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MARCINKIEWICZ INTEGRALS WITH ROUGH KERNELS ON PRODUCT SPACES

By

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Abstract. Suppose that $\Omega(x',y')\in L^1(S^{n-1}\times S^{m-1})$ is a homogeneous function of degree zero satisfying the mean zero property (1.1), and that $h(s,t)$ is a bounded function on $\mathbb{R}\times\mathbb{R}$. The Marcinkiewicz integral operator $\mu_\Omega(f)$ along a continuous surface $\gamma(u,v)$ on the product space $\mathbb{R}^n\times\mathbb{R}^m$ $(n\geq 2,\ m\geq 2)$ is defined by

$$\mu_\Omega f(\xi,\eta,z)=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|F_{t,s}(x,y,z)|^2 2^{-2t-2s}dtds\right)^{1/2}$$

where $F_{t,s}(\xi,\eta,z)$

$$=\int_{\substack{|x|<2^t\\ |y|<2^s}}h(|x|,|y|)|x|^{-n+1}|y|^{-m+1}\Omega(x',y')f(\xi-x,\eta-y,z-\gamma(|x|,|y|))dxdy.$$

We prove that the operator $\mu_\Omega f$ is bounded on $L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})$, $p\in(1,\infty)$, provided that Ω is a function in certain block space $B_q^{0,1}(S^{n-1}\times S^{m-1})$ for some $q>1$ and that two lower dimensional maximal functions related to γ are bounded on L^p. These two lower dimensional maximal functions are natural extension of a well-known maximal function along curves.

1. Introduction

Let $\mathbb{R}^N$ $(N=n$ or $m)$, $N\geq 2$, be the N-dimensional Euclidean space and S^{N-1} be the unit sphere in $\mathbb{R}^N$ equipped with normalized Lebesgue measure $d\sigma=d\sigma(\cdot)$. For nonzero points $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$, we define $x'=x/|x|$ and $y'=y/|y|$. For $n\geq 2$, $m\geq 2$, let $\Omega(x',y')\in L^1(S^{n-1}\times S^{m-1})$ be a homogeneous function of degree zero, and satisfy

$$\int_{S^{n-1}}\Omega(x',y')d\sigma(x')=\int_{S^{m-1}}\Omega(x',y')d\sigma(y')=0. \tag{1.1}$$

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Let $h(s,t)$ be a locally integrable function on $\mathbb{R}\times\mathbb{R}$. The Marcinkiewicz integral operator $T_\Omega f$ on the product space $\mathbb{R}^n\times\mathbb{R}^m$ is defined by

$$T_\Omega f(x,y)=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\beta_{t,s}(x,y)|^2 2^{-2t-2s}dtds\right)^{1/2}, \tag{1.2}$$

for all test functions $f\in S(\mathbb{R}^n\times\mathbb{R}^m)$, where $\beta_{t,s}(\xi,\eta)$

$$=\int_{|x|<2^t}\int_{|y|<2^s}h(|x|,|y|)\Omega(x',y')|x|^{-n+1}|y|^{-m+1}f(\xi-x,\eta-y)dxdy.$$

In the one parameter case, if $h=1$ and Ω satisfies some regularity conditions, then it is known that the operator T_Ω is bounded on $L^p(\mathbb{R}^n)$, $1<p<\infty$ (see [13]). In order to weaken the regularity condition on Ω, the following two theorems were proved.

THEOREM A (see [2]). *Suppose $n\geq 2$, $m\geq 2$ and that Ω is a homogeneous function of degree zero and satisfies* (1.1). *If h is a bounded function, then the operator T_Ω is bounded on $L^2(\mathbb{R}^n\times\mathbb{R}^m)$ provided $\Omega\in L(\operatorname{Log}^+L)^2(S^{n-1}\times S^{m-1})$.*

THEOREM B (see [1]). *Suppose $n\geq 2$, $m\geq 2$ and that Ω is a homogeneous function of degree zero and satisfy* (1.1). *If h is a bounded function, then the operator T_Ω is bounded on $L^p(\mathbb{R}^n\times\mathbb{R}^m)$, $1<p<\infty$, provided $\Omega\in L^q(S^{n-1}\times S^{m-1})$ for some $q>1$.*

On the other hand, recently Jiang and Lu [5] introduced the block spaces $B_q^{0,1}$, $q>1$ on $S^{n-1}\times S^{m-1}$ (see Section 2 for the definition). It was proved by Keitoku and Sato [6] that for any fixed $q>1$, $B_q^{0,1}(S^{n-1})\supset L^r(S^{n-1})$ for all $r>1$ and the inclusion is proper, although, so far, it is still not clear about the relationship between the spaces $B_q^{0,1}$ and $L\operatorname{Log}^+L$ on the sphere.

In this paper, we will study the operator T_Ω along a continuous surface. Precisely, for $(x,y,z)\in\mathbb{R}^{n+m+1}$ we define

$$\nu_\Omega f(x,y,z)=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|F_{t,s}(x,y,z)|^2 2^{-2t-2s}dtds\right)^{1/2}$$

where $F_{t,s}(\xi,\eta,z)$

$$=\int_{\substack{|x|<2^t\\|y|<2^s}}h(|x|,|y|)|x|^{-n+1}|y|^{-m+1}\Omega(x',y')f(\xi-x,\eta-y,z-\gamma(|x|,|y|))dxdy.$$

Let $t,s,z\in\mathbb{R}$, we define the following two maximal functions

$$M_\gamma^1 h(t,s,z)=\sup_{R>0,S>0}R^{-1}S^{-1}\int_0^R\int_0^S|h(t-u,s-v,z-\gamma(u,v))|dudv,$$

$$M_\gamma^2 g(t,z) = \sup_{R>0,S>0} R^{-1}S^{-1}\int_0^R\int_0^S |g(t-u, z-\gamma(u,v))|dudv.$$

The main purpose of this paper is to prove the following theorem.

