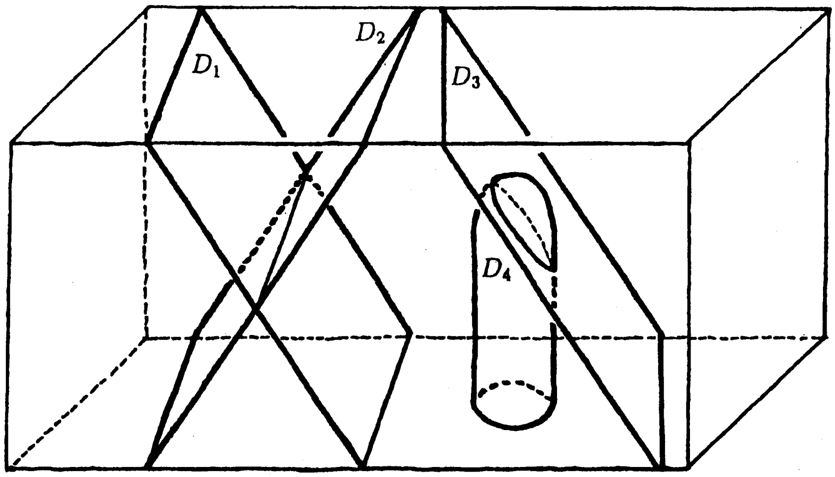


D_i's : proper disks
$\{D_1, D_2, D_3\}$ is admissible, but $\{D_1, D_2, D_3, D_4\}$ is not admissible

Figure 1

Let B and B^* be 3-balls in $\mathbb{R}^3$ such that $B \cap B^* = \partial B \cap \partial B^*$ is a disk, say D^*. Let $B^{**} = Cl(\mathbb{R}^3 - (B \cup B^*))$, where $Cl(\cdots)$ is the closure of $(\cdots)$. Set $S^* = \partial B^*$ and $S^{**} = \partial B^{**}$. The sphere S^{**} is called the *twin* of S^*. The sphere S^* is called a *criterion sphere with a waterdrop* B for the map f provided that

(1) the intersection $f(S) \cap B^*$ (resp. $f(S) \cap B^{**}$) consists of admissible surfaces in B^* (resp. B^{**}),
(2) the intersection $f(S) \cap B$ consists of admissible disks in B,
(3) the set ∂D^* does not contain singularities of the map f, and
(4) $(S^* - D^*) \cap Cl(Z_f) \neq \emptyset$.

Fix a point p^* in $(S^* - D^*) \cap Cl(Z_f)$. The point p^* is called the *star* of the criterion sphere. The closure of each component of the complementary domain of $f^{-1}(S^* \cup S^{**})$ in S is called a *piece*. The image of a piece is called a *wall*.

Remark. The star is not essential but we use the star to make our argument simple.

Next theorem assures us the existence of a criterion sphere.

THEOREM 1. *Any nice immersion of a 2-sphere into* $\mathbb{R}^3$ *possesses a criterion sphere with a waterdrop.*

Let M be an orientable compact connected 3-manifold with ∂M a 2-sphere. The map f is said to *extend* to an immersion $\tilde{f} : M \to \mathbb{R}^3$ if there exists a homeomorphism $h : S \to \partial M$ with $\tilde{f} \circ h = f$ (cf.[3]).

For the nice immersion f, there exists an immersion $\nu : S \times [-1,1] \to \mathbb{R}^3$ with $\nu(x,0) = f(x)$ for all $x \in S$. If the map f extends to an immersion of an orientable compact connected 3-manifold, then we can assume that one of $\nu(S \times [-1,0])$ or $\nu(S \times [0,1])$ does not meet Z_f. Hence throughout this paper we assume that

$$\nu(S \times [0,1]) \cap Z_f = \emptyset.$$

Further for the criterion sphere S^* with the waterdrop, we assume that

(1) $\nu^{-1}(S^*) = f^{-1}(S^*) \times [0,1]$,
(2) $\nu^{-1}(S^{**}) = f^{-1}(S^{**}) \times [0,1]$, where S^{**} is the twin of S^*, and
(3) for each piece Q, the map ν embeds $Q \times [0,1]$ into $\mathbb{R}^3$.

Now for each wall W in B^* (resp. B^{**}), the union $W \cup S^*$ (resp. $W \cup S^{**}$) splits $\mathbb{R}^3$ into three domains, two are bounded, and the other unbounded. We denote by $In(W)$ the closure of the bounded domain that does not contain the star p^*. A wall W is said to be *inner* if $\nu(Q \times [0,1]) \subset In(W)$, where Q is the piece with $f(Q) = W$. A wall is said to be *outer* if the wall is not inner. A wall W is said to *dominate* another wall W' if $W' \subset In(W)$. A set of mutually disjoint walls is called a *room* provided that

(1) there exists only one inner wall (hence the other walls are outer walls),
(2) the inner wall dominates all other walls, and
(3) none of outer walls dominate any walls.

For a room $\mathcal{R}$ in B^* (resp. B^{**}), let W be the inner wall of $\mathcal{R}$. Let

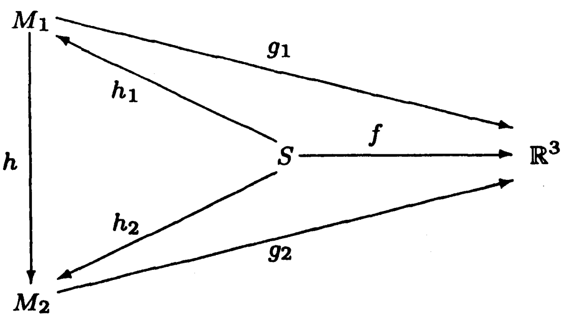
$$R = Cl(In(W) - \bigcup_{W' \in \mathcal{R} - W} In(W')).$$

Each component of the set $R \cap S^*$ (resp. $R \cap S^{**}$) is called a *floor* of the room. For each floor F, each component of the set $F \cap S^* \cap S^{**}$ is called a *tight* floor. A set of rooms is called a *star stair* (resp. a *double-stars stair*) provided that

(1) each wall of the rooms belongs to B^* (resp. B^{**}), and
(2) any wall in B^* (resp. B^{**}) belongs to exactly one room in the set.

The set of the floors of the rooms on a stair is called the *floor diagram* of the stair. The set of the tight floors of the rooms on a stair is called the *tight floor diagram* of the stair. A pair of a star stair and a double-stars stair is called a *house* on the criterion sphere with the waterdrop, if both of the tight floor diagrams are same.

Suppose that the nice immersion f extends to immersions $g_i : M_i \to \mathbb{R}^3$ $(i = 1,2)$. Then the maps g_1 and g_2 are *equivalent* if and only if there exists a homeomorphism $h : M_1 \to M_2$ such that the following diagram commutes (cf. [3]):



where $h_i : S \longrightarrow \partial M_i$ is the homeomorphism with $g_i \circ h_i = f$.

Then we have the following classifying theorem of extensions.

THEOREM 2. *The number of equivalent extension classes of the map is equal to the one of houses on the criterion sphere with the waterdrop.*

§ 2. Quick building of a house

To build a house quickly we investigate the relation between a room and its tight floors. Throughout this section the map f possesses a criterion sphere S^* with a waterdrop B. Let

B^* = the 3-ball bounded by S^*,
$B^{**} = Cl(\mathbb{R}^3 - (B \cup B^*))$,
$D^* = B \cap S^*$,
$S^{**} = \partial B^{**}$, and
p^* be the star of the criterion sphere S^*.

