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A TOPOLOGICAL APPROACH TO THE NIELSEN'S REALIZATION PROBLEM FOR HAKEN 3-MANIFOLDS

By

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Abstract. Let M be a Haken manifold with a toral or empty boundary which is not a closed Seifert manifold. Zimmermann [31] showed that any mapping class c of M of finite order n can be realized by an auto-homeomorphism of period n using algebraic methods. In a special case, Heil and Tollefson [4], and Hong and McCullough [9] used methods from 3-manifold topology to approach the Zimmermann's theorem. This paper provides a complete proof by another topological approach.

1. Introduction

Let us begin by introducing the realization problem and some related work. The *mapping class group* $\mathcal{M}(M)$ of a manifold M is the quotient group of the group of auto-homeomorphisms of M by the subgroup of those which are isotopic to the identity map. If a mapping class $c \in \mathcal{M}(M)$ of order n is represented by an auto-homeomorphism f of period n, we say c is *realized* by f. Similarly, if a finite subgroup $G \subset \mathcal{M}(M)$ is isomorphic to a group F of auto-homeomorphisms each of which is a representative of a corresponding mapping class in G, we say G is *realized* by F.

The Realization Problem. *Is any mapping class in $\mathcal{M}(M)$ of finite order realized by a periodic auto-homeomorphism? Moreover, is any finite subgroup of $\mathcal{M}(M)$ realized by a group of auto-homeomorphisms?*

For the 2-dimensional case, Nielsen [17, p.24] first dealt with the first half of the problem for any compact, orientable surfaces, and Kerckhoff [15] gave the following full solution:

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THEOREM 1.1 (Kerckhoff). *Let M be a compact surface. Then any finite subgroup of $\mathcal{M}(M)$ is realized by a group of auto-homeomorphisms of M.*

The problem for 3-dimensional case is more difficult. A triple $(A, \pi_1 M, \Omega)$ consisting of a finite subgroup A of $\mathcal{M}(M)$ and the fundamental group $\pi_1 M$ of M together with a homomorphism $\Omega\colon A \to Out(\pi_1 M)$ is called an *abstract kernel*, where $Out(\pi_1 M)$ denotes the group of outer automorphisms of $\pi_1 M$. A group E with an exact sequence $1 \longrightarrow \pi_1 M \xrightarrow{i} E \xrightarrow{j} A \longrightarrow 1$ is called an *extension* to the abstract kernel $(A, \pi_1 M, \Omega)$, if for each $a \in A$ and $f \in E$ satisfying $j(f) = a$ the automorphism of $\pi_1 M$ carrying x to $i^{-1}(f^{-1} i(x) f)$ belongs to the class $\Omega(a)$. We say a Seifert manifold has an orbit-manifold X of *hyperbolic type* if X is an orbifold finitely covered by a hyerbolic surface. Zieschang and Zimmermann have proved the following theorem for Seifert 3-manifolds (see [28, Satz 5.7]):

THEOREM 1.2 (Zieschang and Zimmermann). *Let M be a compact Seifert manifold with an orbit-manifold of hyperbolic type with nonempty boundary, and A a finite subgroup of $\mathcal{M}(M)$. If there is an extension to an abstract kernel $(A, \pi_1 M, \Omega)$, A is realized by an isomorphic group of auto-homeomorphisms of M.*

They also proved that the realization problem is not always affirmative for general Seifert manifolds (see [28, Satz 4.1]):

THEOREM 1.3 (Zieschang and Zimmermann). *Let F_{2p+1} be an orientable closed surface with genus $2p+1$ for $p > 0$. Then there is a finite subgroup A of $\mathcal{M}(F_{2p+1} \times S^1)$ which cannot be realized.*

Furthermore, Zimmermann gave a partial answer for Haken manifolds, i.e. orientable, irreducible and sufficiently large 3-manifolds (see [31, Satz 0.1 and Addendum]):

THEOREM 1.4 (Zimmermann). *Let M be a Haken manifold such that ∂M is either empty or a union of some tori. Suppose M is not a closed Seifert manifold. Then for any homeomorphism $f\colon M \to M$ representing a mapping class of finite order n, there is a homeomorphism $g\colon M \to M$ of period n which is isotopic to f.*

It should be noted that the exterior of a non-separable link L in the 3-sphere S^3 is a Haken manifold and that the symmetry group $Sym(S^3, L)$ of L is defined to be the mapping class group of the pair (S^3, L). For the definition of the symmetries of links, we refer the reader to [6, 7]. The following corollary immediately follows from Theorem 1.4:

COROLLARY 1.5. *Let L be a non-separable link in S^3. Suppose the symmetry group $Sym(S^3, L)$ of L has an element of order n. Then L has the following four possibilities:*

(1) *L has cyclic period n,*
(2) *L has free period n,*
(3) *L is strongly invertible, or*
(4) *L is strongly $(\pm)$ amphicheiral.*

It should be noted that the possibility either (3) or (4) in the above corollary occurs only when $n = 2$.

In this paper, we will give a new proof of Theorem 1.4. Let us explain the difference between our approach and the previous work. Zimmermann [31] first proved the theorem using the theory of group actions, extensions and crystallographic groups. Heil and Tollefson [4] used methods from the topology of 3-manifolds in case of closed Haken manifolds. Their main idea is a modification of f such that afterwards f is periodic on a system $\mathcal{T}$ of essential tori in M, which is based on the vanishment of the algebraic obstruction for f to be homotopic to a homeomorphism which has period n on respective pieces obtained by cutting M along $\mathcal{T}$. Hong and McCullough [9] extended the Heil and Tollefson's work to the case of Haken manifolds with toral boundaries, but they required the condition $f^n \simeq id_M$ rel ∂M. Our approach provides a complete proof by topological methods with no argument about the algebraic obstruction. We start with constructing realizations on respective pieces, and modify them according to the twists occured along $\mathcal{T}$.

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2. Preliminary

For fundamental notations on 3-manifolds, we refer the reader to [5], [12] or [13]. Throughout this paper, we will denote by $\mathcal{N}(X, Y)$ the regular neighbourhood of X in Y, by id_X the identity map of X, and by $I = [0, 1]$ the unit interval. Furthermore we will consider S^1 and D^2 parametrized as $\{e^{i\theta} | \theta \in \mathbb{R}\}$ and $\{re^{i\theta} | 0 \leq r \leq 1, \theta \in \mathbb{R}\}$ respectively.

In the following let us prove two lemmas which provide elementary properties of isotopies.

LEMMA 2.1. (1) *Let N be a manifold homeomorphic to $D^2 \times S^1$. Suppose f_1 and f_2 are auto-homeomorphisms of N satisfying $f_1|_{\partial N} = f_2|_{\partial N}$. Then there is an isotopy from f_1 to f_2 relative to ∂N.*

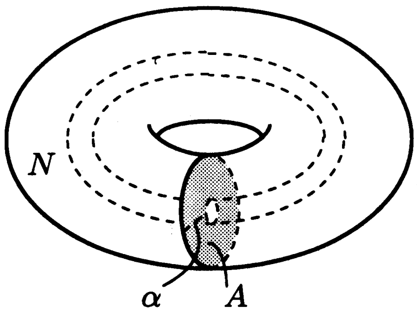


Figure 1

(2) *Let N be a manifold homeomorphic to $S^1 \times S^1 \times I$, α a proper arc in N joining the distinct boundary components of N. Suppose f_1 and f_2 are auto-homeomorphisms of N satisfying $f_1|_{\partial N \cup \alpha} = f_2|_{\partial N \cup \alpha}$. Then there is an isotopy from f_1 to f_2 relative to $\partial N \cup \alpha$.*

Proof. (1) Let D be a meridian disk of N. Modify f_1 by an isotopy relative to ∂N so that $f_1(D) = f_2(D)$. Since $f_1^{-1} \circ f_2|_{\partial D} = id_{\partial D}$, it follows from Alexander's isotopy theorem [1] that there is an isotopy on D from $f_1^{-1} \circ f_2|_D$ to id_D relative to ∂D. We may therefore assume $f_1^{-1} \circ f_2|_{\mathcal{N}(D,N)} = f_2|_{\mathcal{N}(D,N)}$. Consider the ball $B = cl(N - \mathcal{N}(D, N))$. Since $f_1^{-1} \circ f_2|_{\partial B} = id_{\partial B}$, it follows from Alexander's isotopy theorem that there is an isotopy on B from $f_1^{-1} \circ f_2$ to id_B relative to ∂B. Hence there is an isotopy on B from f_1 to f_2 relative to ∂B.

(2) Let A be an essential annulus of N containing α, see Figure 1. Modify f_1 by an isotopy relative to $\partial N \cup \alpha$ so that $f_1(A) = f_2(A)$ and $f_1|_{\mathcal{N}(\alpha,A)} = f_2|_{\mathcal{N}(\alpha,A)}$. By applying Alexander's isotopy theorem to $f_1^{-1} \circ f_2$ on the disk $cl(A - \mathcal{N}(\alpha, A))$, we may assume $f_1|_{\mathcal{N}(A,N)} = f_2|_{\mathcal{N}(A,N)}$. Consider a solid torus $V = cl(N - \mathcal{N}(A, N))$. Then we have $f_1|_{\partial V} = f_2|_{\partial V}$. It follows from (1) that there is an isotopy on V from f_1 to f_2 relative to ∂V. This completes the proof. □

LEMMA 2.2. *Let A be an annulus, $N = A \times I$, $A_t = A \times \{t\}$, and α a proper arc in N joining A_0 and A_1. If two homeomorphisms $f_1 \colon N \to N$ and $f_2 \colon N \to N$ satisfy $f_1|_{A_0 \cup A_1 \cup \alpha} = f_2|_{A_0 \cup A_1 \cup \alpha}$, then f_2 is isotopic to f_1 relative to $A_0 \cup A_1 \cup \alpha$.*

Proof. Let D be a meridian disk of N which contains α such that each of $D \cap A_0$ and $D \cap A_1$ is an arc. Modify f_2 by an isotopy relative to $A_0 \cup A_1 \cup \alpha$ so that $f_1(D) = f_2(D)$. Let I_1 and I_2 denote the distinct components of $\partial D - int(A_0 \cup A_1)$ (see Figure 2). Then $f_1^{-1} \circ f_2(I_i) = I_i$ and $f_1^{-1} \circ f_2|_{\partial I_i} = id_{\partial I_i}$ for each i. It follows from Alexander's isotopy theorem [1] that there is an isotopy on I_i from $f_1^{-1} \circ f_2|_{I_i}$ to id_{I_i} relative to ∂I_i. Therefore by an isotopy of f_2 on N relative to

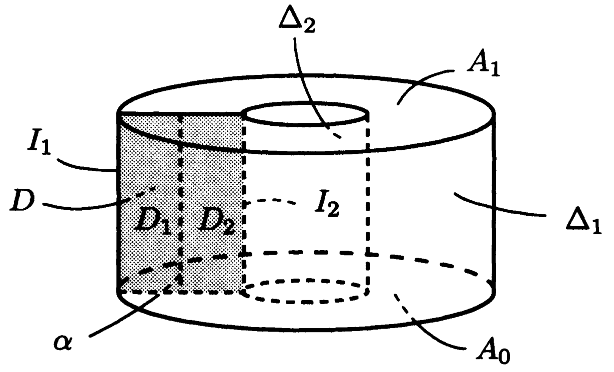


Figure 2

$A_0 \cup A_1 \cup \alpha$, we may assume $f_1 = f_2$ on $A_0 \cup A_1 \cup \partial D \cup \alpha$.

Let D_1 and D_2 denote the disks on D such that $D_1 \cup D_2 = D$ and $D_1 \cap D_2 = \alpha$. Then $f_1^{-1} \circ f_2(D_i) = D_i$ and $f_1^{-1} \circ f_2|_{\partial D_i} = id_{\partial D_i}$ for each i. Let Δ_1 and Δ_2 denote the components of $\partial N - int(A_0 \cup A_1)$ such that $\Delta_i \supset I_i$ for each i. Then we have $f_1^{-1} \circ f_2(\Delta_i) = \Delta_i$ and $f_1^{-1} \circ f_2|_{\partial \Delta_i \cup I_i} = id_{\partial \Delta_i \cup I_i}$ for each i. Note that we obtain a disk by cutting each Δ_i along I_i. Therefore it follows from Alexander's isotopy theorem that there is an isotopy on $\partial N \cup D$ from $f_1^{-1} \circ f_2|_{\partial N \cup D}$ to $id_{\partial N \cup D}$ relative to $A_0 \cup A_1 \cup \partial D \cup \alpha$. Hence by an isotopy of f_2 on N relative to $A_0 \cup A_1 \cup \partial D \cup \alpha$, we may assume $f_1 = f_2$ on $\partial N \cup D$.

Note that we obtain a ball by cutting N along D. Therefore it follows from Alexander's isotopy theorem that there is an isotopy on N from $f_1^{-1} \circ f_2$ to id_N relative to $\partial N \cup D$. This completes the proof. □

3. Hyperbolic 3-Manifolds

As in Theorem 1.4, let M be a compact, connected Haken 3-manifold whose boundary is either empty or a union of some tori. Suppose that M is algebraically simple, and that M is not homeomorphic to neither $D^2 \times S^1$, $S^1 \times S^1 \times I$ nor the twisted I-bundle over the Klein bottle. It follows from Thurston's Uniformization Theorem [16, Theorem B in Chapter V] that M is a complete, finite-volume, hyperbolic 3-manifold.

PROPOSITION 3.1. *Let M be a complete, finite-volume, hyperbolic 3-manifold. Then any mapping class in $\mathcal{M}(M)$ of finite order n is realized by a homeomorphism of period n.*

Proof. The irreducibility of M follows from [2, Proposition D.3.17.] and the incompressibility of ∂M in M follows from [2, Proposition D.3.18.]. Therefore it follows from [27, Corollary 7.5] that $\mathcal{M}(M)$ is naturally isomorphic to $Out(\pi_1 M)$ respecting the peripheral structure. Moreover it follows from Mostow's Rigidity Theorem [16, Corollaries 2 and 3 in Chapter V] that $Out(\pi_1 M)$ is a finite group, and that there is a natural isomorphism $Isom(intM) \cong Out(\pi_1 M)$. Thus any homeomorphism representing a mapping class $c \in \mathcal{M}(intM)$ of order n is isotopic to a unique isometory g of $intM$ of period n. Hence c is realized by a homeomorphism defined as an extension of g to M. □

4. Seifert 3-Manifolds

In this section we will prove the Seifert 3-manifold version of Theorem 1.4.