THEOREM 1. *Suppose that Ω is a homogeneous function of degree zero satisfying (1.1), and that h is a bounded function. Then ν_Ω is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$, provided that $\Omega \in B_q^{0,1}(S^{n-1} \times S^{m-1})$ for some $q > 1$, and for $r \in (1, \infty)$*

(i) $\|M_\gamma^1 h\|_{L^r(\mathbb{R}^3)} \leq C\|h\|_{L^r(\mathbb{R}^3)}$,
(ii) $\|M_\gamma^2 g\|_{L^r(\mathbb{R}^2)} \leq C\|g\|_{L^r(\mathbb{R}^2)}$.

We remark that the above two maximal functions are natural extensions of the maximal functions

$$M_\Gamma h(s,t) = \sup_{R>0} R^{-1}\int_0^R |h(s-u, t-\Gamma(u))|du$$

and

$$\mu_\Gamma g(t) = \sup_{R>0} R^{-1}\int_0^R |g(t-\Gamma(u))|du.$$

The maximal functions M_Γ and μ_Γ play an important role in harmonic analysis and they are extensively studied by many authours. See [14] for the results through 1993.

The surfaces γ satisfying (i) and (ii) are easily available. For example, $\gamma(s,t) = s^\alpha t^\beta$ with $\alpha > 0$ and $\beta > 0$ (see Corollary 3 in [3]). However, it will be more interesting to investigate more general curvature conditions on γ to assert the L^p boundedness of M_γ^1 and M_γ^2, similar to those for M_Γ and μ_Γ.

By Theorem 1 we can also obtain an improvement of Theorem B.

THEOREM 2. *Suppose that Ω and h satisfy the conditions in Theorem 1, then the operator T_Ω is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.*

This paper is organized as follows. In the second section we will review the difinition of the block spaces. After proving the L^p boundedness property for certain maximal functions in Section 3 and obtaining some L^2 estimates in Section 4, we will prove the theorems in Section 5. Our proofs clearly also work for the one parameter case. Thus, even in the one parameter case, our Theorem 2 presents an improvement over the classical result by Stein.

Throughout this paper, we always use letter C to denote positive constants that may vary at each occurrence but is independent of the essential variables.

2. Block Spaces

First we review the definition of the block spaces.

A *q-block* on $S^{n-1} \times S^{m-1}$ is an L^q $(1 < q \le \infty)$ function $b(\cdot,\cdot)$ that satisfies the following conditions (a) and (b).

(a) $\text{supp}(b) \subseteq Q$ where Q is an interval on $S^{n-1} \times S^{m-1}$. Precisely,

$$\begin{aligned} Q &= Q_1(\xi',\alpha) \times Q_2(\eta',\beta), \text{ where} \\ Q_1(\xi',\alpha) &= \{x' \in S^{n-1} : |x'-\xi'| < \alpha \text{ for } \xi' \in S^{n-1} \text{ and } \alpha \in (0,1]\}, \\ Q_2(\eta',\beta) &= \{y' \in S^{m-1} : |y'-\eta'| < \beta \text{ for } \eta' \in S^{m-1} \text{ and } \beta \in (0,1]\}. \end{aligned}$$

(b) $\|b\|_q \le |Q|^{(1/q-1)}$, where $|Q|$ is the volume of Q.

The block spaces $B_q^{0,1}$ on $S^{n-1} \times S^{m-1}$ are defined by

$$B_q^{0,1} = \Big\{\Omega \in L^1(S^{n-1} \times S^{m-1}) : \Omega(x',y') = \sum_\mu C_\mu b_\mu(x',y'), \text{ where each } b_\mu \\ \text{is a } q\text{-block supported in an inverval } Q^\mu, \text{ and } M_q^{0,1}(\{C_\mu\}) < \infty\Big\}$$

where

$$M_q^{0,1}(\{C_\mu\}) = \sum_\mu |C_\mu|\{1 + (\log^+ 1/|Q^\mu|)^2\}. \tag{2. 1}$$

The "norm" $M_q^{0,1}(\Omega)$ of $\Omega \in B_q^{0,1}$ is defined by $M_q^{0,1}(\Omega) = \inf\{M_q^{0,1}(\{C_\mu\})\}$ where the infimum is taken over all q-block decompositions of Ω.

The block spaces were invented by M.H. Taibleson and G. Weiss in the study of the convergence of the Fourier series (see [12]). Later on, these spaces and their applications were studied by many authors [7], [9], [10], [11], etc. For further information, readers may see the book [8]. In particular, it was noted by Keitoku and Sato that, for any fixed $q > 1$, $\bigcup_{r>1} L^r(S^{n-1}) \subsetneq B_q^{0,1}(S^{n-1})$, and the inclusion is proper (see [6]).