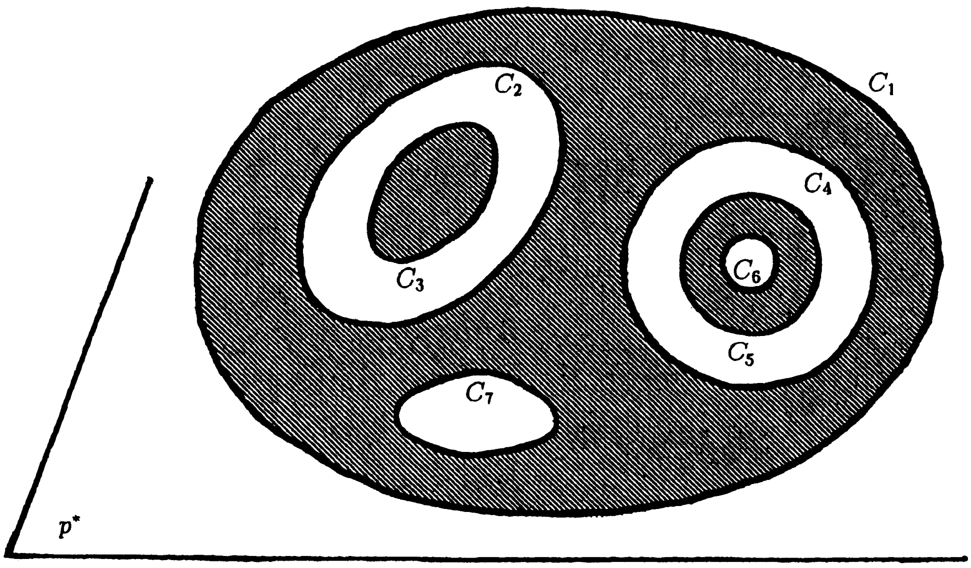
We shall show how to build the stairs and get the (tight) floor diagrams. For each circle C on the sphere S^* (resp. S^{**}), the circle C bounds a disk on S^* (resp. S^{**}) which does not contain the star p^*. We denote the disk by $In(C)$. Let $\mathfrak{C}$ be a set of mutually disjoint circles on the sphere S^* (resp. S^{**}). For each circle C in $\mathfrak{C}$, let *depth*(C) be the number of the circles in the set $\{C' \in \mathfrak{C} | C' \neq C, C \subset In(C')\}$. Hence for the outermost circle C in $\mathfrak{C}$, we have depth$(C) = 0$. For a circle C in $\mathfrak{C}$, let

$$\mathfrak{C}(C) = \{C' \in \mathfrak{C}| \text{ depth } (C') = \text{depth } (C) + 1\}, \text{ and}$$
$$F(C) = Cl(In(C) - \bigcup_{C' \in \mathfrak{C}(C)} In(C')).$$

Let

$$\mathcal{F}(\mathfrak{C}) = \{F(C)|C \in \mathfrak{C}, \text{ depth } (C) \text{ is even }\}, \text{ and}$$
$$F(\mathfrak{C}) = \cup\{F(C)|F(C) \in \mathcal{F}(\mathfrak{C})\} \text{ (see Figure 2).}$$

For a room $\mathcal{R}$, let



$\mathfrak{C} = \{C_1, C_2, \ldots, C_7\}$ $F(\mathfrak{C})$: shaded area

Figure 2

$$\mathcal{S}(\mathcal{R}) = \{C | C \text{ is a boundary component of a wall in } \mathcal{R}\}.$$

We start with the following easy observations.

OBSERVATION 1. *For a room $\mathcal{R}$, $\mathcal{F}(\mathcal{S}(\mathcal{R}))$ is the set of the floors of the room $\mathcal{R}$.* ∎

For a wall W, let

$$\mathcal{S}(W) = \{C | C \text{ is a boundary component of the wall } W\}.$$

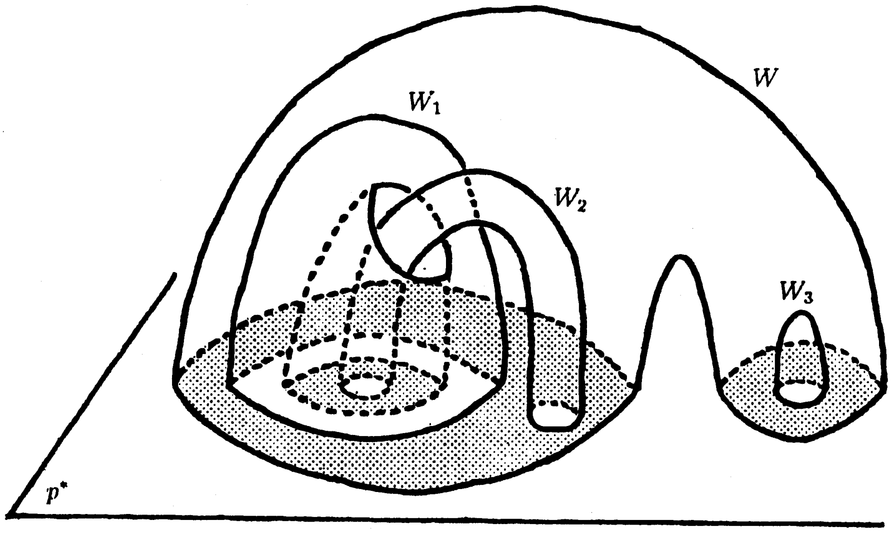
OBSERVATION 2. *For an inner wall W and outer walls $W_1, W_2, \cdots, W_n$ in B^* (resp. B^{**}), the set $\{W, W_1, W_2, \cdots, W_n\}$ is a room if and only if the following two conditions are satisfied:*

(1) *for each $i = 1, 2, \cdots, n$, $F(\mathcal{S}(W_i)) \subset F(\mathcal{S}(W))$, and*
(2) $F(\mathcal{S}(W_i)) \cap F(\mathcal{S}(W_j)) = \emptyset$ $(i \neq j)$. ∎

OBSERVATION 3. *If $\mathcal{R} = \{W, W_1, W_2, \cdots, W_n\}$ is a room with the inner wall W and the outer walls $W_1, W_2, \cdots, W_n$, then*

$$F(\mathcal{S}(\mathcal{R})) = Cl(F(\mathcal{S}(W)) - \bigcup_{i=1}^{n} F(\mathcal{S}(W_i))) \text{ (see Figure 3).} \blacksquare$$

OBSERVATION 4. *Let $\mathcal{R} = \{W, W_1, W_2, \cdots, W_n\}$ be a room as above. Let W'*



$\mathcal{R} = \{W, W_1, W_2, W_3\}$ $F(\mathcal{S}(\mathcal{R}))$: dotted area

Figure 3

be an outer wall in B^* *(resp.* B^{**}*) where* $\mathcal{R}$ *lies. Then* $\mathcal{R}' = \{W, W_1, W_2, \cdots, W_n, W'\}$ *is a room if and only if* $F(\mathcal{S}(W')) \subset F(\mathcal{S}(\mathcal{R}))$*. Furthermore if* $\mathcal{R}'$ *is a room, then*

$$F(\mathcal{S}(\mathcal{R}')) = Cl(F(\mathcal{S}(\mathcal{R})) - F(\mathcal{S}(W'))). \blacksquare$$