Let M be a Seifert 3-manifold whose boundary is a union of some tori. Let B be the orbit-manifold of M and $p \colon M \to B$ the projection map. It should be noted that each of the disk, annulus and Möbius band admits both a hyperbolic and euclidean structure. So we may assume that $intB$ is finitely covered by a hyperbolic surface. We consider the two fibrations $\{\{x\} \times \mathbb{R} | x \in \mathbb{H}^2\}$ and $\{\mathbb{H}^2 \times \{r\} | r \in \mathbb{R}\}$ of the universal covering $\mathbb{H}^2 \times \mathbb{R}$ of $intM$. It follows from [28, §5] that the former fibration induces the Seifert fibration of M and the latter fibraton induces a foliation of M dual to the Seifert fibration, which we call the *dual foliation* of M.

Let i be the least common multiple of the orders of all exceptional fibers in M. Then the i-fold orbifold covering $\tilde{B}$ of B is a surface. If $\tilde{B}$ is orientable, we can regard any dual foliation of M as a $\tilde{B}$-bundle over S^1. Otherwise M has two exceptional leaves each homeomorphic to $\tilde{B}$ and any normal leaf homeomorphic to the orientable double cover of $\tilde{B}$.

4.1 Elementary Seifert 3-manifolds

First we deal with the elementary Seifert 3-manifolds, $D^2 \times S^1$ and $S^1 \times S^1 \times I$, which are exceptional in terms of [26, Satz 10.1].

PROPOSITION 4.1. *Let M be a manifold homeomorphic to either $D^2 \times S^1$ or $S^1 \times S^1 \times I$. For any homeomorphism $f \colon M \to M$ representing a mapping class of order n, there is a homeomorphism $g \colon M \to M$ of period n which is isotopic to f. Moreover, if M is Seifert fibered and f is isotopic to a fiber-preserving homeomorphism, g can be chosen so as to preserve both the Seifert fibration and a dual foliation of M.*

Proof. First we assume that M is homeomorphic to $D^2 \times S^1$. Regard the torus ∂M as the quotient of $\mathbb{R}^2$ by the integer lattice $\mathbb{Z}^2$. We may assume that $\pi_1(\partial M)$ acts on $\mathbb{R}^2$ as the universal covering group of ∂M with the generators $[S^1 \times \{y\}]$ and $[\{x\} \times S^1]$ corresonding to $(1,0)$ and $(0,1)$-translation on $\mathbb{R}^2$ respectively. By the natural homomorphism $\mathcal{M}(\partial M) \cong GL(2,\mathbb{Z})$ (see [20, 3]), $[f|_{\partial M}] \in \mathcal{M}(\partial M)$ corresponds to a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying $A^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It should be noted that $\begin{pmatrix} a \\ c \end{pmatrix} = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ since f carries any meridian of M to a meridian. Then the homeomorphism $g\colon M \to M$ defined by $g(re^{i\theta}, e^{i\varphi}) = (re^{i(\pm\theta + b\varphi)}, e^{id\varphi})$ satisfies $g^n = id_M$, and $g \simeq f$ follows from Lemma 2.1.

Suppose that M is Seifert fibered and that f is isotopic to a fiber-preserving homeomorphism. After a modification of the Seifert fibration of M by an isotopy, we may assume that the fiber passing $(re^{i\theta}, 1) \in D^2 \times S^1$ is the set $\{(re^{i(\theta + p\varphi)}, e^{iq\varphi}) | \varphi \in \mathbb{R}\}$ where p and $q \neq 0$ are coprime integers. Therefore A has the eigen vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} p \\ q \end{pmatrix}$. Hence g preserves both the Seifert fibration and the dual foliation $\{D^2 \times \{y\} | y \in S^1\}$.

Next we assume that M is homeomorphic to $S^1 \times S^1 \times I$. Let $p\colon M \to S^1 \times S^1$ be the natural projection map. Regard the torus $S^1 \times S^1$ as the quotient of $\mathbb{R}^2$ by the integer lattice $\mathbb{Z}^2$. We may assume that $\pi_1(S^1 \times S^1) \cong \pi_1 M$ acts on $\mathbb{R}^2$ as the universal covering group of $S^1 \times S^1$ with the generators $[S^1 \times \{y\}]$ and $[\{x\} \times S^1]$ corresonding to $(1,0)$ and $(0,1)$-translation on $\mathbb{R}^2$ respectively. Let $\bar{f}$ denotes the auto-homeomorphism of $S^1 \times S^1$ induced from $f|_{S^1 \times S^1 \times \{0\}}$ by p. By the homomorphism $\mathcal{M}(\partial M) \cong GL(2,\mathbb{Z})$, $[\bar{f}] \in \mathcal{M}(S^1 \times S^1)$ corresponds to a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $A^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Define a homeomorphism $g\colon M \to M$ by

$$g(e^{i\theta}, e^{i\varphi}, t) = \begin{cases} (e^{i(a\theta + b\varphi)}, e^{i(c\theta + d\varphi)}, t), & \text{if } f \text{ fixes the boundary components,} \\ (e^{i(a\theta + b\varphi)}, e^{i(c\theta + d\varphi)}, 1 - t), & \text{otherwise.} \end{cases}$$

Then we have $g^n = id_M$ and $g \simeq f$ follows from Lemma 2.1.

Suppose that M is Seifert fibered and that f is isotopic to a fiber-preserving homeomorphism. Without loss of generality, we may assume that M has fibers $\{S^1 \times \{(y,t)\} | y \in S^1, t \in I\}$. Then $A = \pm \begin{pmatrix} 1 & b' \\ 0 & d' \end{pmatrix}$. Since $A^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $|\det A| = |d'| = 1$. If $d' = 1$, $A = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ follows from [3, §0]. Hence g preserves both the Seifert fibration and the dual foliation $\{\{x\} \times S^1 \times I | x \in S^1\}$. If $d' = -1$, then $A = \pm \begin{pmatrix} 1 & b' \\ 0 & -1 \end{pmatrix}$ has eigen vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} b' \\ -2 \end{pmatrix}$. Therefore

g preserves both the Seifert fibration and the dual foliation whose leaf passing $(e^{i\theta}, 1, 0)$ is the annulus $\{(e^{i(\theta+b'\varphi)}, e^{-2i\varphi}) \times I | \theta \in \mathbb{R}\}$. This completes the proof. □

4.2 S^1-bundles over surfaces and twists along annulus systems

Let M be a compact, connected, orientable S^1-bundle over a surface such that ∂M is a union of some tori. Suppose that M is homeomorphic to neither $D^2 \times S^1$ nor $S^1 \times S^1 \times I$. Let B be the base manifold of M and $p \colon M \to B$ the projection map. Let us consider a fiber-preserving homeomorphism $f \colon M \to M$ representing a mapping class of order n such that the homeomorphism $\varphi \colon B \to B$ induced from f is periodic.

It follows from [4, Lemma 1] that there is a system $\{\alpha_1, \alpha_2, \dots, \alpha_\mu\}$ of disjoint, essential arcs in B such that $\alpha_i = \varphi^{i-1}(\alpha_1)$ for $1 < i \leq \mu$ and $\alpha_1 = \varphi(\alpha_\mu)$, which builds up to a system $\mathcal{A} = \{A_1, A_2, \dots, A_\mu\}$ of disjoint, saturated, essential annuli in M such that $A_i = p^{-1}(\alpha_i)$. We may assume $f(\mathcal{N}(\bigcup \mathcal{A}, M)) = \mathcal{N}(\bigcup \mathcal{A}, M)$. Suppose that f^μ reverses the orientation of the normal of each A_i. Let A_1' be a component of $cl(\partial \mathcal{N}(A_1, M) - \partial M)$ and $A_i' = f^{i-1}(A_1')$ for $1 < i \leq 2\mu$. Then $A_1' \cup A_2' \cup \cdots \cup A_{2\mu}'$ is a system of disjoint, saturated, essential annuli and $f^{2\mu}$ preserves the orientation of the normal of each A_i'. Thus we may assume that f^μ preserves the orientation of the normal of each A_i. We call $\mathcal{A}$ an *annulus system* for f.

We call each connected component of $E_\mathcal{A} = cl(M - \mathcal{N}(\bigcup \mathcal{A}, M))$ the piece. Suppose that each piece is endowed with the S^1-bundle structure induced from that of M. Let $\mathcal{P}$ denotes the set of all pieces. Then f induces a permutation on a finite set $\mathcal{P}$, which can be written as a product of disjoint cycles as follows:

$$(M_{1,1}\ M_{1,2}\ \cdots\ M_{1,\nu_1})(M_{2,1}\ M_{2,2}\ \cdots\ M_{2,\nu_2})\ \cdots\ (M_{\kappa,1}\ M_{\kappa,2}\ \cdots\ M_{\kappa,\nu_\kappa})$$

We call $O_i = \bigcup_{j=1}^{\nu_i} M_{i,j}$ the i-th *orbit* of f. It should be noted that κ is either one or two according as the arc α_i/φ on the orbit surface B/φ is non-separating or separating. Let $\mathcal{A}_{i,j}$ denotes the set of the annuli in $\mathcal{A}$ each of which has $M_{i,j}$ on a side (i.e. $\mathcal{N}(A_k, M) \cap M_{i,j} \neq \phi$ for any $A_k \in \mathcal{A}_{i,j}$).

In the following, we assume that $f^n|_{E_\mathcal{A}} = id_{E_\mathcal{A}}$ and that $f|_{E_\mathcal{A}}$ preserves both the S^1-bundle structure and a dual foliation of $E_\mathcal{A}$.

LEMMA 4.2. *Suppose that M is not homeomorphic to the twisted I-bundle over the Klein bottle. Then the automorphism $(f^n)_* \colon \pi_1 M \to \pi_1 M$ induced from f^n carries any element $t \in \pi_1 M$ to $\gamma^r t \gamma^{-r}$ where $\gamma \in \pi_1 M$ is the homotopy class represented by any fiber of M and r is an integer.*

Proof. Let us fix a base point $x \in E_\mathcal{A}$ for $\pi_1 M$. It follows from [27, Corollary

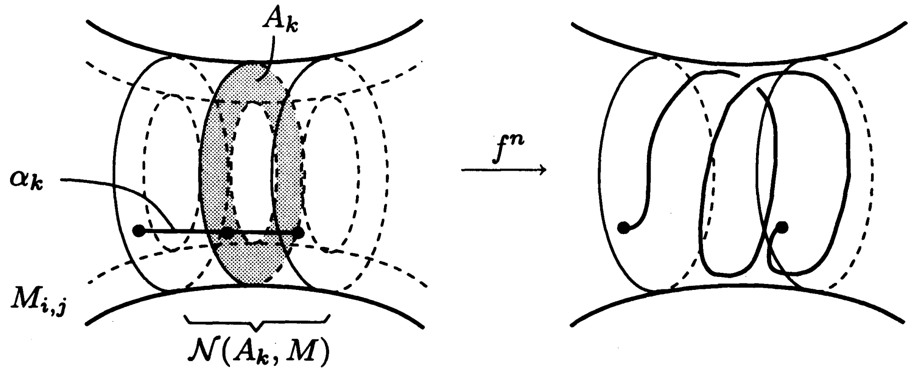


Figure 3

7.5] that $(f^n)_*$ is an inner automorphism. So there is an element $\xi \in \pi_1 M$ such that $(f^n)_*(t) = \xi t \xi^{-1}$ for any $t \in \pi_1 M$. Regard $p(x)$ as the base point for $\pi_1 B$. Put $\eta = p_*(\xi)$ where $p_*\colon \pi_1 M \to \pi_1 B$ is a homomorphism induced from p. Then the automorphism $(\varphi^n)_*\colon \pi_1 B \to \pi_1 B$ induced from φ^n carries any element t to $\eta t \eta^{-1}$. Since φ is periodic, we have $(\varphi^n)_*(t) = t$ for any $t \in \pi_1 B$. Hence η lies in the center of $\pi_1 B$ and therefore $\eta = 1$. Since the kernel of p_* is generated by any fiber, we obtain $\xi = \gamma^r$. $\square$

If B is orientable, Lemma 4.2 implies that $(f^n)_*$ is the identity map.

Let α_k be a proper arc in $\mathcal{N}(A_k, M)$ joining distinct components of $\partial\mathcal{N}(A_k, M) - \partial M$. If the loop $\alpha_k \cup f^n(\alpha_k)$ is homotopic to τ fibers in $\mathcal{N}(A_k, M)$, we call τ the *twist number* of f along $\mathcal{A}$ (see Figure 3). Let $M_{i,j}$ be a piece in $\mathcal{P}$ with an orientable base manifold such that $A_k \in \mathcal{A}_{i,j}$. Assume that $M_{i,j}$ is placed on only one side of A_k and that fibers in $M_{i,j} \cup \mathcal{N}(A_k, M)$ are oriented consistently. We consider the orientation of α_k outward from $M_{i,j}$. If the loop $\alpha_k \cup f^n(\alpha_k)$ endowed with the orinetation determined by $f^n(\alpha_k)$ is homotopic to $\tau_{i,j}^{(k)}$ fibers by an orientation-preserving homotopy, we call $\tau_{i,j}^{(k)}$ the *normalized twist number* of f for $M_{i,j}$ along A_k.

LEMMA 4.3. *If any piece in $\mathcal{P}$ has the nonorientable base manifold (see Figure 4), then the twist number of f along $\mathcal{A}$ is zero.*

Proof. Let $M_{i,j}$ and $M_{i',j'}$ be the pieces in $\mathcal{P}$ each placed on the mutually opposite sides of A_1. We fix $x_1 \in M_{i,j}$ and $x_2 \in M_{i',j'}$, and regard x_1 as the base point for $\pi_1 M$. Let $\gamma \in \pi_1 M$ be the homotopy class of infinite order represented by a fiber passing x_1. It follows from Lemma 4.2 that the automorphism $(f^n)_*\colon \pi_1 M \to \pi_1 M$ induced from f^n carries any element t to $\gamma^r t \gamma^{-r}$ for some integer r.