3. Certain Maximal Functions

Let the functions h and $\Omega = \sum C_\mu b_\mu$ be as in Theorem 1. Let $B_{t,s} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : |x| < 2^t,\ |y| < 2^s\}$ and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. We define the measure $\sigma_{\Omega,t,s}$ by $\sigma_{\Omega,t,s} * f(x,y,z) = 2^{-t-s}F_{t,s}(x,y,z)$. Then it is easy to see that its Fourier transform is

$$\begin{aligned} &\hat{\sigma}_{\Omega,t,s}(\xi,\eta,z) \\ &= \int_{B_{t,s}} 2^{-t-s} h(|x|,|y|)|x|^{-n+1}|y|^{-m+1}\Omega(x',y')e^{-i\{\langle\xi,x\rangle + \langle\eta,y\rangle + \gamma(|x|,|y|)z\}}dxdy. \end{aligned}$$

Similarly, we define the measures $|\sigma_{b_\mu,t,s}|$ and $|\sigma_{\Omega,t,s}|$ by letting their Fourier transforms be

$$
\begin{aligned}
&\widehat{|\sigma_{b_\mu,t,s}|}(\xi,\eta,z)\\
&=\int_{B_{t,s}} 2^{-s-t}|x|^{-n+1}|y|^{-m+1}|b_\mu(x',y')|e^{-i\{\langle x,\xi\rangle+\langle y,\eta\rangle+\gamma(|x|,|y|)z\}}dxdy;\\
&\widehat{|\sigma_{\Omega,t,s}|}(\xi,\eta,z)\\
&=\int_{B_{t,s}} 2^{-s-t}|x|^{-n+1}|y|^{-m+1}|\Omega(x',y')|e^{-i\{\langle x,\xi\rangle+\langle y,\eta\rangle+\gamma(|x|,|y|)z\}}dxdy.
\end{aligned}
$$

Then we define the maximal functions $\sigma^*_{b_\mu}f$ and $\sigma^*_\Omega f$ by

$$
\begin{aligned}
\sigma^*_{b_\mu}f(x,y,z)&=\sup_{t,s\in\mathbb{R}^2}\Big|\ |\sigma_{b_\mu,t,s}|*f(x,y,z)\ \Big|;\\
\sigma^*_{\Omega}f(x,y,z)&=\sup_{t,s\in\mathbb{R}^2}\Big|\ |\sigma_{\Omega,t,s}|*f(x,y,z)\ \Big|.
\end{aligned}
$$

It is easy to see that the total variations of $\sigma_{\Omega,t,s}$ and $\sigma_{b_\mu,t,s}$ satisfy

$$
\begin{aligned}
&\|\ |\sigma_{\Omega,t,s}|\ \|_1=\int_{B_{t,s}}2^{-s-t}|h(|x|,|y|)\Omega(x',y')|\ |x|^{-n+1}|y|^{-m+1}dxdy\le C,\\
&\|\ |\sigma_{b_\mu,t,s}|\ \|_1\le C
\end{aligned}
\tag{3. 1}
$$

uniformly for t,s and b_μ, and both $|\sigma_{\Omega,t,s}|$ and $|\sigma_{b_\mu,t,s}|$ are positive.

Proposition 3.1. *If the surface γ satisfies (i) in Theorem 1, then both σ^*_b and σ^*_Ω are bounded on $L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})$ and the bound of σ^*_b is independent of the block $b(\cdot,\cdot)$.*

Proof. Since $\sigma^*_\Omega f\le\sum|C_\mu|\sigma^*_{b_\mu}f$, clearly we only need to prove the L^p boundedness of σ^*_b. By the definition, for any $f(x)\ge 0$, $\sigma^*_b f(x,y,z)$ is equal to

$$
\begin{aligned}
&\sup_{\substack{t>0\\s>0}} s^{-1}t^{-1}\int_0^s\int_0^t\int_{S^{n-1}\times S^{m-1}}|b(\xi',\eta')|f(x-u\xi',y-v\eta',z-\gamma(u,v))d\sigma(\xi')d\sigma(\eta')dudv\\
&\quad\le C\int_{S^{n-1}\times S^{m-1}}|b(\xi',\eta')|M_{\xi',\eta'}f(x,y,z)d\sigma(\xi')d\sigma(\eta')
\end{aligned}
$$

where

$$
M_{\xi',\eta'}f(x,y,z)=\sup_{s>0,t>0}\int_0^s\int_0^t f(x-u\xi',y-v\eta',z-\gamma(u,v))dudv
$$

is the Hardy-Littlewood maximal function in the space $\mathbb{R} \times \mathbb{R}$ along γ in the direction (ξ', η'). Thus

$$\|\sigma_b^* f\|_p \leq \int_{S^{n-1}\times S^{m-1}} |b(\xi', \eta')| \, \|M_{\xi',\eta'} f\|_p d\sigma(\xi') d\sigma(\eta').$$

Let $1 = (1, 0, 0, \ldots, 0) \in S^{n-1}$ and $\tilde{1} = (1, 0, 0, \ldots, 0) \in S^{m-1}$. For each fixed (ξ', η') choose a rotation $\rho = \rho_1 \otimes \rho_2$ such that $\rho_1 \xi = 1$ and $\rho_2 \eta = \tilde{1}$. Let $\rho^{-1} = \rho_1^{-1} \otimes \rho_2^{-1}$ be the inverse of ρ. We define the function f_ρ by $f_\rho(x, y, z) = f(\rho_1 x, \rho_2 y, z)$. So

$$f(x - u\xi', y - v\eta', z - \gamma(u, v)) = f_{\rho^{-1}}(\rho_1 x - u1, \rho_2 y - v\tilde{1}, z - \gamma(u, v)).$$

By this fact, Condition (i) in Theorem 1, and changing variables, it is easy to see that

$$\|M_{\xi',\eta'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})}$$

where C is independent of (ξ', η'). The proposition is proved.