For a wall W, let

$$\varepsilon_W = \begin{cases} +1 & (\text{if } W \text{ is an inner wall}) \\ -1 & (\text{if } W \text{ is an outer wall}). \end{cases}$$

Let

$$\mathcal{P}^* = \{(F(\mathcal{S}(W)), \varepsilon_W) | W \text{ lies on } B^* \}, \text{ and}$$
$$\mathcal{P}^{**} = \{(F(\mathcal{S}(W)), \varepsilon_W) | W \text{ lies on } B^{**} \}.$$

Using Observation 2 and Observation 4, allot each element $(F(\mathcal{S}(W)), -1)$ in $\mathcal{P}^*$ (resp. $\mathcal{P}^{**}$) to an element $(F(\mathcal{S}(W')), +1)$ in $\mathcal{P}^*$ (resp. $\mathcal{P}^{**}$) next by next so that we have a star stair (resp. a double-stars stair) and a floor diagram by Observation 3 and Observation 4. Once we have a floor diagram, easily we can get a tight floor diagram. Finally find couples with the same tight floor diagram.

§ 3. An Example of Houses

Let $H = \{(0, y, z) \mid 0 \le y \le 1, \ 0 \le z \le 1\}$. Let α be an immersed arc in H, as shown in Figure 4, where boundary points lie on z-axis. We denote by

S_α the immersed sphere in $\mathbb{R}^3$ obtained by rotating the immersed arc α around z-axis. Let $a_1, a_2, \cdots, a_6$ be the double points of the arc α as shown in Figure 4. For each $i = 1, 2, \cdots, 6$, let T_i be a regular neighborhood of the double curve of S_α that contains the point a_i. We assume that $T_i \cap H$ is a proper disk of the solid torus T_i. Let

$$X^+ = \{(x, y, z) \mid x \geq 0\}, \text{ and } X^- = \{(x, y, z) \mid x \leq 0\}.$$

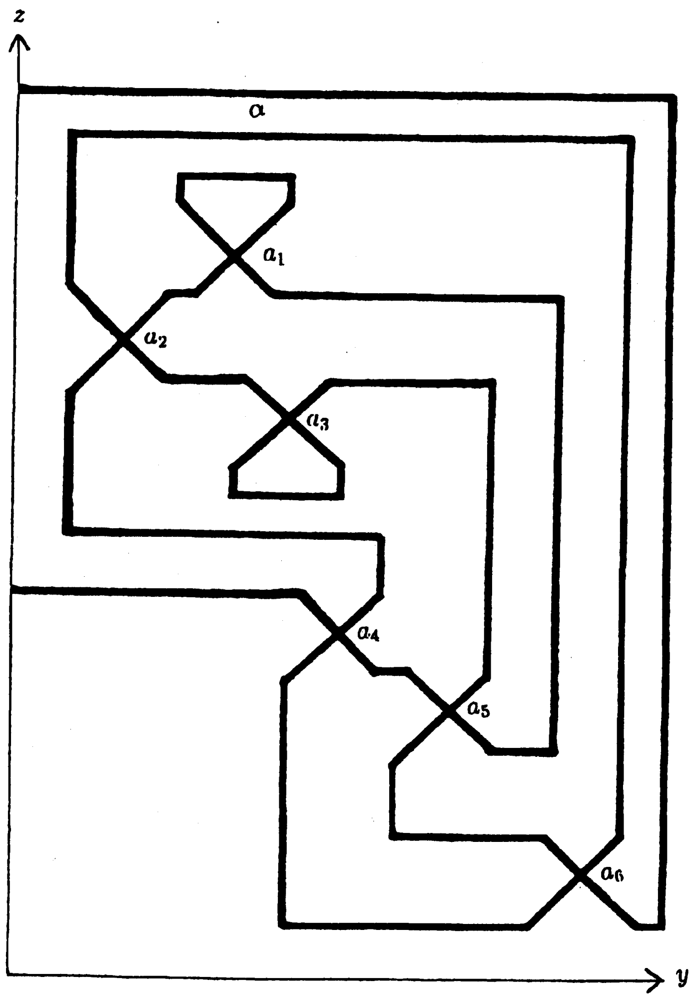


Figure 4

Let N_i be a regular neighborhood of $T_i \cap H$ in the 3-ball $T_i \cap X^-$. For each $i = 1, 2, \cdots, 6$, let A_i be a line segment in H, as shown in Figure 5, between a boundary point of $T_i \cap H$ and a point z_i on z-axis. We denote by D_i^* be the disk obtained by rotating the arc A_i around z-axis. Let N_i^* be a regular neighborhood of D_i^* in $Cl(\mathbb{R}^3 - T_i)$. Let B' be a regular neighborhood of the line segment between z_1 and the point $(0, 0, 2)$.

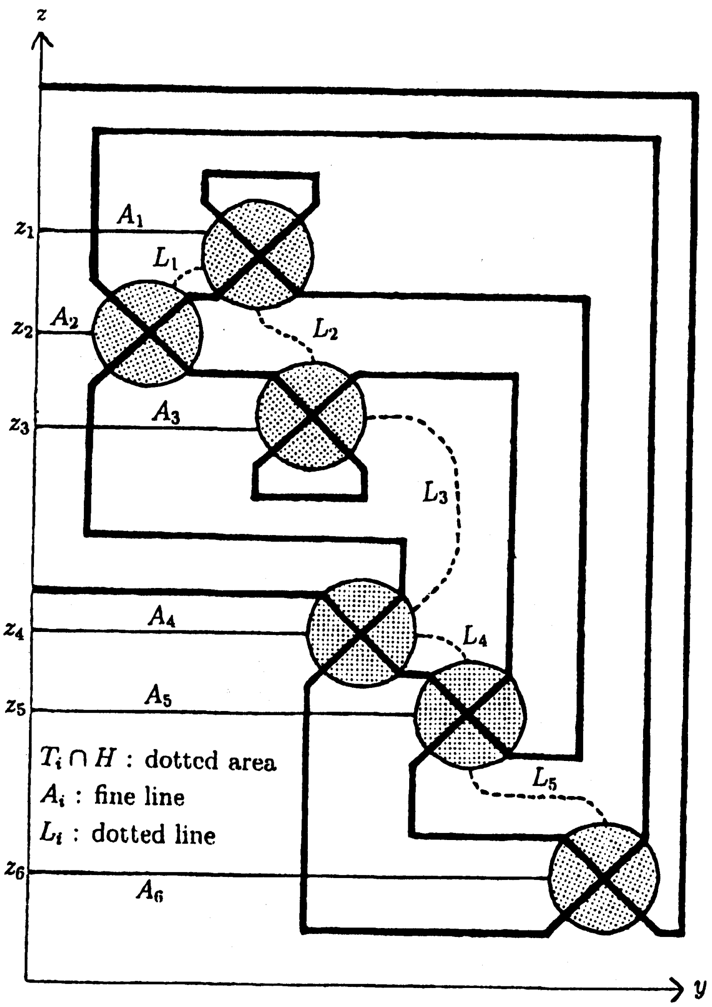


Figure 5

We assume that $N_1^* \cup B'$ is a 3-ball. Let $L_1, L_2, \cdots, L_5$ be simple arcs as shown in Figure 5. For $i = 1, 2, \cdots, 5$, let E_i be a regular neighborhood of L_i in $Cl(\mathbb{R}^3 - \bigcup_{j=1}^{6} T_j)$. We assume that