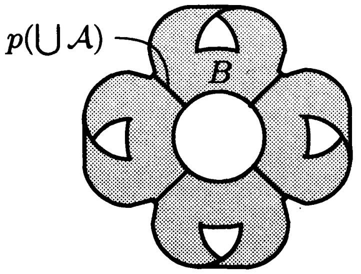


Figure 4

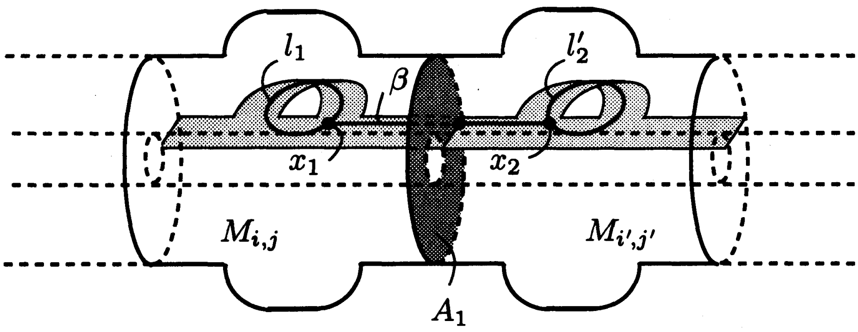


Figure 5

First take a closed path $l_1 \subset M_{i,j}$ with the base point x_1 such that $[l_1]\gamma = \gamma^{-1}[l_1]$ as illustrated in Figure 5. Then $f^n(l_1) = l_1$ implies $\gamma^r[l_1]\gamma^{-r} = \gamma^{2r}[l_1] = [l_1]$. Hence $r = 0$.

Next take a path $\beta \subset M$ from x_1 to x_2 intersecting $\bigcup \mathcal{A}$ in a point on A_1, and a closed path $l'_2 \subset M_{i',j'}$ with the base point x_2 such that the product path $l_2 = \beta l'_2 \beta^{-1}$ satisfies $[l_2]\gamma = \gamma^{-1}[l_2]$. Suppose that the twist number of f along $\mathcal{A}$ is τ. Then $(f^n)_*([l_2]) = \gamma^{\pm\tau}[l_2]\gamma^{\mp\tau} = \gamma^{\pm 2\tau}[l_2] = [l_2]$. Hence $\tau = 0$. $\square$

Suppose that the orbit O_i has an orientable base manifold and that all the fibers in O_i are oriented consistently. Recall that O_i has ν_i pieces. It should be noted that the S^1-bundle structure and the dual foliation of $M_{i,1}$ preserved by $f^{\nu_i}|_{M_{i,1}}$ induces a product bundle structure $p(M_{i,1}) \times S^1$ of $M_{i,1}$. Then we may assume that $f^{\nu_i}(x, e^{i\theta}) = (\varphi^{\nu_i}(x), e^{\varepsilon i(\theta + 2\pi\delta)})$ for any $(x, e^{i\theta}) \in p(M_{i,1}) \times S^1$ where $\varphi\colon B \to B$ is a homeomprhism induced from f, $\varepsilon = \pm 1$ and $\delta \in \mathbb{R}$. Define an isotopy $h'_s\colon M_{i,1} \times I \to M_{i,1}$ by $h'_s(x, e^{i\theta}, t) = (\varphi^{\nu_i}(x), e^{\varepsilon i(\theta + 2\pi(\delta + \frac{\nu_i st}{n}))})$. Here h'_s translates the image of $f^{\nu_i}|_{M_{i,1}}$ in the direction of fibers $\dfrac{\nu_i}{n}\varepsilon s$ times. Moreover define an isotopy $h_s\colon O_i \times I \to O_i$ by $h_s(x,t) = x$ on $O_i - M_{i,\nu_i}$ and $(h_s)_t = (h'_s)_t \circ f^{1-\nu_i}$ on M_{i,ν_i}. Then h_s translates the image of $f^n|_{O_i}$ in the direction of fibers s times. We call h_s the *s-translation* of f on O_i.

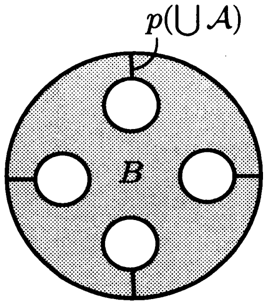


Figure 6

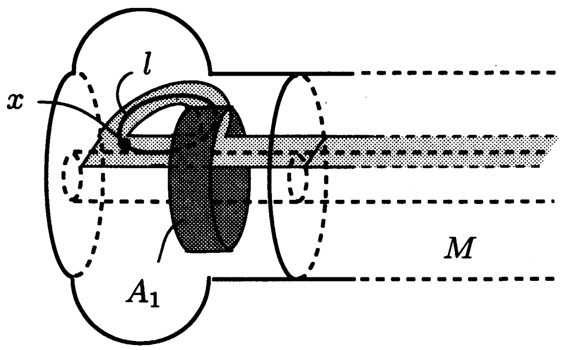


Figure 7

LEMMA 4.4. *If $\mathcal{P}$ has only one piece whose base manifold is orientable, then we can isotope f by an isotopy whose restriction on $E_{\mathcal{A}}$ is an s-translation, $s \in \mathbb{Q}$, so that afterwards the twist number of f along $\mathcal{A}$ is zero.*

Proof. Let τ be the twist number of f along $\mathcal{A}$. Let us fix a base point $x \in E_{\mathcal{A}}$ for $\pi_1 M$ and denote by $\gamma \in \pi_1 M$ the homotopy class of infinite order represented by the fiber passing x.

First assume that B is orientable. Figure 6 shows an example. There is a closed path $l \subset M$ with the base point x which intersects $\mathcal{A}$ at a point on A_1 as illustrated in Figure 7. Then we have $[l]\gamma = \gamma[l]$ and $(f^n)_*([l]) = \gamma^{\pm\tau}[l]$ where $(f^n)_* \colon \pi_1 M \to \pi_1 M$ is the automorphism induced from f^n. It follows from Lemma 4.2 that $(f^n)_*([l]) = \gamma^\tau[l]\gamma^{-\tau} = [l]$. Hence $\tau = 0$.

Next assume that B is nonorientable. Figure 8 shows an example. If $\mathcal{A}$ has only one annulus, then an isotopy of f whose restriction on $E_{\mathcal{A}}$ is either $\frac{\tau}{2}$ or $\left(-\frac{\tau}{2}\right)$-translation cancels the twist number. Suppose that $\mathcal{A}$ has two or more annuli. Then B is not the Möbius band. Let us fix a base point $x \in E_{\mathcal{A}}$. For each i let l_i be a closed path in M with base point x intersecting

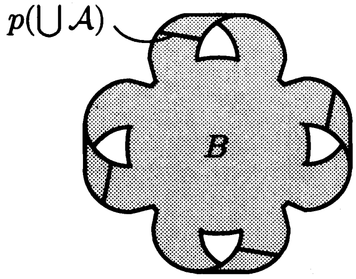


Figure 8

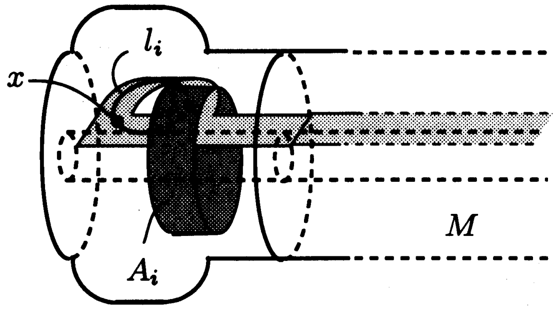


Figure 9

$\bigcup\mathcal{A}$ at a point on A_i as illustrated in Figure 9. Then we have $[l_i]\gamma = \gamma^{-1}[l_i]$. Suppose $(f^n)_*([l_i]) = \gamma^{\varepsilon_i\tau}[l_i]$ where $\varepsilon_i = \pm 1$. It follows from Lemma 4.2 that $(f^n)_*([l_i]) = \gamma^r[l_i]\gamma^{-r} = \gamma^{2r}[l_i]$. Hence $\varepsilon_i\tau = 2r$ for any i. Thus an isotopy whose restriction on $E_{\mathcal{A}}$ is $(-r)$-translation cancels the twist number. This completes the proof. □

LEMMA 4.5. *If f has only one orbit which consists of two or more pieses each having an orientable base manifold (see Figure 10), then we can isotope f by an isotopy which is an s-translation on $E_{\mathcal{A}}$, $s \in \mathbb{Q}$, so that afterwards the twist number of f along $\mathcal{A}$ is zero.*

Proof. Assume that the twist number τ of f along $\mathcal{A}$ is not zero. We denote by $\tau_{1,i}^{(j)}$ the normalized twist number of f for $M_{1,i}$ along $A_j \in \mathcal{A}_{1,i}$.

We claim that, for any piece $M_{1,i}$, f has a common normalized twist number along all annuli in $\mathcal{A}_{1,i}$ (see Figure 11). Assume the converse.

Let us recall that $\mathcal{P}$ contains ν_1 pieces. Let A_j and $A_{j'}$ be annuli in $\mathcal{A}_{1,i}$. Then $f^{j'-j}(A_j) = A_{j'}$. According as $f^{j'-j}(M_{1,i}) = M_{1,i}$ or $f^{j'-j}(M_{1,i}) \neq M_{1,i}$, $j' - j$ is or is not a multiple of ν_1. Then a permutation on $\mathcal{A}_{1,i}$ induced by

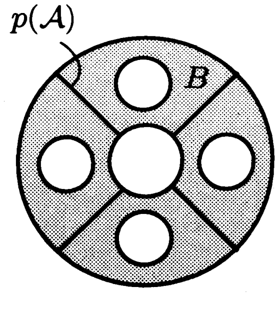


Figure 10

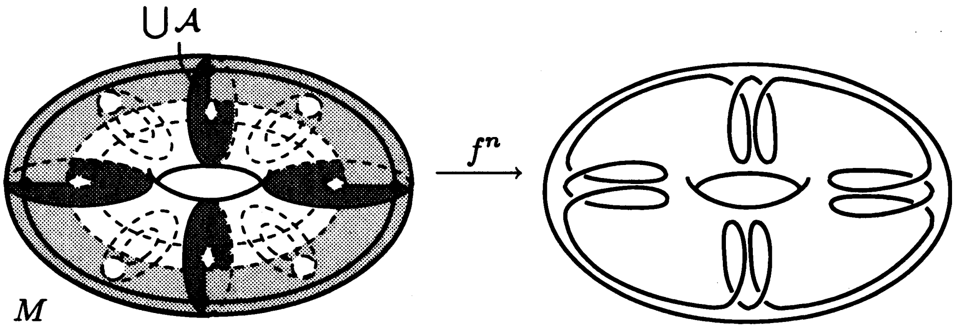


Figure 11

f^{ν_1} is a product of two disjoint cycles $\mathcal{C}_{1,i}^{(1)}$ and $\mathcal{C}_{1,i}^{(2)}$. We regard each $\mathcal{C}_{1,i}^{(k)}$ as a subset of $\mathcal{A}_{1,i}$. Let us consider a decomposition $\mathcal{A}_{1,i} = \mathcal{T}_{1,i}^{(\tau)} \cup \mathcal{T}_{1,i}^{(-\tau)}$ where $\mathcal{T}_{1,i}^{(t)} = \{A_j \in \mathcal{A}_{1,i} | \tau_{1,i}^{(j)} = t\}$. If f^{ν_1} preserves the orientation of fibers of $M_{1,i}$, we have $\{\mathcal{C}_{1,i}^{(1)}, \mathcal{C}_{1,i}^{(2)}\} = \{\mathcal{T}_{1,i}^{(\tau)}, \mathcal{T}_{1,i}^{(-\tau)}\}$. Otherwise we have both $\mathcal{C}_{1,i}^{(k)} \cap \mathcal{T}_{1,i}^{(\tau)} \neq \phi$ and $\mathcal{C}_{1,i}^{(k)} \cap \mathcal{T}_{1,i}^{(-\tau)} \neq \phi$ for each k.

Let us fix a piece M_{1,i_1} in $\mathcal{P}$. There are two annuli $A_{j_0} \in \mathcal{C}_{1,i_1}^{(1)}$ and $A_{j_1} \in \mathcal{C}_{1,i_1}^{(2)}$ along which f has different normalized twist number for M_{1,i_1}. Take a path α from a point $x \in A_{j_0}$ to $f^{j_1-j_0}(x) \in A_{j_1}$ through M_{1,i_1} such that $\alpha \cap (\bigcup \mathcal{A}) = \partial\alpha$. It should be noted that $f^{j_1-j_0}$ carries α to a path on the opposite side of A_{j_1}. Let σ be the minimal positive integer such that $(f^{j_1-j_0})^{\sigma}(A_{j_0}) = A_{j_0}$. We put $\alpha_k = (f^{j_1-j_0})^{k-1}(\alpha)$ and take a path β from $(f^{j_1-j_0})^{\sigma-1}(x)$ to x through $(f^{j_1-j_0})^{\sigma-1}(M_{1,i_1})$ such that $\beta \cap (\bigcup \mathcal{A}) = \partial\beta$. Then the product $l = \alpha_1\alpha_2\cdots\alpha_{\sigma-1}\beta$ is a closed path. Regard x as a base point for $\pi_1 M$. As we start from x and go along l, we obtain sequences $(A_{j_1}, A_{j_2}, \dots, A_{j_\sigma})$ of annuli in $\mathcal{A}$ where $j_\sigma = j_0$ and $(M_{1,i_1}, M_{1,i_2}, \dots, M_{1,i_\sigma})$ of pieces in $\mathcal{P}$ as illustrated in Figure 12. Then $\tau_{1,i_k}^{(j_{k-1})} = -\tau_{1,i_k}^{(j_k)}$ for any k. In other words, l is twisted by f^n along $A_{j_{k-1}}$ and A_{j_k} in the same direction with respect to the orientations of l and fibers of M_{1,i_k} for each k. This situation occurs only when $[l]\gamma = \gamma[l]$ where γ is the homotopy class of infinite order represented by the fiber passing x. Let

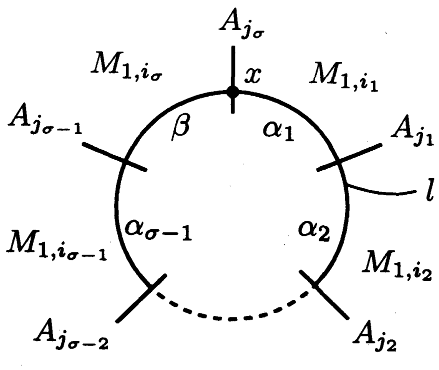


Figure 12

$(f^n)_*\colon \pi_1 M \to \pi_1 M$ be the isomorphism induced from f^n. Then $(f^n)_*([l]) = \gamma^{\pm\sigma\tau}[l]$. It follows from Lemma 4.2 that $(f^n)_*([l]) = [l]$. Therefore $\tau = 0$, contradiction. Hence our claim is proved.