We also need to study two more maximal functions. For any block function b, we define $A_{b,t,s}$ by $A_{b,t,s} * f(\xi, \eta, z) =$

$$2^{-t-s} \int_{B_{t,s}} h(|x|, |y|) b(x', y') |x|^{-n+1} |y|^{-m+1} f(\xi - x, \eta, z - \gamma(|x|, |y|)) dx dy.$$

Then it is easy to see that the Fourier transform of $A_{b,t,s}$ is

$$\begin{aligned} &\hat{A}_{b,t,s}(\xi, \eta, z) \\ &= 2^{-s-t} \int_{B_{t,s}} h(|x|, |y|) |x|^{-n+1} |y|^{-m+1} b(x', y') e^{-i\{\langle x, \xi\rangle + \gamma(|x|,|y|) z\}} dx dy. \end{aligned}$$

Similarly we define $\Lambda_{b,t,s'}$, $\tau_{b,t,s}$ and $\Sigma_{b,t,s}$ by

$$\begin{aligned} &\hat{\Lambda}_{t,s}(\xi, \eta, z) \\ &= 2^{-s-t} \int_{B_{t,s}} h(|x|, |y|) |x|^{-n+1} |y|^{-m+1} b(x', y') e^{-i\{\langle y, \eta\rangle + z\gamma(|x|,|y|)\}} dx dy, \end{aligned}$$

and

$$\hat{\tau}_{b,t,s}(\xi, \eta, z) = \hat{\sigma}_{b,t,s}(\xi, \eta, z) - \hat{A}_{b,t,s}(\xi, \eta, z)$$

$$\hat{\Sigma}_{b,t,s}(\xi, \eta, z) = \hat{\sigma}_{b,t,s}(\xi, \eta, z) - \hat{\Lambda}_{b,t,s}(\xi, \eta, z).$$

Then for any non-negative function f

$$|A_{b,t,s} * f(\xi, \eta, z)| \leq C|D_{b,t,s} * f(\xi, \eta, z)| \tag{3.2}$$

$$|\Lambda_{b,t,s} * f(\xi,\eta,z)| \le |G_{b,t,s} * f(\xi,\eta,z)| \tag{3.3}$$

where both $D_{b,t,s}$ and $G_{b,t,s}$ are positive and

$$\begin{aligned} &D_{b,t,s} * f(\xi,\eta,z)\\ &\quad = 2^{-t-s}\int_{B_{t,s}} |x|^{-n+1}|y|^{-m+1}|b(x',y')f(\xi - x,\eta,z-\gamma(|x|,|y|))|dxdy,\\ &G_{b,t,s} * f(\xi,\eta,z)\\ &\quad = 2^{-t-s}\int_{B_{t,s}} |x|^{-n+1}|y|^{-m+1}|b(x',y')|f(\xi,\eta - y,z-\gamma(|x|,|y|))dxdy. \end{aligned}$$

PROPOSITION 3.2. *Let*

$$G_b^* f = \sup_{(t,s)\in\mathbb{R}^2} |G_{b,t,s} * f|, \qquad D_b^* f = \sup_{(t,s)\in\mathbb{R}^2} |D_{b,t,s} * f|.$$

If the surface γ satisfies (ii) *in Theorem 1, then both G_b^* and D_b^* are L^p bounded.*

Proof. The proof for the proposition is exactly the same as that in proving Proposition 1, but using (ii) instead of (i) in Theorme 1. We omit the detail.

It is easy to see that $\tau_{b,t,s}$ and $\Sigma_{b,t,s}$ are bounded by positive measures. More precisely, for any non-negative function f

$$|\tau_{b,t,s} * f| \le \{|\sigma_{b,t,s}| + D_{b,t,s}\} * f \tag{3.4}$$

$$|\Sigma_{b,t,s} * f| \le \{|\sigma_{b,t,s}| + G_{b,t,s}\} * f. \tag{3.5}$$

Thus by Propositions 3.1 and 3.2, we have

$$\| \sup_{(t,s)\in\mathbb{R}^2} |\tau_{b,t,s} * f| \,\|_p \le C\|\sigma_b^* f\|_p + \|D_b^* f\|_p \le C\|f\|_p; \tag{3.6}$$

$$\| \sup_{(t,s)\in\mathbb{R}^2} |\Sigma_{b,t,s} * f| \,\|_p \le C\|\sigma_b^* f\|_p + \|G_b^* f\|_p \le C\|f\|_p; \tag{3.7}$$

where C is independent of the block $b(\cdot,\cdot)$.

4. L^2 Estimates

The main purpose of this section is to obtain the following lemma.