$$E_i^+ = E_i \cap X^+, \text{ and } E_i^- = E_i \cap X^- \text{ are 3-balls.}$$

Then $N = T_1 \cup T_3 \cup E_2^+$ is an unkotted handle body. Set

$$B^* = Cl(N - (N_1 \cup N_3)) \cup N_1^* \cup N_3^* \cup B'.$$

Now $B = (\bigcup_{i=1}^{6} N_i) \cup (\bigcup_{j=1}^{5} E_j^-)$ is a waterdrop (see Figure 6). Then ∂B^* is a criterion sphere with the waterdrop B for the immersed sphere S_α. Let p^* be a point on $\partial B^* \cap$ z-axis whose z-coordinate is larger than 2. Then p^* is the star of the criterion sphere ∂B^*. For the immersed sphere S_α, we shall count the number of houses with respect to the criterion sphere ∂B^* with the waterdrop B and the star p^*.

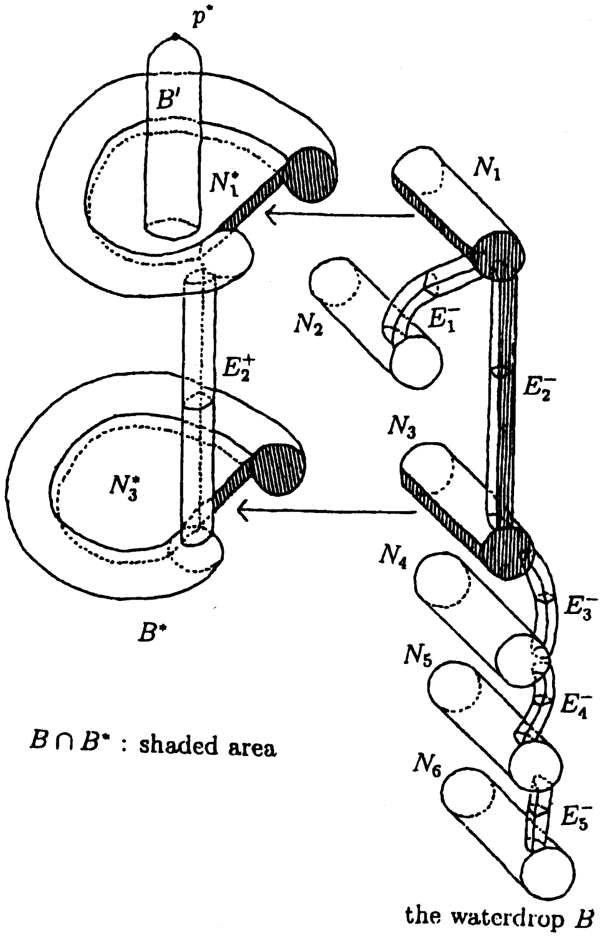


Figure 6

Now $\alpha \cap B^*$ consists of seven simple arcs. Let $v_1, v_2, \cdots, v_7$ be the seven arcs as shown in Figure 7. For each $i = 1, 2, \cdots, 7$, let V_i be the wall in B^* which contains the arc v_i. Similarly $Cl(\alpha - B^*)$ consists of seven simple arcs $w_1, w_2, \cdots, w_7$ as shown in Figure 7. Let $B^{**} = Cl(\mathbb{R}^3 - (B^* \cup B))$. For each $i = 1, 2, \cdots, 7$, let W_i be the wall in B^{**} which contains the arc w_i. Now $\bigcup_{i=1}^{7} \partial V_i = (\bigcup_{i=1}^{7} V_i) \cap \partial B^*$ consists of nine circles as shown in Figure 8. And $\bigcup_{i=1}^{7} \partial W_i = (\bigcup_{i=1}^{7} W_i) \cap \partial B^{**}$ consists of 21 circles as shown in Figure 9, eight circles of which are contained in $Cl(\partial B - \partial B^*)$.

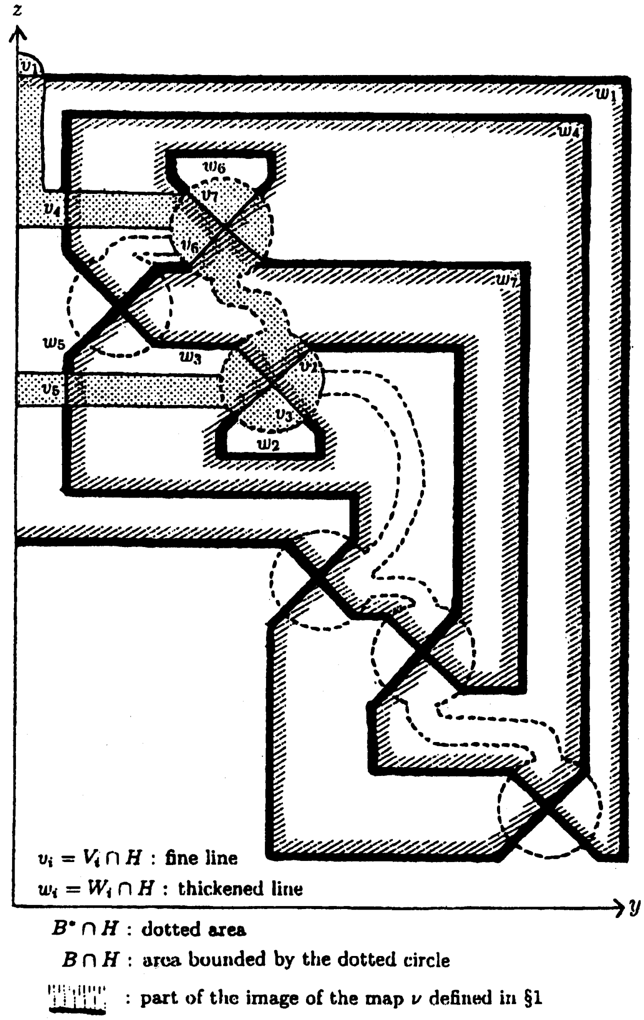


Figure 7

Our problem, how to construct houses, is reduced to the problem, how to find extensions of oriented circles in two planes, which is investigated by many people. We investigate area near a cirlce $\partial B^* \cap H'$, where $H' = \{(\varepsilon, y, z) \mid -1 \leq y \leq 1, 0 \leq z \leq 1\}$ for some small $\varepsilon > 0$. Note that $\partial B^* \cap H' = \partial B^{**} \cap H'$.

Now V_1, V_4, V_6, W_1, W_4, and W_7 are only the inner walls. Observing Figure 9, we have

(1) W_1 never dominates W_2,
(2) W_7 never dominates W_6, and
(3) W_3 and W_5 must be dominated by different walls.