Let A_j be any annulus in $\mathcal{A}$. Suppose that A_j is placed between $M_{1,i}$ and $M_{1,i'}$. We may assume that fibers in $M_{1,i} \cup A_j \cup M_{1,i'}$ are oriented consistently. Then $\tau_{1,i}^{(j)} = -\tau_{1,i'}^{(j)}$. Therefore, if $\tau \neq 0$, $f^{i'-i}|_{M_{1,i}}$ reverses the orientation of fibers. So s-translation of f on $E_{\mathcal{A}}$ isotopes the images of $f^n|_{M_{1,i}}$ and $f^n|_{M_{1,i'}}$ along fibers in the opposite directions s-times. Therefore an isotopy of f which is either $\frac{\tau}{2}$ or $\left(-\frac{\tau}{2}\right)$-translation on $E_{\mathcal{A}}$ cancels the twist number. This completes the proof. □

LEMMA 4.6. *Suppose that $\mathcal{P}$ contains a piece with an orientable basemanifold. If f has two orbits O_1 and O_2 , then either*

(1) *Any piece in O_1 has an orientable base manifold and we can isotope f by an isotopy which is s-translation on O_1, $s \in \mathbb{Q}$, and invariant on O_2 so that afterwards the twist number of f along $\mathcal{A}$ is zero, or*
(2) *Any piece in O_2 has an orientable base manifold and we can isotope f by an isotopy which is s-translation on O_2, $s \in \mathbb{Q}$, and invariant on O_1 so that afterwards the twist number of f along $\mathcal{A}$ is zero.*

Proof. Assume that the twist number τ of f along $\mathcal{A}$ is not zero. Without loss of generality, we may assume that any piece in O_1 has an orientable base manifold.

First assume that any piece in O_2 has the nonorientable base manifold (see Figure 13). We claim that, for any piece $M_{1,i}$ in O_1, f has a common normalized twist number along all annuli in $\mathcal{A}_{1,i}$. Let us fix a base point $x_{1,i} \in M_{1,i}$ for $\pi_1 M$. Let A_j be any annulus in $\mathcal{A}_{1,i}$, and $M_{2,i'}$ the piece placed on the side of A_j opposite to $M_{1,i}$. Let us take a path α_j from $x_{1,i}$ to a point $x_{2,i'} \in M_{2,i'}$ intersecting $\bigcup \mathcal{A}$ in a point on A_j. Since $M_{2,i'}$ has the nonorientable

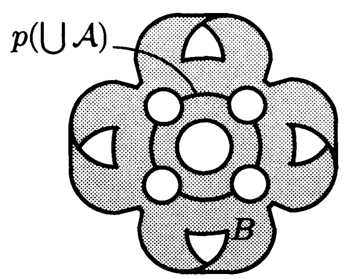


Figure 13

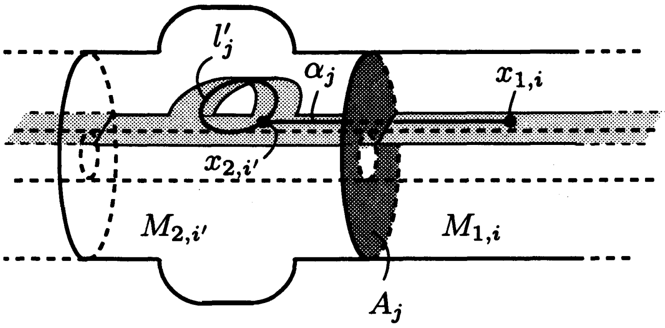


Figure 14

base manifold, there is a closed path $l'_j \subset M_{2,i'}$ with the base point $x_{2,i'}$ such that the product path $l_j = \alpha_j l'_j \alpha_j^{-1}$ satisfies $[l_j]\gamma = \gamma^{-1}[l_j]$ where γ is the homotopy class of infinite order represented by the fiber passing $x_{1,i}$ (see Figure 14). Let $\tau_{1,i}^{(j)}$ be the normalized twist number of f for $M_{1,i}$ along A_j. Then we have $(f^n)_*([l_j]) = \gamma^{\tau_{1,i}^{(j)}}[l_j]\gamma^{-\tau_{1,i}^{(j)}} = \gamma^{2\tau_{1,i}^{(j)}}[l_j]$ where $(f^n)_* \colon \pi_1 M \to \pi_1 M$ is the isomorphism induced from f^n. It follows from Lemma 4.2 that $(f^n)_*([l_j]) = \gamma^r[l_j]\gamma^{-r} = \gamma^{2r}[l_j]$. Therefore $\tau_{1,i}^{(j)} = r$ for any $A_j \in \mathcal{A}_{1,i}$. Hence our claim is proved. Then an isotopy of f which is $(-r)$-translation on O_1 and invariant on O_2 cancels the twist number.

Next assume that any piece in O_2 has an orientable base manifold (see Figure 15). We denote by $\tau_{i,j}^{(k)}$ the normalized twist number of f for $M_{i,j}$ along $A_k \in \mathcal{A}_{i,j}$. The following argument is essentially based on the same method used in the proof of Lemma 4.5.

We claim that, for any piece $M_{i,j}$, f has a common normalized twist number along all annuli in $\mathcal{A}_{i,j}$ (see Figure 16). Assume the converse. For each piece $M_{i,j}$ in $\mathcal{P}$ we consider a decomposition $\mathcal{A}_{i,j} = \mathcal{T}_{i,j}^{(\tau)} \cup \mathcal{T}_{i,j}^{(-\tau)}$ where $\mathcal{T}_{i,j}^{(t)} = \{A_k \in \mathcal{A}_{i,j} | \tau_{i,j}^{(k)} = t\}$. Let us fix a piece M_{1,i_1}. We take two annuli $A_{j_0} \in \mathcal{T}_{1,i_1}^{(-\tau)}$ and $A_{j_1} \in \mathcal{T}_{1,i_1}^{(\tau)}$. Let M_{2,i_2} be the piece placed on the side of A_{j_1} opposite to M_{1,i_1}.

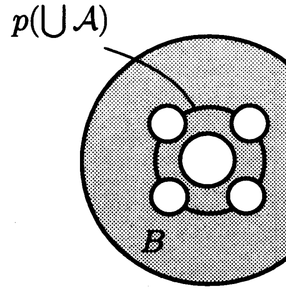


Figure 15

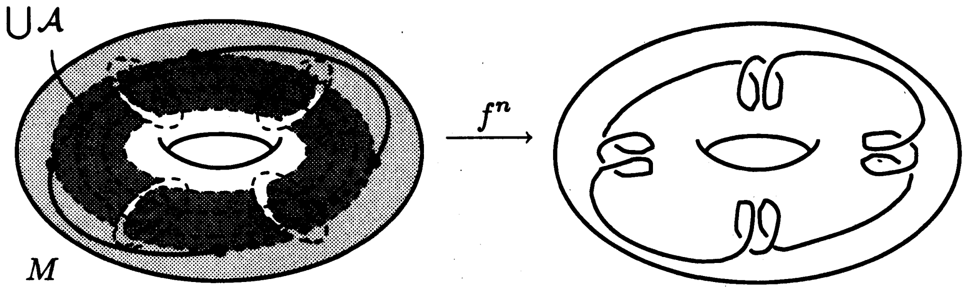


Figure 16

If $A_{j_1} \in \mathcal{T}^{(-\tau)}_{2,i_2}$, we take an annulus $A_{j_2} \in \mathcal{T}^{(\tau)}_{2,i_2}$. Otherwise $A_{j_2} \in \mathcal{T}^{(-\tau)}_{2,i_2}$. Let α be a path from a point $x \in A_{j_0}$ to $f^{j_2-j_0}(x) \in A_{j_2}$ through M_{1,i_1} and M_{2,i_2} such that $\alpha \cap (\bigcup\mathcal{A}) = \{x, f^{j_2-j_0}(x), \text{a point on } A_{j_1}\}$. Note that $f^{j_2-j_0}$ carries α to a path on the opposite side of A_{j_2}. Let σ be the minimal positive integer such that $(f^{j_2-j_0})^{\sigma}(A_{j_0}) = A_{j_0}$. We put $\alpha_k = (f^{j_2-j_0})^{k-1}(\alpha)$ and take a path β from $(f^{j_2-j_0})^{\sigma-1}(x)$ to x through $(f^{j_2-j_0})^{\sigma-1}(M_{1,i_1})$ and $(f^{j_2-j_0})^{\sigma-1}(M_{2,i_2})$ such that $\beta \cap (\bigcup\mathcal{A}) = \{x, (f^{j_2-j_0})^{\sigma-1}(x), \text{a point on } (f^{j_2-j_0})^{\sigma-1}(A_{j_1})\}$. Then the product $l = \alpha_1\alpha_2\cdots\alpha_{\sigma-1}\beta$ is a closed path. Regard x as a base point for $\pi_1 M$. As we start from x and go along l, we obtain sequences $(A_{j_1}, A_{j_2}, \ldots, A_{j_{2\sigma}})$ of annuli in $\mathcal{A}$ where $j_{2\sigma} = j_0$ and $(M_{1,i_1}, M_{2,i_2}, \ldots, M_{2,i_{2\sigma}})$ of pieces in $\mathcal{P}$ (see Figure 17). Then $\tau^{(j_{2k-2})}_{1,i_{2k-1}} = -\tau^{(j_{2k-1})}_{1,i_{2k-1}}$ and $\tau^{(j_{2k-1})}_{2,i_{2k}} = -\tau^{(j_{2k})}_{2,i_{2k}}$ for $1 \le k \le \sigma$. This situation occurs only when $[l]\gamma = \gamma[l]$ where γ is the homotopy class of infinite order represented by the fiber passing x. We obtain $(f^n)_*([l]) = \gamma^{\pm 2\sigma\tau}[l]$ where $(f^n)_*\colon \pi_1 M \to \pi_1 M$ is the isomorphism induced from f^n. It follows from Lemma 4.2 that $(f^n)_*([l]) = [l]$. Therefore $\tau = 0$, contradiction. Hence our claim is proved.

Consequently an isotopy of f which is τ or $(-\tau)$-translation on O_1 and invariant on O_2 cancels the twist number. This completes the proof. □

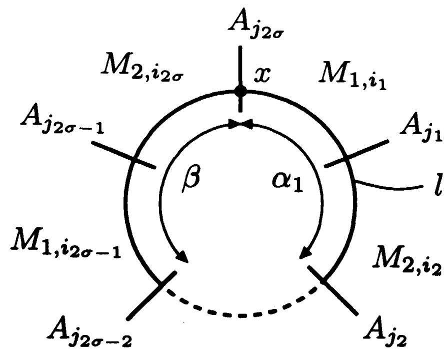


Figure 17

4.3 The realization theorem for S^1-bundles over surfaces

Let M be a compact, connected, orientable 3-manifold whose boundary is a union of some tori. Suppose that M is endowed with an S^1-bundle structure $\mathcal{S}$ and a dual foliation $\mathcal{F}$. First suppose that the base manifold B of M is orientable. Then $\mathcal{S}$ and $\mathcal{F}$ induce a product bundle structure $B \times S^1$ of M. So we define $\Phi_{\mathcal{F}}\colon M \to S^1$ as the natural projection map. Next suppose that B is nonorientable. Let $\tilde{B}$ be an orientable double cover of B and $T\colon \tilde{B} \to \tilde{B}$ a covering transformation. Then M is covered by $\tilde{B} \times S^1$ with a covering transformation $(x, y) \to (T(x), \bar{y})$ for $(x, y) \in \tilde{B} \times S^1$, and this product bundle structure induces $\mathcal{S}$ and $\mathcal{F}$. So we define $\Phi_{\mathcal{F}}\colon M \to S^1_+ = \{x \in S^1 | \mathrm{Im}\ x \geq 0\}$ as the map induced from natural projection map $\tilde{B} \times S^1 \to S^1$. In both cases, we call $\Phi_{\mathcal{F}}$ the *projection map* of $\mathcal{F}$.

The following proposition is the S^1-bundle version of Theorem 1.4.

PROPOSITION 4.7. *Let M be a compact, orientable S^1-bundle over a surface such that ∂M is a union of some tori. For any homeomorphism $f\colon M \to M$ representing a mapping class of order n, there is a homeomorphism $g\colon M \to M$ of period n which is isotopic to f. Moreover, if f is isotopic to a fiber-preserving homeomorphism, g can be chosen so as to preserve both the S^1-bundle structure and a dual foliation of M.*

Proof. Let us consider a permutation on the set of all connected components of M induced by f which is written as a product of disjoint cycles as follows:

$$(M_{1,1}\ M_{1,2}\ \cdots\ M_{1,\nu_1})(M_{2,1}\ M_{2,2}\ \cdots\ M_{2,\nu_2})\ \cdots\ (M_{\kappa,1}\ M_{\kappa,2}\ \cdots\ M_{\kappa,\nu_\kappa})$$

Suppose that $f^{\nu_i}|_{M_{i,1}}$ represents a mapping class of order n_i. Assume that there is a homeomorphism $g_i\colon M_{i,1} \to M_{i,1}$ of period n_i which is isotopic to $f^{\nu_i}|_{M_{i,1}}$. Then we obtain a required periodic homeomorphism $g\colon M \to M$ isotopic to f

as follows:

$$g=\begin{cases} f & \text{on } M-\bigcup_{i=1}^{\kappa} M_{i,\nu_i} \\ g_i \circ f^{1-\nu_i} & \text{on } M_{i,\nu_i} \end{cases}$$

Here g has period n on each $M_{i,j}^*$ as is verified as follows:

$$\begin{aligned} g^n|_{M_{i,j}} &= \left(g|_{M_{i,j-1}} \circ \cdots \circ g|_{M_{i,1}} \circ g|_{M_{i,\nu_i}} \circ g|_{M_{i,\nu_i-1}} \circ \cdots \circ g|_{M_{i,j}}\right)^{\frac{n}{\nu_i}} \\ &= \left(f|_{M_{i,j-1}} \circ \cdots \circ f|_{M_{i,1}} \circ g_i \circ (f|_{M_{i,1}})^{-1} \circ \cdots \circ (f|_{M_{i,j-1}})^{-1}\right)^{\frac{n}{\nu_i}} \\ &= f|_{M_{i,j-1}} \circ \cdots \circ f|_{M_{i,1}} \circ g_i^{\frac{n}{\nu_i}} \circ (f|_{M_{i,1}})^{-1} \circ \cdots \circ (f|_{M_{i,j-1}})^{-1} \\ &= id_{M_{i,j}}. \end{aligned}$$

Moreover assume that f is fiber-preserving and each g_i preserves both the Seifert fibration and a dual foliation of $M_{i,1}$. We consider each $M_{i,j}$ endowed with the dual foliation induced from that of $M_{i,1}$ by g^{j-1}. These foliations define a dual foliation of M which is preserved by g. Hence it is sufficient to prove the proposition when M is connected.