LEMMA 4.1. *Let $\Omega = \sum C_\mu b_\mu$ be a block function in Theorem 1, where each $b = b_\mu$ is a q-block with* supp(b) $\subseteq Q$. *Let q' be the conjugate exponent to q.*

Then

(i) $|\hat{\sigma}_{\Omega,t,s}(\xi,\eta,z)| \le C|2^t\xi|\ |2^s\eta|;$

(ii) $|\hat{\tau}_{b,t,s}(\xi,\eta,z)| \le C|2^t\xi|^{1/\log|Q|}\ |2^s\eta|$ *if* $|Q| < e^{q/1-q};$

(iii) $|\hat{\tau}_{b,t,s}(\xi,\eta,z)| \le C|2^t\xi|^{1/q'}\ |2^s\eta|$ *if* $|Q| \ge e^{q/1-q};$

(iv) $|\hat{\Sigma}_{b,t,s}(\xi,\eta,z)| \le C|2^t\xi|\ |2^s\eta|^{1/\log|Q|}$ *if* $|Q| < e^{q/1-q};$

(v) $|\hat{\Sigma}_{b,t,s}(\xi,\eta,z)| \le C|2^t\xi|\ |2^s\eta|^{1/q'}$ *if* $|Q| \ge e^{q/1-q};$

(vi) $|\hat{\sigma}_{b,t,s}(\xi,\eta,z)| \le C\{|2^t\xi|\ |2^s\eta|\}^{1/\log|Q|}$ *if* $|Q| < e^{q/1-q};$

(vii) $|\hat{\sigma}_{b,t,s}(\xi,\eta,z)| \le C\{|2^t\xi|\ |2^s\eta|\}^{1/q'}$ *if* $|Q| \ge e^{q/1-q};$

where C *is a constant independent of* $t, s, z \in \mathbb{R}$, $(\xi,\eta) \in \mathbb{R}^{n+m}$ *and the block* $b(\cdot,\cdot)$.

For the sake of simplicity, we prove the case $n > 2$ and $m > 2$ only. The proof for other cases are similar, with only minor modifications.

By the mean zero property (1.1) of Ω, we have

$$\begin{aligned}|\hat{\sigma}_{\Omega,t,s}(\xi,\eta,z)| &\le C2^{-t-s}\int_0^{2^t}\int_0^{2^s}|h(u,v)\int_{S^{n-1}\times S^{m-1}}\Omega(x',y')\\ &\qquad\times\{e^{-iu\langle\xi,y'\rangle}-1\}\{e^{-iv\langle\eta,x'\rangle}-1\}d\sigma(x')d\sigma(y')|dudv\\ &\le C\|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}|\xi|\ |\eta|2^{-s-t}\int_0^{2^t}\int_0^{2^s}uv|h(u,u)|dudv.\end{aligned}$$

So we obtain (i).

We turn to prove (ii). Fixing any $\xi \ne 0$ and $\eta \ne 0$, by the method of rotation, without loss of generality, we may write

$$\begin{aligned}&|\hat{\tau}_{b,t,s}(\xi,\eta,z)|\\ &\le C2^{-s-t}\int_0^{2^t}\int_0^{2^s}\Big|h(v,v)\int_{S^{n-1}\times S^{m-1}}b(x',y')\\ &\qquad\times e^{-iu|\xi|\langle 1,x'\rangle}\{e^{-iv|\eta|\langle\tilde{1},y'\rangle}-1\}d\sigma(x')d\sigma(y')\Big|dudv\\ &\le C2^{-t-s}\int_0^{2^t}|\eta|v\int_{S^{m-1}}\int_0^{2^s}\Big|\int_{S^{n-1}}b(x',y')e^{-ix_1'|\xi|u}d\sigma(x')\Big|d\sigma(y')dudv.\end{aligned}$$

Thus $|\hat{\tau}_{b,t,s}(\xi,\eta,z)|$ is dominated by

$$C|2^s\eta|\int_{s^{m-1}}\int_0^{2^t|\xi|}2^{-t}|\xi|^{-1}\Big|\int_{\mathbb{R}}\phi_{y'}(x_1')e^{-ix_1'u}dx_1'\Big|dud\sigma(y')$$

where

$$\phi_{y'}(\zeta) = (1-\zeta^2)^{(n-3)/2}\chi_{\{|\zeta|<1\}}(\zeta)\int_{S^{n-2}} b(\zeta,(1-\zeta^2)^{1/2}\tilde{x},y')d\sigma(\tilde{x})$$

is a one dimension function. Therefore,

$$|\hat{\tau}_{b,t,s}(\xi,\eta,z)| \le C|2^s\eta|\int_{S^{m-1}}\int_0^{2^t|\xi|} 2^{-t}|\xi|^{-1}|\hat{\phi}_{y'}(u)|dud\sigma(y').$$

Pick a number ω in the interval (1,2) such that $\omega < q$. By Hölder's inequality we have

$$|\hat{\tau}_{b,t,s}(\xi,\eta,z)| \le C|2^s\eta|\int_{S^{m-1}}(2^t|\xi|)^{-1/\omega'}\|\hat{\phi}_{y'}\|_{L^{\omega'}}d\sigma(y')$$

where ω' is the conjugate exponent to ω. Thus by the Hausdorff-Young inequality, we find that $|\hat{\tau}_{b,t,s}(\xi,\eta,z)|$ is dominated by

$$C|2^s\eta|\;|2^t\xi|^{-1/\omega'}\int_{S^{m-1}}\|\phi_{y'}\|_\omega d\sigma(y'). \tag{4.1}$$