Observing, in Figure 8, the line segment P_1P_2 in $Cl(\partial B^* - F(\partial V_3))$, where $F(\cdots)$ is the one defined in §2, we have

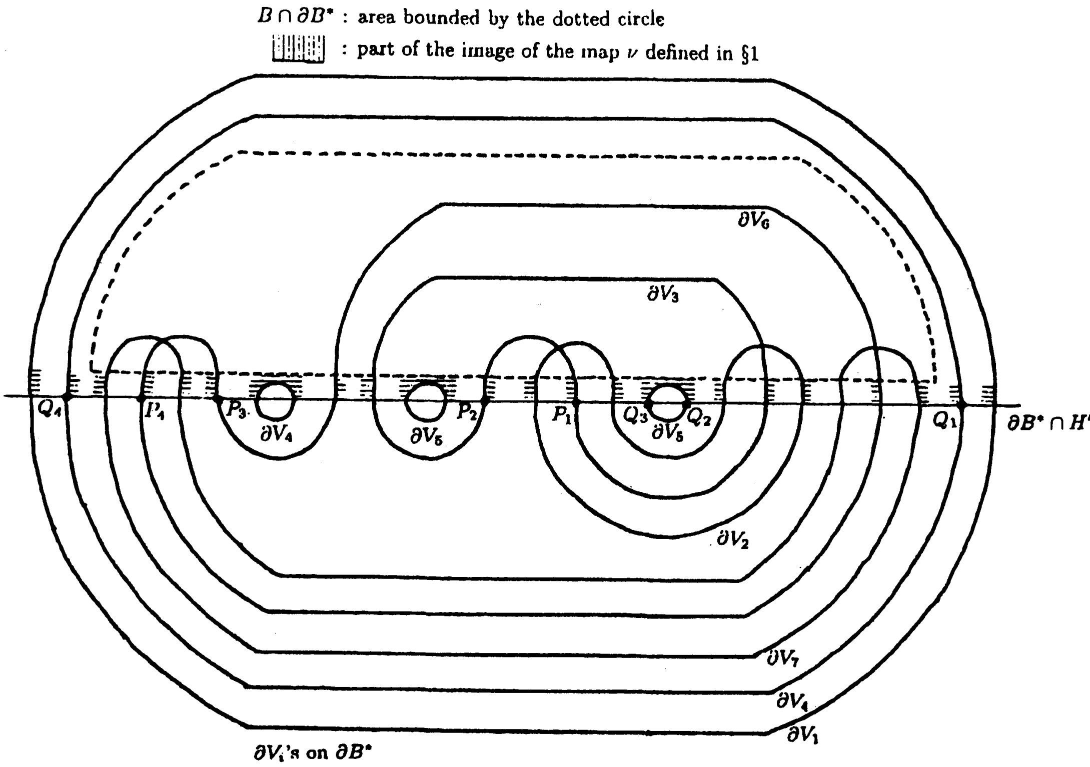


Figure 8

(4) W_2 and W_3 must be dominated by the same wall (see the line segment P_1P_2 in Figur 9). Hence W_1 never dominates W_3 by (1).

Observing, in Figure 8, the line segment P_3P_4 in $F(\partial V_6)$, we have

(5) W_5 and W_6 must be dominated by the same wall (see the line segment P_3P_4 in Figur 9). Hence W_7 never dominates W_5 by (2).

Thus only three cases are possible. Namely,

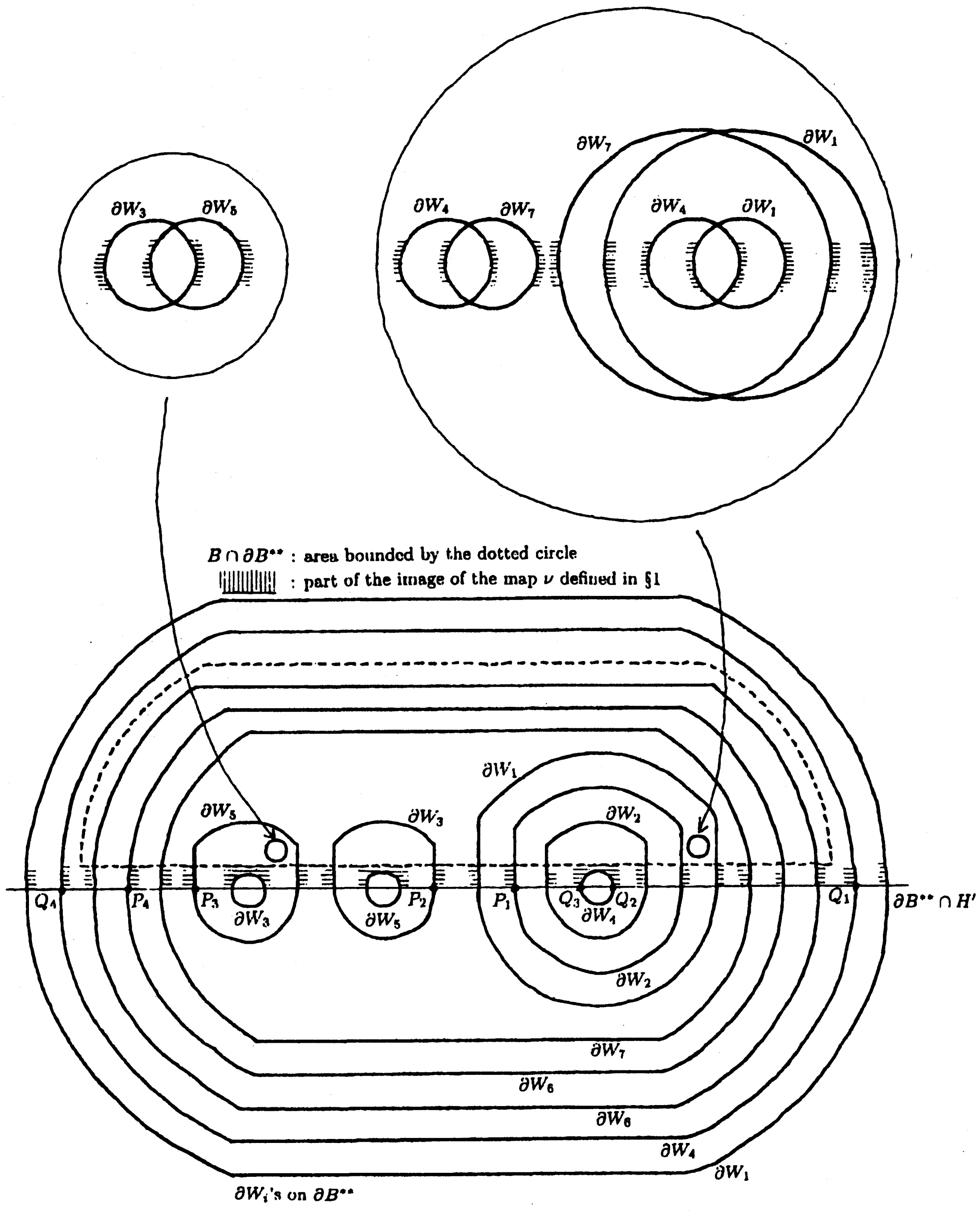
Case I. W_1 dominates W_5 and W_6, W_4 dominates W_2 and W_3, and W_7 dominates none.

Case II. W_1 dominates W_5 and W_6, W_4 dominates none, and W_7 dominates W_2 and W_3.

Case III. W_1 dominates none, W_4 dominates W_5 and W_6, and W_7 dominates W_2 and W_3.