The case in which M is homeomorphic to either $D^2 \times S^1$ or $S^1 \times S^1 \times I$ follows from Proposition 4.1. In the other case, it follows from [26, Satz 10.1] that f is isotopic to a fiber-preserving homeomorphism. We therefore assume that f preserves the S^1-bundle structure $\mathcal{S}$ of M. Moreover it follows from [27, Remark in P.85] that f^n is isotopic to id_M by a fiber-preserving isotopy. Let B be the base manifold of M and $\varphi\colon B \to B$ the homeomorphism induced from f. Then φ represents the mapping class of order n' where n' is a divisor of n. By Theorem 1.1 we may assume that φ has period n'.

We carry out the proof by induction on the minimal length of a hierarchy of M defined by saturated, essential annuli reducing M to solid tori.

The basic step follows from Proposition 4.1.

We assume that M is not a solid torus. Let $\mathcal{A} = \{A_1, \dots, A_\mu\}$ be an annulus system for f. We put $N_i = \mathcal{N}(A_i, M)$ for $1 \le i \le \mu$. Suppose that $E_{\mathcal{A}} = cl(M - \mathcal{N}(\bigcup \mathcal{A}, M))$ is endowed with the S^1-bundle structure $\mathcal{S}_{\mathcal{A}}$ induced from $\mathcal{S}$. By the induction hypothesis, we assume that $f|_{E_{\mathcal{A}}}$ is periodic and preserves both $\mathcal{S}_{\mathcal{A}}$ and a dual foliation $\mathcal{F}_{\mathcal{A}}$ of $E_{\mathcal{A}}$. Moreover, by Section 4.2, we may assume that the twist number of f along $\mathcal{A}$ is zero.

First we parametrize N_1. Let us consider the i-th orbit O_i of f. Suppose that any piece in O_i has orientable (resp. nonorientable) base manifold. Let $\Phi_i\colon M_{i,1} \to S^1$ (resp. S^1_+) be a projection map of the foliation of $M_{i,1}$ induced by $\mathcal{F}_{\mathcal{A}}$. By modifying the parametrization of S^1 (resp. S^1_+) if necessary, we assume that the periodic auto-homeomorphism of S^1 (resp. S^1_+) induced from $f^{\nu_i}|_{M_{i,1}}$ by Φ_i is isometric. Let $\Psi_i\colon O_i \to S^1$ (resp. S^1_+) be a map such that $\Psi_i = \Phi_i \circ f^{1-j}$ on $M_{i,j}$ for $1 \le j \le \nu_i$. Suppose A_1 is placed between M_{i_0,j_0} and M_{i_1,j_1}. We consider N_1 parametrized as $S^1 \times I \times I$ so that:

(1) $S^1 \times I \times \{t\} = M_{i_t,j_t} \cap N_1$ for $t \in \partial I$,
(2) $\mathcal{S}$ induces the fibration $\{S^1 \times \{(s,t)\} | (s,t) \in I \times I\}$ of N_1,
(3) if M_{i_t,j_t} has an orientable base manifold, $\Psi_{i_t}(x,s,t) = x$ or $\bar{x}$ for $(x,s,t) \in S^1 \times I \times \partial I$, and
(4) if M_{i_t,j_t} has the nonorientable base manifold, $\Phi_{i_t}(x,s,t) = x$ for $(x,s,t) \in S^1_+ \times I \times \partial I$ and $\Psi_{i_t}(x,s,t) = \bar{x}$ for $(x,s,t) \in (S^1 - S^1_+) \times I \times \partial I$.

Next we extend $f|_{E_\mathcal{A}}$ to a periodic map on M. Since $\mathcal{A}$ has μ annuli, we have $f^\mu(N_1) = N_1$. Let us recall that f^μ does not exchange the sides of A_1. Since $f|_{E_\mathcal{A}}$ preserves both $\mathcal{S}_\mathcal{A}$ and $\mathcal{F}_\mathcal{A}$, we may assume that $f^\mu(x,s,t) = (\xi_t(x), \eta_t(s), t)$ for $(x,s,t) \in S^1 \times I \times \partial I$ where $\xi_t \colon S^1 \to S^1$ and $\eta_t \colon I \to I$ are homeomorphisms. Note that ξ_0 and ξ_1 are isometries and that the twist number of f along $\mathcal{A}$ is zero. Therefore $\xi_0 = \xi_1$. We define a periodic homeomorphism $g_{N_1} \colon N_1 \to N_1$ by $g_{N_1}(x,s,t) = (\xi_0(x), (1-t)\eta_0(s) + t\eta_1(s), t)$. It follows from Lemma 2.2 that $g_{N_1} \simeq f^\mu|_{N_1}$ relative to $cl(\partial N_1 - \partial M)$. Moreover we define a homeomorphism $g \colon M \to M$ by

$$g = \begin{cases} f & \text{on } M - f^{-1}(N_1), \\ g_{N_1} \circ f^{1-\mu} & \text{on } f^{-1}(N_1). \end{cases}$$

Then g is a periodic homeomorphism which is isotopic to f and preserves $\mathcal{S}$. Let us consider the D^2-fibration $\{\{x\} \times I \times I | x \in S^1\}$ of N_1 and those of N_i induced by f^{1-i} for $1 < i \leq \mu$. These D^2-fibrations together with $\mathcal{F}_\mathcal{A}$ determines a dual foliation $\mathcal{F}$ of M preserved by g.

Although g is periodic, there is a possibility that the period of g is not n. Note that the auto-homeomorphism of B induced from g has period n'. If B is orientable, the $\mathcal{S}$ and $\mathcal{F}$ induce a product bundle structure of M. Therefore some s-translation of g on M makes g a homeomorphism of period n. Otherwise g does either preserve or exchange the two exceptional leaves of $\mathcal{F}$, and we have either $n = n'$ or $n = lcm\{2, n'\}$ respectively. Then the period of g is just n. This completes the proof. □

4.4 The realization theorem for Seifert 3-manifolds

The following theorem is the Seifert 3-manifold version of Theorem 1.4, which is proved by Zimmermann using algebraic methods (see Satz 2.1, Lemma 2.2, Lemma 2.3, and Step C in the proof of Satz 0.1 in [31]). Now we prove this theorem using topological methods.

THEOREM 4.8 (Zimmermann). *Let M be a compact Seifert manifold whose boundary is a union of some tori. For any homeomorphism $f \colon M \to M$ representing a mapping class of order n, there is a homeomorphism $g \colon M \to M$ of period n which is isotopic to f. Moreover, if f is isotopic to a fiber-preserving*

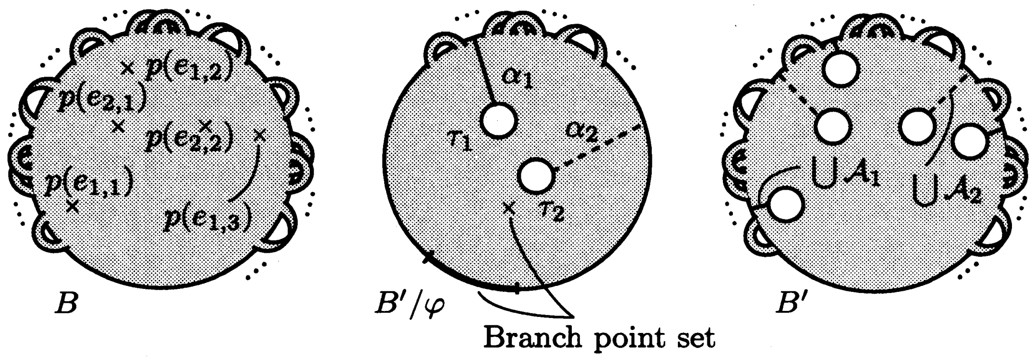


Figure 18

homeomorphism, g can be chosen so as to preserve both the Seifert fibration and a dual foliation of M.

Proof. As in the proof of Proposition 4.7, it is sufficient to prove the theorem when the Seifert manifold M is connected. If M is homeomorphic to either $D^2 \times S^1$ or $S^1 \times S^1 \times I$, the theorem follows from Proposition 4.1. Otherwise it follows from [26, Satz 10.1] that f is isotopic to a fiber-preserving homeomorphism. We therefore assume that f preserves the Seifert fibration $\mathcal{S}_M$ of M. It should be noted that by [27, Remark in p.85] f^n is isotopic to id_M by a fiber-preserving isotopy. Let B be the orbit-manifold of M, $p\colon M \to B$ the projection map, and $\varphi\colon B \to B$ the homeomorphism induced from f. By Theorem 1.1, we assume that φ is periodic.

Let $\mathcal{E}$ denotes the set of all exceptional fibers of M. Then f induces a permutation on $\mathcal{E}$ which can be written as a product of disjoint cycles as follows:

$$(e_{1,1}\ e_{1,2}\ \cdots\ e_{1,\nu_1})(e_{2,1}\ e_{2,2}\ \cdots\ e_{2,\nu_2})\ \cdots\ (e_{\kappa,1}\ e_{\kappa,2}\ \cdots\ e_{\kappa,\nu_\kappa})$$

We denote by o_i the common order of $e_{i,1}, e_{i,2}, \dots, e_{i,\nu_i}$ for each i. Suppose that $M' = M - int\mathcal{N}(\bigcup\mathcal{E}, M)$ is a saturated submanifold endowed with the S^1-bundle structure $\mathcal{S}_{M'}$ induced from $\mathcal{S}_M$. We may assume $f(M') = M'$. By Proposition 4.7, we assume that f is periodic on M' and preserves both $\mathcal{S}_{M'}$ and a dual foliation $\mathcal{F}^{(0)}_{M'}$ of M'. Let $\Phi_{M'}$ be a projection map of $\mathcal{F}^{(0)}_{M'}$. We put $V_{i,j} = \mathcal{N}(e_{i,j}, M)$ and $B' = p(M')$.

First we construct a new foliation of M'. Let us consider the orbit surface B'/φ. Then each $p(\partial V_{i,j}) \subset \partial B'$ descends to an arc or a loop $\tau_i \subset \partial(B'/\varphi)$. Take disjoint arcs $\alpha_1, \alpha_2, \dots, \alpha_\kappa$ on B'/φ such that each α_i joins τ_i and $\partial(B'/\varphi) - \bigcup_{i=1}^{\kappa}\tau_i$ and misses any branch points. Then we obtain an annulus system $\mathcal{A}_i$ for $f|_{M'}$ for each i (see Figure 18). Suppose that $\partial V_{i,1}$ meets σ_i annuli in $\mathcal{A}_i$ and $\mathcal{A}_i = \{A_{i,1}, A_{i,2}, \dots, A_{i,\sigma_i\nu_i}\}$ where $A_{i,j} = f^{j-1}(A_i, 1)$ for $1 < j \le \sigma_i\nu_i$ and

$A_{i,1} = f(A_{i,\sigma_i\nu_i})$. We may assume that each $N_{i,j} = \mathcal{N}(A_{i,j}, M')$ is saturated and satisfies $f(\bigcup_{j=1}^{\sigma_i\nu_i} N_{i,j}) = \bigcup_{j=1}^{\sigma_i\nu_i} N_{i,j}$. For each i we consider $N_{i,1}$ parametrized as $S^1 \times I \times I$ so that:

(1) $S^1 \times I \times \partial I = cl(\partial N_{i,1} - \partial M')$,
(2) $\mathcal{S}_{M'}$ induces a fibration $\{S^1 \times \{(s,t)\} | (s,t) \in I \times I\}$ of $N_{i,1}$,
(3) if B' is orientable, $\Phi_{M'}(x,s,t) = x$ for $(x,s,t) \in S^1 \times I \times I$, and
(4) if B' is nonorientable, $\Phi_{M'}(x,s,t) = x$ for $(x,s,t) \in S^1_+ \times I \times I$ and $\Phi_{M'}(x,s,t) = \bar{x}$ for $(x,s,t) \in (S^1 - S^1_+) \times I \times I$.

Note that f^{ν_i} preserves three fibrations of $\partial V_{i,1}$; the one induced from $\mathcal{S}_M$, the one induced from $\mathcal{F}^{(0)}_{M'}$, and the meridians. Therefore f^{ν_i} does either preserve or reverse the orientations of all these fibrations. Let us consider a D^2-fibration of $N_{i,1}$ whose leaf passing $(x,0,0)$ is $\{(xe^{\frac{2\pi i}{o_i\sigma_i}t}, s, t) \in S^1 \times I \times I | s,t \in I\}$, and those of $N_{i,j}$ induced by f^{j-1} for $1 < j \leq \sigma_i\nu_i$. These fibrations together with the foliation of $M' - \bigcup_{i=1}^{\kappa}\bigcup_{j=1}^{\sigma_i\nu_i} N_{i,j}$ induced by $\mathcal{F}^{(0)}_{M'}$ determines a new foliation $\mathcal{F}^{(1)}_{M'}$ of M'. Let $o = lcm\{o_1, o_2, \ldots, o_\kappa\}$. It should be noted that each leaf of $\mathcal{F}^{(1)}_{M'}$ is an o-fold cover of the corresponding leaf of $\mathcal{F}^{(0)}_{M'}$.