By the definition of $\phi_{y'}$ and Hölder's inequality again, we have

$$\begin{aligned}\int_{S^{m-1}}\|\phi_{y'}\|_{L^\omega(\mathbb{R})}d\sigma(y') &\le C\|b\|_{L^\omega(S^{n-1}\times S^{m-1})}\\ &\le C\|b\|_{L^q(S^{n-1}\times S^{m-1})}|Q|^{1/\omega-1/q} \le C|Q|^{-1/\omega'}.\end{aligned} \tag{4.2}$$

Now combining (4.1) and (4.2) and taking $\omega = \log|Q|/(1+\log|Q|)$, we easily obtain (ii). Switching the variables ξ and η in the proof of (ii), we obtain the estimate (iv). If $|Q| \ge e^{q/(1-q)}$, taking $\omega = q$ in the proofs of (4.1) and (4.2), then we obtain that

$$|\hat{\tau}_{b,t,s}(\xi,\eta,z)| \le C|2^s\eta|\;|2^t\xi|^{-1/q'}|Q|^{-1/q'} \le C_q|2^s\eta|\;|2^t\xi|^{-1/q'},$$

where the constant C depends only on $q > 1$. Thus (iii) is proved. Similarly we can prove (v). Since the proof of (vi) and (vii) are similar, we will prove (vi) only. By the method of rotation

$$|\hat{\sigma}_{b,t,s}(\xi,\eta,z)| \le C|2^t\xi|^{-1}|2^s\eta|^{-1}\int_0^{2^t|\xi|}\int_0^{2^s|\eta|}|\hat{\mathcal{F}}_b(s,t)|dudv$$

where

$$\mathcal{F}_b(s,t) = (1-s^2)^{(n-3)/2}(1-t^2)^{(m-3)/2}\chi_{\{|s|<1,|t|<1\}}(s,t)\Theta(s,t)$$

and

$$\Theta(s,t)=\int\int_{S^{n-2}\times S^{m-2}}|b(s,(1-s^2)^{1/2}\tilde{x},t,(1-t^2)^{1/2}\tilde{y})|d\sigma(\tilde{x})d\sigma(\tilde{y}).$$

Again we use Hölder's inequality and the Hausdoff-Young inequality to obtain

$$|\hat{\sigma}_{b,t,s}(\xi,\eta,z)|\le C|2^t\xi|^{-1}|2^s\eta|^{-1}\Big\{\int_0^{2^t|\xi|}\int_0^{2^s|\eta|}dudv\Big\}^{1/\omega}\|\mathcal{F}_b\|_{\omega}.$$

Using the proof in (4.2), we obtain $\|\mathcal{F}_b\|_{\omega}\le C|Q|^{-1/\omega'}$. Therefore

$$|\hat{\sigma}_{b,t,s}(\xi,\eta,z)|\le C|Q|^{-1/\omega'}|2^k\xi|^{-1/\omega'}|2^j\eta|^{-1/\omega'}.$$

Letting $\omega=\log|Q|/\{\log|Q|+1\}$, we obtain (vi).

5. Proof of Theorem

Our proof is based on the method used in [3]. For a given block function $\Omega=\sum C_\mu b_\mu$, by Lemma 4.1, without loss of generality, we assume that the supports Q_μ of b_μ are uniformly small such that

$$|Q_\mu|<e^{q/(1-q)}\quad\text{and}\quad\log(\log(1/|Q_\mu|))\ge 1.$$

Take two radial Schwartz functions, $\Phi\in S(\mathbb{R}^n)$, $\Psi\in S(\mathbb{R}^m)$ such that the values of their Fourier transforms are between 0 and 1 and satisfy

$$\int_{\mathbb{R}}\widehat{\Phi(2^t)}dt=\int_{\mathbb{R}}\widehat{\Psi(2^s)}ds=1,$$
$$\operatorname{supp}(\hat{\Phi})\subseteq\{x\in\mathbb{R}^n;2^{-1}\le|x|<2\},$$
$$\operatorname{supp}(\hat{\Psi})\subseteq\{y\in\mathbb{R}^m;2^{-1}<|y|\le 2\}.$$

Let $\Phi_t(x)=2^{-nt}\Phi(x/2^t)$ and $\Psi_s(y)=2^{-ms}\Psi(y/2^s)$. Then by checking the Fourier transforms, it is easy to see that for any test function f,

$$f\simeq\int_{\mathbb{R}}\int_{\mathbb{R}}(\Phi_t\otimes\Psi_s\otimes\delta)*f\,dsdt. \tag{5.1}$$

Define the g-function by

$$g(f)(x,y,z)=\Big(\int_{\mathbb{R}}\int_{\mathbb{R}}|(\Phi_t\otimes\Psi_s\otimes\delta)*f(x,y,z)|^2dtds\Big)^{1/2}$$

where (also in (5.1)) δ is the Dirac δ function. By [4], we know that

$$\|g(f)\|_{L^p(\mathbb{R}^{n+m+1})}\le C\|f\|_{L^p(\mathbb{R}^{n+m+1})} \tag{5.2}$$

for any $p \in (1, \infty)$.