For Case I, we have a house

$$\{\{W_1, W_5, W_6\}, \{W_4, W_2, W_3\}, \{W_7\}\}, \{\{V_1, V_7\}, \{V_4, V_3\}, \{V_6, V_2, V_5\}\}.$$



∂W_3
∂W_5
∂W_4
∂W_7
∂W_7
∂W_1
∂W_4
∂W_1
$B \cap \partial B^{**}$: area bounded by the dotted circle
: part of the image of the map ν defined in §1
∂W_1
∂W_5
∂W_3
∂W_2
Q_4
P_4
P_3
∂W_3
∂W_5
P_2
P_1
Q_3
Q_2
∂W_4
Q_1
$\partial B^{**} \cap H'$
∂W_2
∂W_7
∂W_6
∂W_6
∂W_4
∂W_1
∂W_i's on ∂B^{**}

Figure 9

For Case III, we have a house

$$\{\{W_1\},\{W_4,W_5,W_6\},\{W_7,W_2,W_3\}\},\{\{V_1,V_2\},\{V_4,V_3,V_7\},\{V_6,V_5\}\}.$$

For Case II, since W_4 dominates none, V_4 dominates V_5 by observing the line segment Q_1Q_2 in $F(\partial W_4)$. Since V_4 dominates V_5, we can not get the line segment Q_3Q_4 in $Cl(F(\partial V_4) - F(\partial V_5))$, but we must have the line segment Q_3Q_4 in $F(\partial W_4)$. This is a contradiction.Therefore we have exactly two houses.

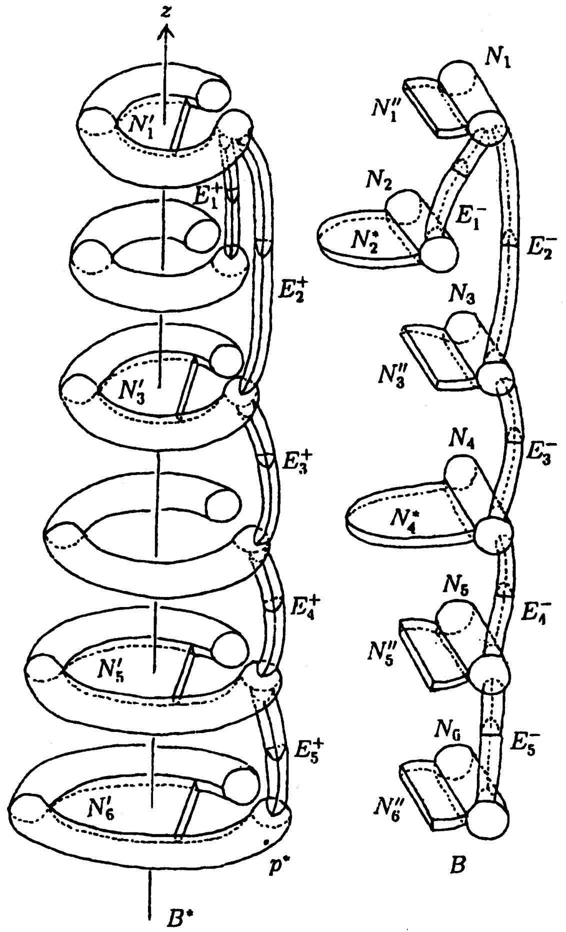


Figure 10

Note: For $i = 1,3,5,6$, split the 3-ball N_i^* into two balls N_i' and N_i'' with $N_i' \cap N_i = \emptyset$ and $N_i'' \cap S_\alpha = \emptyset$. Set

$$B^* = Cl(\bigcup_{i=1}^{6}(T_i - N_i)) \cup (\bigcup_{i=2}^{5} E_j^+) \cup N_1' \cup N_3' \cup N_5' \cup N_6', \text{ and}$$

$$B = Cl(\bigcup_{i=1}^{6} N_i) \cup (\bigcup_{i=2}^{5} E_j^-) \cup N_1'' \cup N_2^* \cup N_3'' \cup N_4^* \cup N_5'' \cup N_6'.$$

The images of B and B^* are shown in Figure 10. Then ∂B^* will be a criterion sphere with the water drop B obtained by the method of the proof of Theorem 1 in the next section.

§ 4. Proof of Theorem 1

Let $f : S \to \mathbb{R}^3$ be the nice immersion. Let $L_1, L_2, \cdots, L_m$ be the closures of the double curves of the map. Hence the union $X = \bigcup_{i=1}^{m} L_i$ is the set of singularities of the map f. Using an isotopy of $\mathbb{R}^3$, for the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$, we may assume that $p(X)$ is a connected graph such that the degree of each vertex is 2, 4 or 6, where all the vertices of degree 6 are the images of triple points of the map f. For each vertex of degree 4, choose mutually disjoint vertical line segments $L_{m+1}, L_{m+2}, \cdots, L_n$ with

$$L_i \cap X = \partial L_i \ \ (i = m+1, m+2, \cdots, n).$$

We may assume that

the set $L_i - \partial L_i$ pierces $f(S)$ $(i = m+1, m+2, \cdots, n)$.

Then $Y = \bigcup_{i=1}^{n} L_i$ is connected. Let N be a regular neighborhood of Y.Since N is ambient isotopic to a regular neighborhood of the planar graph $p(Y) = p(X)$, N is an unkotted handle body. Note that $Cl(f(S) - N)$ consists of mutually disjoint surfaces. Let k be the genus of the surface ∂N. We can find mutually disjoint proper disks $D_1, D_2, \cdots, D_k$ of the handle body N such that

(1) for each $i = 1, 2, \cdots, k$, let N_i be a regular neighborhood of D_i in N, then $Cl(N - \bigcup_{i=1}^{k} N_i)$ is a 3-ball,
(2) for each $i = 1, 2, \cdots, k$, the disk D_i is pierced once by one of $L_1, L_2, \cdots, L_n$, and
(3) for each $i = 1, 2, \cdots, k$, the meet $D_i \cap f(S)$ is empty or the letter $\times$, i.e. the union of two arcs which intersect each other at one point.

Let N^* be the closure of the complementary domain of N in $\mathbb{R}^3$. We can find mutually disjoint proper disks $D_1^*, D_2^*, \cdots, D_k^*$ of N^* such that

(4) $D_i^* \cap D_j = \emptyset$ $(i \neq j)$,
(5) for $i = 1, 2, \cdots, k$, on the surface ∂N, the circle ∂D_i^* transversely intersects the circle ∂D_i by one point.

If there exists a circle in $D_i^* \cap f(S)$, then choose an arc connecting a point in an outermost circle and a point in $\partial D_i^* - N_i$. Let Q be a small regular neighborhood of the union of the arc and the disk bounded by the outermost circle (see Figure 11). Then the union $N \cup Q$ is still an unkotted handle body. We use the same notation N and D_i^* for $N \cup Q$ and $Cl(D_i^* - Q)$ respectively. Repeat this modification of N to $N \cup Q$, then we can assume that

(6) for $i = 1, 2, \cdots, k$, the meet $D_i^* \cap f(S)$ consists of mutually disjoint proper arcs of D_i^* where D_i^* and $f(S) \cap N^*$ transversely intersect.