Next we modify $\mathcal{F}^{(1)}_{M'}$ so that any leaf has a meridional boundary component on each $\partial V_{i,j}$. Assume that $\partial V_{i,1}$ meets $N_{i,1}$ and fibers on $V_{i,1} \cup N_{i,1}$ are oriented consistently. Let us consider the fibration of $\partial V_{i,1}$ induced by $\mathcal{F}^{(1)}_{M'}$. Suppose that any fiber goes p_i times in the direction of $e_{i,1}$. It should be noted that any fiber meets annuli in $\mathcal{A}_i$ in $o_i\sigma_i$ points and that the loop $\partial V_{i,1} \cap A_{i,1}$ goes o_i times in the direction of $e_{i,1}$. Consider a new D^2-fibration of $N_{i,1}$ whose leaf passing $(x,0,0)$ is $\{(xe^{\frac{2\pi i}{o_i\sigma_i}t(1-\frac{p_i}{o_i})}, s, t) \in S^1 \times I \times I | s,t \in I\}$, and those of $N_{i,j}$ induced by f^{j-1} for $1 < j \leq \sigma_i\nu_i$. Then the foliation $\mathcal{F}^{(2)}_{M'}$ of M' determined by these D^2-fibrations together with the foliation of $M' - \bigcup_{i=1}^{\kappa}\bigcup_{j=1}^{\sigma_i\nu_i} N_{i,j}$ induced by $\mathcal{F}^{(1)}_{M'}$ has the required property.

Finally we construct the required realization. Suppose that $\mathcal{S}_M$ induces the fibration of $V_{i,1}$ of type (o_i, ρ_i). We consider $V_{i,1}$ parametrized as $D^2 \times S^1$ so that:

(1) $\mathcal{S}_M$ induces the fibration of $V_{i,1}$ whose fiber passing $(re^{i\theta}, 1) \in D^2 \times S^1$ is the loop $\{(re^{i(\theta+\rho_i\varphi)}, e^{io_i\varphi}) \in D^2 \times S^1 | \varphi \in \mathbb{R}\}$, and
(2) $\mathcal{F}^{(2)}_{M'}$ induces the fibration $\{S^1 \times \{y\} | y \in S^1\}$ of $\partial V_{i,1}$.

Since $f|_{M'}$ preserves both $\mathcal{S}_{M'}$ and $\mathcal{F}^{(2)}_{M'}$, we may suppose $(f^{\nu_i}|_{\partial V_{i,1}})(e^{i\theta}, e^{i\varphi}) = (e^{i\xi(\theta)}, e^{i\zeta(\varphi)})$ for $(e^{i\theta}, e^{i\varphi}) \in \partial D^2 \times S^1$ where $\xi\colon \mathbb{R} \to \mathbb{R}$ and $\zeta\colon \mathbb{R} \to \mathbb{R}$ are homeomorphisms. Define a homeomorphism $g_i\colon V_{i,1} \to V_{i,1}$ by $g_i(re^{i\theta}, e^{i\varphi}) = (re^{i\xi(\theta)}, e^{i\zeta(\varphi)})$. Then we obtain a homeomorphism $g\colon M \to M$ of period n as follows:

$$g = \begin{cases} f & \text{on } M - \bigcup_{i=1}^{\kappa} N_{i,\sigma_i\nu_i} \text{ and} \\ g_i \circ f^{1-\sigma_i\nu_i} & \text{on } N_{i,\sigma_i\nu_i}. \end{cases}$$

It follows from Lemma 2.1 that g is isotopic to f. Let us consider the D^2-fibration $\{D^2 \times \{x\} | x \in S^1\}$ of each $N_{i,1}$ and those of $N_{i,j}$ induced by g^{j-1} for $1 < j \leq \sigma_i\nu_i$. Then these D^2-fibrations together with $\mathcal{F}^{(2)}_{M'}$ determine a dual foliation of M preserved by g. This completes the proof. □

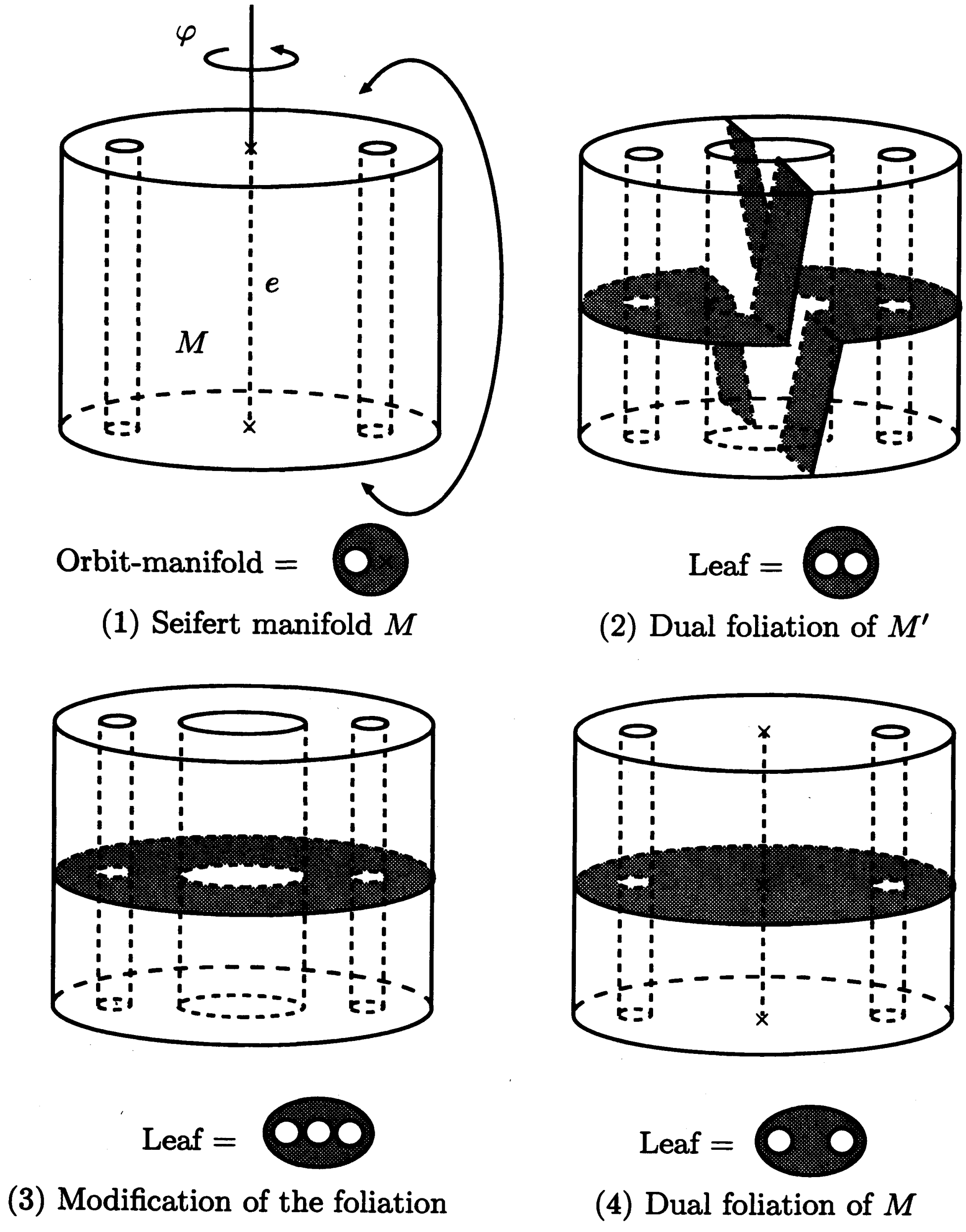
REMARK 4.9. Figure 19 shows an example of the modifications of foliations in the above proof. Let V be a disk with two holes and $\varphi\colon V \to V$ an orientation preserving homeomorphism of period two exchanging the holes. We consider a Seifert manifold M which is the quotient of $V \times I$ via the identification $(x, 0) = (\varphi(x), 1)$. Note that M has an exceptional fiber e of order two as illustrated in (1). For the sake of simplicity, we suppose that $f\colon M \to M$ is a π-rotation around e. Then (2) shows a dual foliation of M'. In this figure two leaves are illustrated. Next we construct a new foliation of M' such that each leaf is a double cover of the original one, as illustrated in (3). Finally this foliation extends to a dual foliation of M, as illustrated in (4).

5. Haken 3-Manifolds

Let M be a Haken manifold such that ∂M is either empty or a union of some tori. Let $\mathcal{T}$ be a finite set of disjoint, non-parallel, essential tori embedded in M obtained by Jaco, Shalen and Johannson's Canonical Torus Decomposition Theorem (see [13] and [14]). Then the closure of each component of $M - int\mathcal{N}(\bigcup \mathcal{T}, M)$, which we call a piece, is either a Seifert manifold or a simple manifold. We consider any piece homeomorphic to a Seifert manifold Seifert fibered. In particular, we assume that any piece homeomorphic to the twisted I-bundle over the Klein bottle has orbit-manifold a disk and two exceptional fibers. We call a Seifert fibered piece simply a *Seifert piece.* We may assume by [16, Theorem B in Chapter V] that each of the pieces other than Seifert pieces is endowed with a complete hyperbolic structure of finite volume in its interior, which we call a *hyperbolic piece.*

We consider M oriented and each piece endowed with the orientation induced from that of M. For each Seifert piece with an orientable oribit-manifold, we consider the oribit-manifold oriented and each fiber endowed with the orientation induced from those of the piece and the orbit-manifold.

Let $\mathcal{P}$ denotes the set of all pieces. Suppose that f is an arbitrary auto-homeomorphism of M representing a mapping class of finite order n. Isotope f



(1) Seifert manifold M

(2) Dual foliation of M'

(3) Modification of the foliation

(4) Dual foliation of M

Figure 19

so that afterwards $f(\bigcup \mathcal{T}) = \bigcup \mathcal{T}$ and $f(\mathcal{N}(\bigcup \mathcal{T}, M)) = \mathcal{N}(\bigcup \mathcal{T}, M)$. Then f induces a permutation on a finite set $\mathcal{P}$ which can be written as a product of disjoint cycles as follows:

$$(M_{1,1}\ M_{1,2}\ \cdots\ M_{1,\nu_1})(M_{2,1}\ M_{2,2}\ \cdots\ M_{2,\nu_2})\ \cdots\ (M_{\kappa,1}\ M_{\kappa,2}\ \cdots\ M_{\kappa,\nu_\kappa})$$

We call $O_i = \bigcup_{j=1}^{\nu_i} M_{i,j}$ the i-th *orbit* of f.

5.1 Fundamental groups

Suppose that a piece $M_{i,j} \in \mathcal{P}$ is placed on a side of a torus $T \in \mathcal{T}$. Since T is homotopic to a component of $\partial M_{i,j}$, we may regard $\pi_1 T$ as a subgroup of $\pi_1 M_{i,j}$. The incompressibility of T implies that $\pi_1 M_{i,j}$ has a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ corresponding to T. Therefore a nontrivial loop l on T generates an infinite cyclic subgroup of $\pi_1 M_{i,j}$. We will study conditions for this subgroup to lie in the center of $\pi_1 M_{i,j}$.

LEMMA 5.1. *Let $M_{i,j}$ be a hyperbolic piece in $\mathcal{P}$ with nonempty boundary, l a nontrivial loop on $\partial M_{i,j}$. Then $[l] \in \pi_1 M_{i,j}$ does not lies in the center of $\pi_1 M_{i,j}$.*

Proof. Assume the converse. It follows from [2, Proposition D.3.18.] that $\partial M_{i,j}$ is incompressible. Therefore $[l]$ generates an infinite cyclic subgroup of the center of $\pi_1 M_{i,j}$. Hence it follows from [12, Theorem VI.24.] that $M_{i,j}$ is homeomorphic to a Seifert manifold. Recall that hyperbolic pieces admit no Seifert fibration, contradiction. □

LEMMA 5.2. *Let $M_{i,j}$ be a Seifert piece in $\mathcal{P}$ which is homeomorphic to neither $D^2 \times S^1$ nor $S^1 \times S^1 \times I$. Let l be a nontrivial loop on $\partial M_{i,j}$. Then $[l] \in \pi_1 M_{i,j}$ lies in the center of $\pi_1 M_{i,j}$ if and only if $M_{i,j}$ has an orientable orbit-manifold and $[l]$ lies in the subgroup generated by any regular fiber.*

Proof. Our argument will require three steps; (1) the existence of a Seifert fibration of $M_{i,j}$ which satisfy the necessity of the lemma, (2) the consistency with the originally given Seifert fibration of $M_{i,j}$, and (3) the sufficiency of the lemma.

Step 1. We claim that if $[l]$ lies in the center of $\pi_1 M_{i,j}$, then $M_{i,j}$ admits a Seifert fibration such that the orbit-manifold is orientable and $[l]$ lies in the group generated by any regular fiber.

It follows from [22, Corollary 3.3] that $\partial M_{i,j}$ is incompressible. Therefore $[l]$ generates an infinite cyclic normal subgroup of $\pi_1 M_{i,j}$. Hence it follows from [5, Corollary 12.8] that $M_{i,j}$ admits a Seifert fibration such that $[l]$ lies in the group generated by any regular fiber. It should be noted that at this step this Seifert fibration is not necessarily isotopic to the originally given one.

It suffices to show that $M_{i,j}$ has an orientable orbit-manifold. Assume the converce. Suppose that $M_{i,j}$ has p exceptional fibers and a nonorientable orbit-manifold of genus q with r boundary components. Then [12, VI.10] shows that the fundamental group of $M_{i,j}$ is presented as follows:

$$\begin{aligned}\pi_1 M_{i,j} \cong \quad & \langle \gamma_1, \dots, \gamma_q, \delta_1, \dots, \delta_p, \varepsilon_1, \dots, \varepsilon_r, \lambda \mid \\ & \qquad \gamma_i \lambda \gamma_i^{-1} = \lambda^{-1}, [\delta_j, \lambda] = 1, [\varepsilon_k, \lambda] = 1, \\ & \qquad \delta_j^{s_j} = \lambda^{t_j}, \lambda^u = \gamma_1^2 \cdots \gamma_q^2 \delta_1 \cdots \delta_p \varepsilon_1 \cdots \varepsilon_r \rangle\end{aligned}$$

where each s_j is the index of the j-th exceptional fiber, $0 < t_j < s_j$ for $1 \le j \le p$, and u is an integer. Moreover λ is represented by any regular fiber and generates an infinite cyclic normal subgroup of $\pi_1 M_{i,j}$ by [12, VI.11]. We therefore have $[\lambda, \gamma_i] \neq 1$ and hence $[[l], \gamma_i] \neq 1$, contradiction. Hence our claim was proved.