By (5.1) and the Minkowski inequality, we have that

$$\begin{aligned}\nu_\Omega f(x,y,z) &= \left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}(\Phi_{t+u}\otimes\Psi_{s+v}\otimes\delta)*\sigma_{\Omega,t,s}*f(x,y,z)dudv\right)^2 dtds\right)^{1/2}\\ &\le \int_{\mathbb{R}^2} I_{u,v}f(x,y,z)dudv\end{aligned}$$

where

$$I_{u,v}f(x,y,z) = \left(\int_{\mathbb{R}^2}|(\Phi_{t+u}\otimes\Psi_{s+v}\otimes\delta)*\sigma_{\Omega,t,s}*f(x,y,z)|^2 dsdt\right)^{1/2}$$

By Minkowski's inequality, for any $p \in (1, \infty)$

$$\|\nu_\Omega f\|_{L^p} \le C\int_{\mathbb{R}^2}\|I_{u,v}f\|_{L^p}dudv.$$

Now we prove that there exists a constant C independent of u and v such that

$$\|I_{u,v}f\|_{L^p(\mathbb{R}^{n+m+1})} \le C\|f\|_{L^p(\mathbb{R}^{n+m+1})} \quad \text{for all} \quad p\in(1,\infty). \tag{5.3}$$

Note that

$$\sigma_{\Omega,t,s} = \sum c_\mu \sigma_{b_\mu,t,s} \quad \text{and} \quad \sum |c_\mu| < \infty.$$

Let

$$I_{u,v,b}f(x,y,z) = \left(\int_{\mathbb{R}^2}|(\Phi_{t+u}\otimes\Psi_{s+v}\otimes\delta)*\sigma_{b,t,s}*f(x,y,z)|^2 dsdt\right)^{1/2}$$

To prove (5.3), it suffices to show that

$$\|I_{u,v,b}f\|_{L^p} \le C\|f\|_{L^p} \tag{5.4}$$

where the constant C is independent of u, v and the blocks b. We define a linear operator T on any function $\mathcal{F}(x,y,z,t,s)$ by $T\mathcal{F}(x,y,z,t,s) = \sigma_{b,t,s} * \mathcal{F}(x,y,z,t,s)$ and want to prove the mixed norm inequality

$$\big\| \|T\mathcal{F}\|_{L^2(\mathbb{R}^2)}\big\|_{L^p(\mathbb{R}^{n+m+1})} \le C\big\| \|\mathcal{F}\|_{L^2(\mathbb{R}^2)}\big\|_{L^p(\mathbb{R}^{n+m+1})}. \tag{5.5}$$

for $p \in (1, \infty)$.

By duality, we only need to show the cases $p \in (1, 2]$. By the definition and (3.1), it is easy to see that

$$\big\| \|T\mathcal{F}\|_{L^1(\mathbb{R}^2)}\big\|_{L^1(\mathbb{R}^{n+m+1})} \le C_1\big\| \|\mathcal{F}\|_{L^1(\mathbb{R}^2)}\big\|_{L^1(\mathbb{R}^{n+m+1})},$$

By Proposition 3.1, we have

$$\Big\| \, \|T\mathcal{F}\|_{L^\infty(\mathbb{R}^2)} \Big\|_{L^q(\mathbb{R}^{n+m+1})} \le C_2 \Big\| \, \|\mathcal{F}\|_{L^\infty(\mathbb{R}^2)} \Big\|_{L^q(\mathbb{R}^{n+m+1})}$$

for any $q > 1$. Clearly the above constants C_1 and C_2 are independent of the functions b. Thus by interpolation we obtain (5.5). In particular, letting $\mathcal{F}(x,y,t,s) = (\Phi_{t+u} \otimes \Psi_{s+v} \otimes \delta) * f(x,y)$, then we obtain that

$$\|I_{u,v,b} f\|_{L^p} = \Big\| \, \|T\mathcal{F}\|_{L^2} \Big\|_{L^p} \le C \Big\| \, \|\mathcal{F}\|_{L^2} \Big\|_{L^p} = C\|g(f)\|_{L^p} \le C\|f\|_{L^p}.$$

(5.4) is proved.

We are in a position to prove Theorem 1. If $u > 0$ and $v > 0$, by Plancherel's theorem

$$\|I_{u,v} f\|_2^2 \le C \int_{\mathbb{R}^2} \int_{\Delta_{t,s,u,v}} |\hat{\sigma}_{\Omega,t,s}(\xi,\eta,z)|^2 \, |\hat{f}(\xi,\eta,z)|^2 \, d\xi \, d\eta \, dz \, dt \, ds$$

where

$$\begin{aligned} \Delta_{t,s,u,v} = \big\{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m : {} & 2^{-t-u-1} \le |\xi| < 2^{-t-u+1}, \\ & 2^{-s-v-1} \le |\eta| < 2^{-s-v+1}\big\} \times \mathbb{R}. \end{aligned}$$

Thus by (i) of Lemma 4.1, we know that if $(\xi,\eta,z) \in \Delta_{t,s,u,v}$ then

$$|\hat{\sigma}_{\Omega,t,s}(\xi,\eta,z)| \le C|2^t\xi| \, |2^s\eta| \le C2^{-u}2^{-v}.$$

Thus

$$\|I_{u,v} f\|_2^2 \le C2^{-2(u+v)} \int_{\mathbb{R}^2} \int_{\Delta_{t,s,u,v}} |\hat{f}(\xi,\eta,z)|^2 d\xi \, d\eta \, dz \, dt \, ds.$$