Then the boundary of $Cl((N - \bigcup_{i=1}^{k} N_i))$, say S^*, is a sphere. For each $i = 1, 2, \cdots, k$, let N_i^* be a regular neighborhood of D_i^* in N^*. Now the union $N_i \cup N_i^*$ is a 3-ball such that

(1) $(N_i \cup N_i^*) \cap S^*$ is a disk on $\partial(N_i \cup N_i^*)$, and
(2) the meet $(N_i \cup N_i^*) \cap f(S)$ consists of admissible disks in $(N_i \cup N_i^*)$.

Connecting the balls $(N_1 \cup N_1^*), (N_2 \cup N_2^*), \cdots$, and $(N_k \cup N_k^*)$ to get a waterdrop (see Figure 11). This completes the proof of Theorem 1.

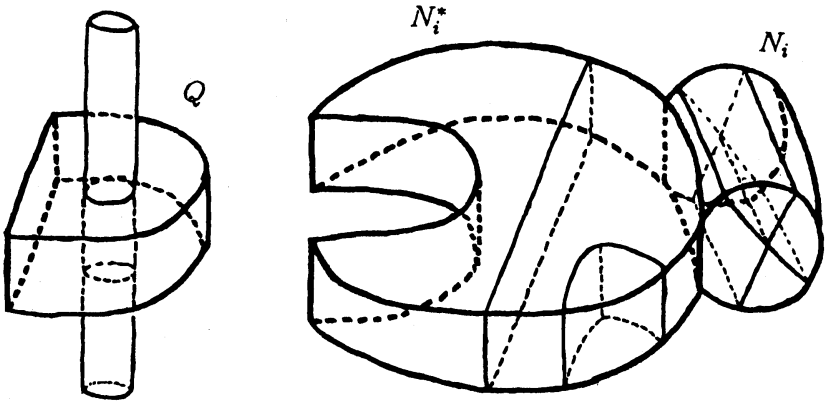


Figure 11

§ 5. Proof of Theorem 2

First we need the following two key lemmata.

Lemma 1. *Let B be a 3-ball. Let $g : M \to B$ be an immersion of an orientable compact connected 3-manifold M into B such that $g^{-1}(\partial B)$ is a surface on ∂M and that the map g embeds the closure of each component of $\partial M - g^{-1}(\partial B)$. Then the map g is an embedding.*

Proof. Let $F_1, F_2, \cdots, F_m$ be the closures of the components of $\partial M - g^{-1}(\partial B)$. For each $i = 1, 2, \cdots, m$, the image $g(F_i)$ splits the 3-ball B into two domains. Let G_i be the closure of one of the two domains such that any regular neighborhood of $g(F_i)$ in G_i does not contain the image of a regular neighborhood of F_i in M. Let $\widetilde{G_1}, \widetilde{G_2}, \cdots, \widetilde{G_m}$ be mutually disjoint copies of $G_1, G_2, \cdots, G_m$. Using the map g, glue $\widetilde{G_1}, \widetilde{G_2}, \cdots, \widetilde{G_m}$ to M. Let $\widetilde{M}$ be the resulting 3-manifold. Then the map g extends to a covering map $\tilde{g} : \widetilde{M} \to B$ with $\tilde{g}(\widetilde{G_i}) = G_i$ $(i = 1, 2, \cdots, m)$. Since B is a 3-ball, the map $\tilde{g}$ must be a homeomorphism. Therefore the restriction $g = \tilde{g}|M$ is an embedding. ■

Lemma 2. *Let $h : G \to B$ be an immersion of a closed connected surface G into a 3-ball B such that*

(1) *the closures of the complementary domains of $h^{-1}(\partial B)$ in G are mutually disjoint disks, say $D_1, D_2, \cdots, D_n$, and*
(2) *the images $h(D_1), h(D_2), \cdots, h(D_n)$ are admissible disks in B.*

Then G is a sphere and the map h is an embedding.

Proof. Let $G' = h^{-1}(\partial B)$. For each $i = 1, 2, \cdots, n$, the image $h(\partial D_i)$ splits the sphere ∂B into two disks. Let D'_i be the one of the two disks which does not contain the image of any boundary collar of ∂D_i in G' by h. Let $h' : G \to \partial B$ be an immersion with $h'|G' = h$ and $h'(D_i) = D'_i$ $(i = 1, 2, \cdots, n)$. Then h' is a covering map. Hence h' is a homeomorphism. Thus G is a sphere. The map h embeds G' into ∂B. This means that $h(\partial G') = \bigcup_{i=1}^{n} h(\partial D_i)$ consists of mutually disjoint circles. Hence $h(D_i) \cap h(D_j) = \emptyset$ $(i \neq j)$ by Condition (2). Therefore the map h is an embedding. ■

Let $f : S \to \mathbb{R}^3$ be the nice immersion possessing a criterion sphere S^* with a waterdrop B. Let S^{**}, B^*, B^{**}, D^*, and the star p^* as ones in §2. Let $\mathfrak{E}$ be the set of the equivalent extension classes of the map f and $\mathfrak{H}$ be the set of houses on the criterion sphere with the waterdrop. By four steps, we shall prove Theorem 2 by constructing two maps $\mathfrak{h} : \mathfrak{E} \to \mathfrak{H}$ and $\mathfrak{g} : \mathfrak{H} \to \mathfrak{E}$ with $\mathfrak{h} \circ \mathfrak{g} = id$ and $\mathfrak{g} \circ \mathfrak{h} = id$, where id is the identity map.

In the following two steps, we define the map $\mathfrak{h} : \mathfrak{E} \to \mathfrak{H}$ as follows.
Step 1. Suppose that the nice immersion f extends to an immersion $g : M \to \mathbb{R}^3$ of an orientable compact connected 3-manifold M into $\mathbb{R}^3$. Let $[g]$ be the equivalent extension class of g. To make argument simple, we assume that $\partial M = S$ and $g|\partial M = f$. We shall construct a house $\mathfrak{H}(g)$ with a star stair $\mathfrak{P}^*(g)$ and a double-stars stair $\mathfrak{P}^{**}(g)$. In this step, we shall construct $\mathfrak{P}^*(g)$.

Let $M_1^*, M_2^*, \cdots, M_m^*$ be the components of $g^{-1}(B^*)$. Then the restriction $g_i = g|M_i^* : M_i^* \to B^*$ is an embedding by Lemma 1. Let

$$\mathfrak{R}(M_i^*) = \text{the set of the walls on } g(M_i^*).$$

Recall the immersion $\nu : S \times [0,1] \to \mathbb{R}^3$ whose image does not meet the unbounded complementary domain of $f(S)$ in $\mathbb{R}^3$. Since $g^{-1}(\nu(S \times [0,1]))$ contains a boundary collar of the 3-manifold M, it can be easily checked that $\mathfrak{R}(M_i^*)$ is a room. For $i = 1, 2, \cdots, m$, let

$$\mathfrak{F}(M_i^*) = \text{the set of the components of } g(M_i^*) \cap S^*.$$

And set

$$\mathfrak{P}^*(g) = \{\mathfrak{R}(M_1^*), \mathfrak{R}(M_2^*), \cdots, \mathfrak{R}(M_m^*)\}.$$

Then $\mathfrak{P}^*(g)$ is a star stair and $\bigcup_{i=1}^{m} \mathfrak{F}(M_i^*)$ is the floor diagram of $\mathfrak{P}^*(g)$.