Step 2. Next we claim that the Seifert fibration of $M_{i,j}$ obtained in Step 1 is isotopic to the originally given one.

If $M_{i,j}$ is not homeomorphic to the twisted I-bundle over the Klein bottle, our claim follows from [26, Satz 10.1].

Assume that $M_{i,j}$ is homeomorphic to the twisted I-bundle over the Klein bottle. It follows from [12, VI.5] that the twisted I-bundle over the Klein bottle is a Sefiert manifold in two ways:

(1) it has an orbit-manifold the disk and two exceptional fibers, each of index two, or

(2) it has an orbit-manifold the Möbius band and no exceptional fibers.

Recall that any piece homeomorphic to the twisted I-bundle over the Klein bottle is assumed to be endowed with a Seifert fibration of type (1). Therefore two Seifert fibrations of $M_{i,j}$ are both of type (1). Hence our claim follows from [26, Satz 10.1].

Step 3. Finally we claim that if $M_{i,j}$ has an orientable orbit-manifold and $[l]$ lies in the group generated by any regular fiber, then $[l]$ lies in the center of $\pi_1 M_{i,j}$. Suppose that $M_{i,j}$ has p exceptional fibers and the orbit-manifold of genus q with r boundary components. Then it follows from [12, VI.9] that the fundamental group of $M_{i,j}$ is presented as follows:

$$\begin{aligned}\pi_1 N \cong \quad & \Big\langle \gamma_1, \xi_1, \dots, \gamma_q, \xi_q, \delta_1, \dots, \delta_p, \varepsilon_1, \dots, \varepsilon_r, \lambda \;\Big| \\ & \qquad [\gamma_i, \lambda] = 1, [\xi_i, \lambda] = 1, [\delta_j, \lambda] = 1, [\varepsilon_k, \lambda] = 1, \\ & \qquad \delta_j^{s_j} = \lambda^{t_j}, \lambda^u = \left(\prod_{i=1}^{q} [\gamma_i, \xi_i] \right) \delta_1 \cdots \delta_p \varepsilon_1 \cdots \varepsilon_r \Big\rangle\end{aligned}$$

where each s_j is the index of the j-th exceptional fiber, $0 < t_j < s_j$ for $1 \le j \le p$, and u is an integer. Moreover λ is represented by any regular fiber. Thus λ lies

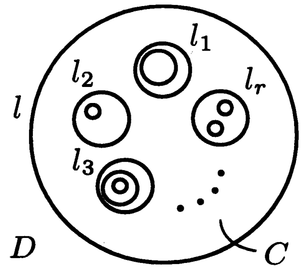


Figure 20

in the center of $\pi_1 M_{i,j}$. Hence our claim was proved and this completes the proof of the lemma. □

LEMMA 5.3. *Let $M_{i,j}$ be a piece in $\mathcal{P}$, $\psi\colon M_{i,j} \hookrightarrow M$ an inclusion map, and $\psi_*\colon \pi_1 M_{i,j} \to \pi_1 M$ a homomorphism induced from ψ. Then ψ_* is injective.*

Proof. It suffices to prove that any loop $l \subset intM_{i,j}$ representing a homotopy class in the kernel of ψ_* is null-homotopic in $M_{i,j}$.

We may assume that l is an embedded loop. Since l is null-homotopic in M, there is a map from D^2 into M carrying ∂D^2 to l. It follows from Dehn's lemma ([19, 8]) that l bounds an embedded disk D in M. Isotope D so that D and $\partial M_{i,j} - \partial M$ meet transversally in a union of loops. Note that these loops are null-homotopic on D. Let $l_1, l_2, \ldots, l_r$ be loops in $D \cap (\partial M_{i,j} - \partial M)$ such that the component C of $D - (\partial M_{i,j} - \partial M)$ containing l is bounded by $l \cup l_1 \cup l_2 \cup \cdots \cup l_r$, as illustrated in Figure 20. Note that $l \subset intM_{i,j}$ implies $C \subset M_{i,j}$. Since $\partial M_{i,j} - \partial M$ is incompressible in M, each l_i is null-homotopic on $\partial M_{i,j} - \partial M$. Therefore each l_i bounds a disk D_i on $\partial M_{i,j} - \partial M$, and hence $C \cup D_1 \cup D_2 \cup \ldots \cup D_r$ forms a disk in $M_{i,j}$ bounded by l. Thus l is null-homotopic in $M_{i,j}$ and the lemma follows. □

It should be noted that, for a piece $M_{i,j} \in \mathcal{P}$ placed on a side of a torus $T \in \mathcal{T}$, any loop on $\partial M_{i,j}$ is homotopic to a loop on T. We summarize with the following proposition.

PROPOSITION 5.4. *Let $M_{i,j}$ be a piece in $\mathcal{P}$ which is homeomorphic to neither $D^2 \times S^1$ nor $S^1 \times S^1 \times I$. Let $\psi_*\colon \pi_1 M_{i,j} \to \pi_1 M$ be a homomorphism induced from the inclusion map $\psi\colon M_{i,j} \hookrightarrow M$. Suppose that $M_{i,j}$ is placed on a side of a torus $T \in \mathcal{T}$. Then a homotopy class $[l] \in \pi_1 M$ of a nontrivial loop l on T lies in the center of $i_*(\pi_1 M_{i,j})$ if and only if $M_{i,j}$ is a Seifert piece with an orientable orbit-manifold and $[l]$ lies in the group generated by any regular fiber of $M_{i,j}$.*

5.2 Realization on each orbit

We recall that f preserves any orbit O_i. We will construct a partial realization on O_i, i.e. a periodic auto-homeomorphism of O_i which is isotopic to $f|_{O_i}$ by an isotopy on O_i. The following lemma shows that $f^n|_{O_i}$ represents a mapping class in $\mathcal{M}(O_i)$ of finite order.

LEMMA 5.5. *$f^n|_{O_i} \simeq id_{O_i}$ by an isotopy on O_i for each i.*

Proof. The homeomorphisms f^n and $f^n|_{O_i}$ respectively induce the automorphisms of $\pi_1 M$ and $\pi_1 O_i$ each of which respects the peripheral structure. It follows from [27, Corollary 7.5] that f^n induces the identity map on $Out(\pi_1 M)$ by $f^n \simeq id_M$. It follows from Lemma 5.3 that for any piece $M_{i,j} \subset O_i$ the homomorphism $(\psi_{i,j})_*\colon \pi_1 M_{i,j} \to \pi_1 M$ induced from the inclusion map $\psi_{i,j}\colon M_{i,j} \to M$ is injective. Then $f^n|_{M_{i,j}}$ induces the identity map on $Out(\pi_1 M_{i,j})$ and therefore $f^n|_{O_i}$ induces the identity map on $Out(\pi_1 O_i)$. Hence $f^n|_{O_i}$ is homotopic to id_{O_i} by a homotopy on O_i. Since $f^n|_{O_i}$ is orientation-preserving by $f^n \simeq id_M$, the required isotopy therefore follows from [27, Theorem 7.1]. □

Suppose that $f|_{O_i}$ represents a mapping class in $\mathcal{M}(O_i)$ of order n_i. We isotope f to make an auto-homeomorphism of M whose restriction on O_i has period n_i for each i. It is sufficient to show the following lemma because any isotopy on $\bigcup_{i=1}^{\kappa} O_i$ extends to an isotopy on M.

LEMMA 5.6. *For each i there is a periodic homeomorphism $f_i\colon O_i \to O_i$ of order n_i which is isotopic to $f|_{O_i}$ by an isotopy on O_i. Moreover, if O_i consists of Seifert pieces, then f_i can be chosen so as to preserve both the Seifert fibration and a dual foliation of O_i.*

Proof. It follows from Proposition 3.1 and Theorem 4.8 that there is a homeomorphism $f_i'\colon M_{i,1} \to M_{i,1}$ of period n_i/ν_i which is isotopic to f^{ν_i} by an isotopy on $M_{i,1}$. Define a homeomorphism $f_i\colon O_i \to O_i$ as follows:

$$f_i = \begin{cases} f & \text{on } O_i - M_{i,\nu_i} \\ f_i' \circ f^{1-\nu_i} & \text{on } M_{i,\nu_i} \end{cases}$$

It should be noted that $f_i' \simeq f^{\nu_i}|_{M_{i,1}}$ implies $f_i \simeq f|_{O_i}$ and that f_i' has period n_i. Therefore f_i is a required periodic homeomorphism.

When $M_{i,1}$ is a Seifert piece, it follows from Theorem 4.8 that f_i' can be chosen so as to preserve both the Seifert fibration and a dual foliation $\mathcal{F}_{i,1}$ of $M_{i,1}$. By [26, Satz 10.1], we may assume that $f_i'|_{O_i - M_{i,\nu_i}} = f|_{O_i - M_{i,\nu_i}}$ is fiber-preserving. Let us consider a foliation $\mathcal{F}_{i,j}$ of $M_{i,j}$ induced from $\mathcal{F}_{i,1}$ by f_i^{j-1} for $1 < j \leq \nu_i$.

Then f_i preserves the dual foliation of O_i defined by $\mathcal{F}_{i,1}, \mathcal{F}_{i,2}, \dots, \mathcal{F}_{i,\nu_i}$. □

5.3 Twists between orbits

In the following we assume by Lemma 5.6 that $f|_{O_i}$ has period n_i, a divisor of n, for $1 \leq i \leq \kappa$. We remark that modifying f only on $\mathcal{N}(\bigcup \mathcal{T}, M)$ is possibly not enough to make f a homeomorphism of peirod n. So we will measure this obstruction by studying twists in $\mathcal{N}(\bigcup \mathcal{T}, M)$ caused by f^n.

Let T be a torus in $\mathcal{T}$. Although we cannot specify a meridian and a longitude on T (because T is essential in M), we need them to describe the type of the twist in $\mathcal{N}(T, M)$ caused by f^n. So we take a pair of simple closed curves m and l on T intersecting each other in a single point so that the homotopy classes $[m]$ and $[l]$ generate the fundamental group $\pi_1 T$. We call m and l a meridian and a longitude of T respectively. We assume that T, m, and l are oriented consistently.

Take a proper arc $\alpha \subset \mathcal{N}(T, M)$ joining the distinct components of $\partial\mathcal{N}(T, M)$ with an orientation determined by those of T and M. Let $k = \alpha \cup f^n(\alpha)$ be a loop with an orientation determined by that of $f^n(\alpha)$. If k is homotopic to a link on T of type (p, q), i.e. it goes longitudinal direction p times and meridional direction q times algebraically, we say that the *twist type* of f^n along T is (p, q).

First let us investigate the twist types of f^n along the tori in $\mathcal{T}$ according to the types of pieces placed on the both sides.

LEMMA 5.7. *Let T be a torus in $\mathcal{T}$. Suppose that M_{i_1,j_1} and M_{i_2,j_2} are pieces in $\mathcal{P}$ placed on the mutually oposite sides of T. Then one of the following cases occurs:*

1. *Each of M_{i_1,j_1} and M_{i_2,j_2} is either a hyperbolic piece or a Seifert piece with a nonorientable orbit-manifold, and the twist type of f^n along T is $(0, 0)$.*
2. *One of M_{i_1,j_1} and M_{i_2,j_2} is a Seifert piece with an orientable orbit-manifold which induces the fibration of T of type (p, q), and the other is either a hyperbolic piece or a Seifert piece with a nonorientable orbit-manifold. The twist type of f^n along T is (kp, kq) where k is an integer.*
3. *Both M_{i_1,j_1} and M_{i_2,j_2} are Seifert pieces each having an orientable orbit-manifold. These Seifert pieces induce the fibrations of T of type (p_1, q_1) and (p_2, q_2) respectively, and the twist type of f^n along T is $(k_1p_1 + k_2p_2, k_1q_1 + k_2q_2)$ where k_1 and k_2 are integers.*

Proof. Let us fix a base point $x_1 \in M_{i_1,j_1}$ for $\pi_1 M$. Recall $f^n \simeq id_M$. It follows from [27, Corollary 7.5] that f^n induces an inner automorphism of $\pi_1 M$ by $\xi \in \pi_1 M$.

Let $l_1 \subset M_{i_1,j_1}$ be an arbitrary nontrivial closed path with the base point

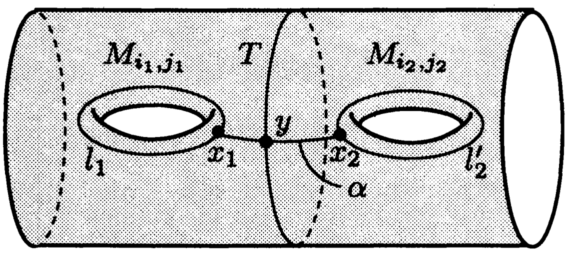


Figure 21

x_1 as illustrated in Figure 21. Since $f^n|_{M_{i_1,j_1}} = id_{M_{i_1,j_1}}$, we have $[\xi, [l_1]] = 1$. Assume $\xi \neq 1$. It follows from [13, Lemma VI.1.5] that we can homotope l_1 and any loop representing ξ into the same Seifert piece. Therefore by Proposition 5.4 M_{i_1,j_1} is a Seifert piece with an orientable orbit-manifold and ξ lies in the subgroup generated by any regular fiber of M_{i_1,j_1}.

Let $l'_2 \subset M_{i_2,j_2}$ be an arbitrary nontrivial closed path with a base point $x_2 \in M_{i_2,j_2}$. Take a path α from x_1 to x_2 intersecting $\bigcup \mathcal{T}$ in a point y on T. Then the product $l_2 = \alpha l'_2 \alpha^{-1}$ is a closed path with the base point x_1 as illustrated in Figure 21. Suppose that the twist type of f^n along T is (p', q'). Let λ be a closed path on T with the base point y homotopic to a link on T of type (p', q'). We put $\eta = [\alpha_1 \lambda \alpha_1^{-1}]$ where $\alpha_1 = \alpha \cap N_1$. Then $[f^n(l_2)] = \xi[l_2]\xi^{-1} = \eta[l_2]\eta^{-1}$. Therefore $[\xi^{-1}\eta, [l_2]] = 1$. Assume $\xi^{-1}\eta \neq 1$. It follows from [13, Lemma VI.1.5] that we can homotope l_2 and any loop representing $\xi^{-1}\eta$ into the same Seifert piece. Therefore by Proposition 5.4 M_{i_2,j_2} is a Seifert piece with the orientable orbit-manifold and $\xi^{-1}\eta$ lies in the subgroup generated by any regular fiber of M_{i_2,j_2}. Thus we have the following four possibilities:

(A) $\eta = 1$.