Thus by the definition of $\Delta_{t,s,u,v}$, it is easy to see that

$$\|I_{u,v} f\|_2^2 \le C2^{-2(u+v)} \|\hat{f}\|_2^2 \simeq C2^{-2(u+v)} \|f\|_2^2,$$

which shows that

$$\|I_{u,v}\|_{L^2 \to L^2} \le C2^{-u}2^{-v}. \tag{5.6}$$

We now use interpolation between (5.3) and (5.6) to obtain a $\theta > 0$ such that

$$\|I_{u,v} f\|_p \le C^{-v\theta} 2^{-u\theta} \|f\|_p. \tag{5.7}$$

Thus we have

$$\int_0^\infty \int_0^\infty \|I_{u,v} f\|_p du dv \le C\|f\|_p. \tag{5.8}$$

For $u < 0$ and $v \geq 0$, by the cancellation condition of Ω and the definition of $\tau_{\Omega,t,s}$, it is easy to see that $\sigma_{\Omega,t,s} = \tau_{\Omega,t,s}$. So we have

$$I_{u,v}f = \Big(\int_{\mathbb{R}^2} |\tau_{\Omega,t,s} * (\Phi_{t+u} \otimes \Psi_{s+v} \otimes \delta) * f|^2 dtds\Big)^{1/2}.$$

Thus

$$\int_{-\infty}^{0}\int_{0}^{\infty} \|I_{u,v}f\|_p dudv \leq \sum_{\mu} |c_\mu| \int_{-\infty}^{0}\int_{0}^{\infty} \|J_{b_\mu,u,v}f\|_p dudv$$

where

$$J_{b_\mu,u,v}f = \Big(\int_{\mathbb{R}^2} |\tau_{b_\mu,t,s} * (\Phi_{t+r} \otimes \Psi_{s+v} \otimes \delta) * f|^2 dsdt\Big)^{1/2}.$$

Using the exactly same argument in proving (5.4), by (3.6) we can prove

$$\|J_{b_\mu,u,v}f\|_{p_0} \leq C\|f\|_{p_0} \quad \text{for any} \quad 1 < p_0 < \infty, \tag{5.9}$$

where C is independent of b_μ, u and v. On the other hand, by Plancherel's theorem

$$\|J_{b_\mu,u,v}f\|_2 \leq \int_{\mathbb{R}^2}\int_{\Delta_{t,s,u,v}} |\hat{\tau}_{b_\mu,t,s}(\xi,\eta,z)|^2 |\hat{f}(\xi,\eta,z)|^2 d\xi\, d\eta\, dz\, du\, dv.$$

Thus by (ii) of Lemma 4.1, we know that if $(\xi,\eta,z) \in \Delta_{t,s,u,v}$ then

$$|\hat{\tau}_{b_\mu,t,s}(\xi,\eta,z)| \leq C|2^t\xi|^{1/\log|Q_\mu|}|2^s\eta| \leq C2^{-u/\log|Q_\mu|}2^{-v}.$$

Therefore, it is easy to see

$$\|J_{b_\mu,u,v}\|_{L^2\to L^2} \leq C2^{-u/\log|Q_\mu|}2^{-v}. \tag{5.10}$$

We now use interpolation to obtain

$$\|J_{b_\mu,u,v}f\|_p \leq C2^{-v\theta}2^{-u\theta/\log|Q_\mu|}\|f\|_p \tag{5.11}$$

for some $\theta > 0$. This shows that

$$\begin{aligned}
&\int_{-\infty}^{0}\int_{0}^{\infty} \|I_{u,v}f\|_p dudv \\
&\leq C\int_{u<0}\int_{v\geq 0} \sum_{\mu} |c_\mu| 2^{-u\theta/\log|Q_\mu|}2^{-\theta v}\|f\|_p dudv \\
&\leq C\|f\|_p \sum_{\mu} |C_\mu| \log(1/|Q_\mu|).
\end{aligned} \tag{5.12}$$

Clearly, the constant C above is independent of the essential variables. Similarly, by (iv) in Lemma 4.1, we can prove

$$\int_{u\geq 0}\int_{v<0}\|I_{u,v}f\|_p dudv \leq C\|f\|_p \sum_{\mu}|C_\mu|\log(1/|Q_\mu|). \tag{5.13}$$

Finally, using (vi) in Lemma 4.1 and the same argument in (5.12), we find

$$\int_{u<0}\int_{v<0}\|I_{u,v}f\|_p dudv \leq C\|f\|_p \sum_{\mu}|C_\mu|(\log(1/|Q_\mu|))^2. \tag{5.14}$$

Now Theorem 1 follows by (5.8), (5.12), (5.13) and (5.14).

Finally we will prove Theorem 2. In fact, let $\gamma(u,v)\equiv 1$, then clearly γ satisfies (i) and (ii) in Theorem 1. For any function $f\in S(\mathbb{R}^{n+m+1})$, we let h be a function on $S(\mathbb{R})$ such that $\|h\|_p\neq 0$. Then it is easy to see, by the definition and Theorem 1, that

$$\|h\|_{L^p(\mathbb{R})}\|T_\Omega f\|_{L^p(\mathbb{R}^{n+m})} = \|\mu_\Omega(f\otimes h)\|_{L^p(\mathbb{R}^{n+m+1})} \leq C\|f\|_{L^p(\mathbb{R}^{n+m})}\|h\|_{L^p(\mathbb{R})},$$

where $(f\otimes h)(x,y,z)=f(x,y)h(z)$. The theorem is proved.

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