<u>Step 2</u>. In this step, we shall construct $\mathfrak{P}^{**}(g)$.

Let E be a 3-ball whose interior contains the set $g(M) \cup B \cup B^*$. Tunnel the 3-manifold $B^{**} \cap E$ from the p^* to a point of ∂E to get a 3-ball B' which contain the set $g(M) \cap B^{**}$. Use B' instead of B^* in the step 1 to get a double-stars stair. Let $M_1^{**}, M_2^{**}, \cdots, M_n^{**}$ be the components of $g^{-1}(B')$. Set

$$\mathfrak{R}(M_j^{**}) = \text{the set of the walls on } g(M_j^{**}),$$
$$\mathfrak{F}(M_j^{**}) = \text{the set of the components of } g(M_j^{**}) \cap S^{**}, \text{ and}$$
$$\mathfrak{P}^{**}(g) = \{\mathfrak{R}(M_1^{**}), \mathfrak{R}(M_2^{**}), \cdots, \mathfrak{R}(M_n^{**})\}.$$

Then $\mathfrak{P}^{**}(g)$ is a double-stars stair and $\bigcup_{j=1}^{n} \mathfrak{F}(M_j^{**})$ is the floor diagram of $\mathfrak{P}^{**}(g)$. Therefore the star stair $\mathfrak{P}^*(g)$ and the double-stars stair $\mathfrak{P}^{**}(g)$ make a house $\mathfrak{H}(g)$. The house $\mathfrak{H}(g)$ is uniquely determined by the equivalent extension class $[g]$. Set $\mathfrak{h}([g]) = \mathfrak{H}(g)$.

<u>Step 3</u>. In this step, we shall define the map $\mathfrak{g} : \mathfrak{H} \to \mathfrak{E}$.

Let $\mathcal{H}$ be a house with a star stair $\mathcal{P}^* = \{\mathcal{R}_1^*, \mathcal{R}_2^*, \cdots, \mathcal{R}_m^*\}$ and a double-stars stair $\mathcal{P}^{**} = \{\mathcal{R}_1^{**}, \mathcal{R}_2^{**}, \cdots, \mathcal{R}_n^{**}\}$. For each $i = 1, 2, \cdots, m$, let

$$W_i = \text{the inner wall of the room } \mathcal{R}_i^*, \text{ and}$$
$$M(\mathcal{R}_i^*) = Cl(In(W_i) - \bigcup_{W' \in \mathcal{R}_i^* - W_i} In(W')),$$

where $In(\cdots)$ is the one defined in §1 for walls.

Then it is easy to show that

$$\mathfrak{R}(M(\mathcal{R}_i^*)) = \mathcal{R}_i^*, \text{ where } \mathfrak{R}(\cdots) \text{ is the one defined in Step 1 },$$

$\mathfrak{F}(M(\mathcal{R}_i^*)) = \mathcal{F}(\mathcal{S}(\mathcal{R}_i^*))$,
where $\mathfrak{F}(\cdots)$ is the one defined in Step 1, and
$\mathcal{F}(\mathcal{S}(\cdots))$ is the one difined in Observation 1 in §2.

Similarly we have orientable compact connected 3-maifolds $M(\mathcal{R}_1^{**}), M(\mathcal{R}_2^{**}), \cdots, M(\mathcal{R}_n^{**})$ with $\mathfrak{R}(M(\mathcal{R}_j^{**})) = \mathcal{R}_j^{**}$, and $\mathfrak{F}(M(\mathcal{R}_j^{**})) = \mathcal{F}(\mathcal{S}(\mathcal{R}_j^{**}))$ $(j = 1, 2, \cdots, n)$.
Let

$$\mathcal{F} = \bigcup_{i=1}^{m} \mathfrak{F}(M(\mathcal{R}_i^*)) \cup (\bigcup_{j=1}^{n} \mathfrak{F}(M(\mathcal{R}_j^{**}))), \text{ and}$$

$$\mathcal{W} = \{Q | \text{ for some } F \in \mathcal{F}, \ \ Q \text{ is a component of } F \cap \partial B\}.$$

Recall the circle $C = \partial D^*$ on the waterdrop B. For each configuration Q in $\mathcal{W}$, let

$$\mathcal{E}(Q) = \text{the set of components of } \partial Q \cap C.$$

Let $\sim$ be a relation on the set $\mathcal{W}$ defined by

$Q \sim Q'$ if and only if there exists a sequence of configurations $Q_1, Q_2, \cdots, Q_q$ in $\mathcal{W}$ with $Q_1 = Q$, $Q_q = Q'$ and $\mathcal{E}(Q_i) \cap \mathcal{E}(Q_{i+1}) \neq \emptyset$ $(i = 1, 2, \cdots, q-1)$.

Then the relation $\sim$ is an equivalence relation. Let $\mathcal{W}_1, \mathcal{W}_2, \cdots, \mathcal{W}_r$ be the equivalence classes of $\mathcal{W}$. For $k = 1, 2, \cdots, r$, the union $F_k = \bigcup_{Q \in \mathcal{W}_k} Q$ is an immersed connected surface by the condition that the two tight floor diagrams coincide. Now $f(S) \cap B$ consists of admissible disks in B. Glue some of the disks to the surface F_k so that we get an immersed closed connected surface G_k. Note that any two surfaces G_i and G_j never have common disks of $f(S) \cap B$. Since $f(S) \cap B$ consists of admissible disks in B, each surface G_k is a sphere by Lemma 2. Let B_k be the 3-ball bounded by the sphere G_k. Let $M_1^*, M_2^*, \cdots, M_m^*, M_1^{**}, M_2^{**}, \cdots, M_n^{**}$, and $B_1^*, B_2^*, \cdots, B_r^*$ be mutually disjoint copies of $M(\mathcal{R}_1^*), M(\mathcal{R}_2^*), \cdots, M(\mathcal{R}_m^*), M(\mathcal{R}_1^{**}), M(\mathcal{R}_2^{**}), \cdots, M(\mathcal{R}_n^{**})$, and $B_1, B_2, \cdots, B_r$ respectively. Using floor diagrams, glue together the copies to get an orientable compact connected 3-manifold $M_{\mathcal{H}}$ and an immersion $g_{\mathcal{H}} : M_{\mathcal{H}} \to \mathbb{R}^3$ with $g_{\mathcal{H}}(M_i^*) = M(\mathcal{R}_i^*)$, $g_{\mathcal{H}}(M_j^{**}) = M(\mathcal{R}_j^{**})$, and $g_{\mathcal{H}}(B_k^*) = B_k$. Then the map $g_{\mathcal{H}}$ is an extension of the map f. Set $\mathfrak{g}(\mathcal{H}) = [g_{\mathcal{H}}]$.

Step 4. In the sence of Step 1 and Step 2, we have

$\mathfrak{P}^*(g_{\mathcal{H}}) = \mathcal{P}^*$, and
$\mathfrak{P}^{**}(g_{\mathcal{H}}) = \mathcal{P}^{**}$.

Hence $\mathfrak{h} \circ \mathfrak{g}(\mathcal{H}) = \mathcal{H}$. Also the two extentions $g_{\mathfrak{H}(g)}$ and g are equivalent. Hence $\mathfrak{g} \circ \mathfrak{h}([g]) = [g]$. Therefore $\mathfrak{h}$ is bijective. This completes the proof of Theorem 2.

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