(B) $\eta = \xi \neq 1$, M_{i_1,j_1} is a Seifert piece with an orientable orbit-manifold, and $\eta = \xi$ lies in the subgroup generated by any regular fiber of M_{i_1,j_1}.

(C) $\eta \neq \xi = 1$, M_{i_2,j_2} is a Seifert piece with an orientable orbit-manifold, and η lies in the subgroup generated by any regular fiber of M_{i_2,j_2}.

(D) $\eta \neq \xi$, $\eta \neq 1$, $\xi \neq 1$, both M_{i_1,j_1} and M_{i_2,j_2} are Seifert pieces each having an orientable orbit-manifold and ξ and $\xi^{-1}\eta$ lie in the subgroup generated by any regular fiber of M_{i_1,j_1} and M_{i_2,j_2} respectively.

Let us study these cases in turn. If M_{i_1,j_1} is a Seifert piece, we denote by (p_1, q_1) the fibration type of T induced from the Seifert fibration of M_{i_1,j_1} and by γ_1 the homotopy class represented by a regular fiber passing x_1. Similarly if M_{i_2,j_2} is a Seifert piece, we denote by (p_2, q_2) the fibration type of T induced from the Seifert fibration of M_{i_2,j_2} and by γ_2 the homotopy class represented by $\alpha\rho\alpha^{-1}$ where ρ is a regular fiber passing x_2.

Case (A) obviously implies that the twist type of f^n along T is $(0, 0)$, and therefore corresponds to either Case (1), (2) with $k = 0$, or (3) with $k_1 = k_2 = 0$. In Case (B), the twist type of f^n along T is a multiple of (p_1, q_1). If M_{i_2,j_2} also is a Seifert piece with an orientable orbit-manifold, the present case corresponds to Case (3) with $k_2 = 0$, and otherwise Case (2). Case (C) is similar to Case (B). If M_{i_1,j_1} also is a Seifert piece with an orientable orbit-manifold, the present case corresponds to Case (3) with $k_1 = 0$, and otherwise Case (2). In Case (D), suppose $\xi = \gamma_1^{r_1}$ and $\xi^{-1}\eta = \gamma_2^{r_2}$ where r_1 and r_2 are nonzero integers. It should be noted that we have $\xi \neq \xi^{-1}\gamma$ by the assumption of the canonical torus decomposition of M. Then the twist type of f^n along T is $r_1(p_1, q_1) + r_2(p_2, q_2)$, and the present case results to be in Case (3). This completes the proof. □

REMARK 5.8. In the case $M_{i_1,j_1} = M_{i_2,j_2}$ for Lemma 5.7, we have only two possibilities (1) and (3). Although T is placed between the same Seifert piece, the fibration types (p_1, q_1) and (p_2, q_2) of T induced from the Seifert fibration of the piece from mutually oposite sides are different. In other words, we have $\gamma_1 \neq \gamma_2$, though γ_1 is conjugate to γ_2.

Lemma 5.7 implies that f^n may cause twists only around the Seifert pieces in $\mathcal{P}$ each having an orientable orbit-manifold along their fibers. So we will investigate the relation between the twists caused by f^n along distinct tori.

Let $M_{i,j}$ be a Seifert piece in $\mathcal{P}$ with an orientable orbit-manifold. Suppose that $M_{i,j}$ is placed on a side S of $T \in \mathcal{T}$. We denote by $M_{i',j'}$ the piece in $\mathcal{P}$ placed on the side S' of T other than S. Suppose that the twist type of f^n along T is (p_0, q_0) and that the fibration type of T induced from the Seifert fibration of $M_{i,j}$ from S is (p, q). Let ν be a normal of T consistent with the orientations of T and M. We put $\varepsilon = +1$ or -1 according as ν points inwards to or outwards from S respectively. We define the *twist number* of f^n for (T, S) as follows:

(1) If $(p_0, q_0) = k(p, q)$ for an integer k, the twist number of f^n for (T, S) is εk.
(2) Otherwise $M_{i',j'}$ is also a Seifert piece with an orientable orbit-manifold, and we have $(p_0, q_0) = k(p, q) + k'(p', q')$ where k and k' are integers and (p', q') is the fibration type of T induced from the Seifert fibration of $M_{i',j'}$ from S'. Then the twist number of f^n for (T, S) is εk.

It should be noted that the twist number of f^n for (T, S) depends not on the orientation of T but on those of fibers of N.

LEMMA 5.9. *Let $M_{i,j}$ be any Seifert piece in $\mathcal{P}$ with an orientable orbit-manifold. Suppose that $T_1, T_2, \cdots, T_\sigma$ are tori in $\mathcal{T}$ such that $M_{i,j}$ is placed on a side S_k of T_k for each k. Then the twist numbers of f^n for (T_k, S_k) for $1 \leq k \leq \sigma$ are the same.*

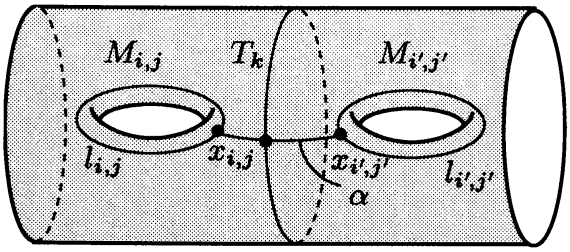


Figure 22

Proof. Let us fix T_k. We denote by $M_{i',j'}$ the piece in $\mathcal{P}$ placed on the side S'_k of T_k other than S_k. It should be noted that we have the possibility of $M_{i,j} = M_{i',j'}$. Let us fix a base point $x_{i,j} \in M_{i,j}$ for $\pi_1 M$ so that any exceptional fiber of $M_{i,j}$ misses $x_{i,j}$. Since $f^n \simeq id_M$, it follows from [27, Corollary 7.5] that f^n induces an inner automorphism of $\pi_1 M$ by $\xi \in \pi_1 M$.

Let us take an arbitrary nontrivial closed path $l_{i,j} \subset M_{i,j}$ with the base point $x_{i,j}$, as illustrated in Figure 22. Since $f^n|_{M_{i,j}} = id_{M_{i,j}}$, we have $[\xi, [l_{i,j}]] = 1$. Assume $\xi \neq 1$. It follows from [13, Lemma VI.1.5] that we can homotope $l_{i,j}$ and any loop representing ξ into the same Seifert piece. Therefore it follows from Proposition 5.4 that ξ lies in the subgroup generated by any regular fiber of $M_{i,j}$. Thus we suppose $\xi = \gamma^r$ where γ is a homotopy class of infinite order represented by the regular fiber of $M_{i,j}$ passing $x_{i,j}$.

Let us take an arbitrary nontrivial closed path $l'_{i',j'} \subset M_{i',j'}$ with a base point $x_{i',j'} \in M_{i',j'}$. Take a path α from $x_{i,j}$ to $x_{i',j'}$ such that α intersects $\bigcup \mathcal{T}$ in a point on T_k. Then the product $l_{i',j'} = \alpha l'_{i',j'} \alpha^{-1}$ is a closed path with the base point $x_{i,j}$ as illustrated in Figure 22. Let t_k be the twist number of f^n for (T_k, S_k).

First we assume that $M_{i',j'}$ is not a Seifert piece with an orientable orbit-manifold. Then $[f^n(l_{i',j'})] = \gamma^{t_k}[l_{i',j'}]\gamma^{-t_k} = \gamma^r[l_{i',j'}]\gamma^{-r}$. Therefore $[\gamma^{t_k - r}, [l_{i',j'}]] = 1$. Hence $t_k = r$ for any k follows from Proposition 5.4.

Next we assume that $M_{i',j'}$ is a Seifert piece with an orientable orbit-manifold. Suppose that any exceptional fiber misses $x_{i',j'}$. Let ρ be the regular fiber passing $x_{i',j'}$. We denote by γ_k the homotopy class represented by the product closed path $\alpha\rho\alpha^{-1}$, and by t'_k the twist number of f^n for (T_k, S'_k). Then $[f^n(l_{i',j'})] = \gamma^{t_k}\gamma_k^{-t'_k}[l_{i',j'}]\gamma_k^{t'_k}\gamma^{-t_k} = \gamma^r[l_{i',j'}]\gamma^{-r}$. Therefore $[\gamma^{t_k - r}\gamma_k^{-t'_k}, [l_{i',j'}]] = 1$. Hence $t_k = r$ for any k follows from Proposition 5.4. This completes the proof. □

Let $M_{i,j}$ be a Seifert piece with an orientable orbit-manifold, T_i, $1 \leq i \leq \sigma$, tori in $\mathcal{T}$ such that $M_{i,j}$ is placed on a side S_i of T_i. In view of Lemma 5.9 the twist number of f^n for (T_i, S_i) may be regarded as independent of (T_i, S_i). This justifies us to call it simply the twist number of f^n for $M_{i,j}$.

5.4 Proof of Theorem 1.4

Let O_i be an orbit of f which consists of Seifert pieces each having an orientable orbit-manifold. Assume that f preserves both the Seifert fibration $\mathcal{S}_i$ and a dual foliation $\mathcal{F}_i$ of O_i. We denote by $\mathcal{E}_i$ the set of all exceptional fibers in O_i. Suppose that $O_i' = O_i - int\mathcal{N}(\bigcup \mathcal{E}_i, O_i)$ is saturated. Then O_i' has an S^1-bundle structure induced from $\mathcal{S}_i$. Let us isotope f by an isotopy on O_i whose restriction on O_i' is s-translation. We further modify f on $\mathcal{N}(\bigcup \mathcal{E}_i, O_i)$ as we constructed g_i in the proof of Theorem 4.8 so that afterwards f preserves both $\mathcal{S}_i$ and $\mathcal{F}_i$. Lemma 2.1 enables us to realize these modifications of f by an isotopy on O_i, which we call *s-translation* of f on O_i.

Suppose that, for $1 \leq i \leq \kappa$, $f|_{O_i}$ represents a mapping class in $\mathcal{M}(O_i)$ of order n_i. Then each n_i is a divisor of n. We may assume by Lemma 5.6 that $f|_{O_i}$ has period n_i for $1 \leq i \leq \kappa$. If O_i consists of Seifert pieces $M_{i,1}, M_{i,2}, \dots , M_{i,\nu_i}$ each having an orientable orbit-manifold, we denote by k_i the common twist number of f^n for them. Otherwise we put $k_i = 0$. We consider an isotopy of f whose restriction on O_i is invariant if $k_i = 0$ and $(-k_i)$-translation otherwise. Then this isotopy cancels the twists along tori in $\mathcal{T}$ caused by f^n.

It suffices to isotope f on $\mathcal{N}(\bigcup \mathcal{T}, M)$ so as to produce the required periodic homeomorphism $g\colon M \to M$.

We regard any torus T in $\mathcal{T}$ as a quotient of $\mathbb{R}^2$ by the integer lattice $\mathbb{Z}^2$. Let $p_T\colon \mathbb{R}^2 \times I \to \mathcal{N}(T, M)$ be a universal covering map and $\tilde{f}_T\colon \mathbb{R}^2 \times I \to \mathbb{R}^2 \times I$ a lift of $f|_{\mathcal{N}(T,M)}$ as the following commutative diagram shows:

$$\begin{array}{ccc} \mathbb{R}^2 \times I & \xrightarrow{\tilde{f}_T} & \mathbb{R}^2 \times I \\ {\scriptstyle p_T}\big\downarrow & & \big\downarrow{\scriptstyle p_{f(T)}} \\ \mathcal{N}(T, M) & \xrightarrow{f} & \mathcal{N}(f(T), M) \end{array}$$

Suppose $\Gamma_T^{(0)}\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $\Gamma_T^{(1)}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be homeomorphisms satisfying

$$\begin{cases} \tilde{f}_T(x, y, 0) = (\Gamma_T^{(0)}(x, y), \varepsilon) \\ \tilde{f}_T(x, y, 1) = (\Gamma_T^{(1)}(x, y), 1 - \varepsilon) \end{cases}$$

where $\varepsilon = 0$ or 1 according as $\tilde{f}_T$ preserves the boundary component of $\mathbb{R}^2 \times I$ or not. Define a homeomorphism $\tilde{g}_T\colon \mathbb{R}^2 \times I \to \mathbb{R}^2 \times I$ by

$$\tilde{g}_T(x, y, t) = ((1 - t)\Gamma_T^{(0)}(x, y) + t\Gamma_T^{(1)}(x, y), (1 - t)\varepsilon + t(1 - \varepsilon)).$$

Let $g_T\colon \mathcal{N}(T, M) \to \mathcal{N}(f(T), M)$ be a homeomorphism induced from $\tilde{g}_T$, α a proper arc in $\mathcal{N}(T, M)$ joining the distinct boundary components. Since the twist type of f^n along T is $(0,0)$, we can modify f on $\mathcal{N}(T,M)$ by an isotopy relative to $\partial\mathcal{N}(T,M)$ so that $f(\alpha) = g_T(\alpha)$. It follows from Lemma 2.1 that there is an isotopy on $\mathcal{N}(T,M)$ from f to g_T relative to $\partial\mathcal{N}(T,M) \cup \alpha$, and hence the twist type of g_T^n along T is $(0,0)$. Moreover $f^n|_{\partial\mathcal{N}(T,M)} = id_{\partial\mathcal{N}(T,M)}$ implies $g_T^n = id_{\mathcal{N}(T,M)}$. Define an auto-homeomorphism g of M as follows:

$$g = \begin{cases} f & \text{on } M - int\mathcal{N}(\bigcup\mathcal{T}, M), \\ g_T & \text{on } \mathcal{N}(T,M) \text{ for any } T \in \mathcal{T}. \end{cases}$$

Then g is isotopic to f and we have $g^n = id_M$. Since g represents a mapping class of order n, the period of g is exactly n. Hence g has the required property. This completes the proof of Theorem 1.4. □

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