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COLORING SOLITAIRE TILINGS

By

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Abstract. Many self-affine tilings have tiles whose boundaries are fractal sets. We show an algorithm to construct finite graphs (or finite automata) which represent the boundaries of tiles in a special class of self-affine tilings which we call the solitaire tilings. We apply these graphs to construct colorings of these tilings.

1. Introduction

By a *tile* we mean a compact subset of $\mathbf{R}^n$ which is the closure of its interior. A *tiling* of $\mathbf{R}^n$ is a collection of tiles, whose union is $\mathbf{R}^n$ and which have pairwise disfoint interiors. Let $\mathcal{T}$ be a tiling of $\mathbf{R}^n$. $\mathcal{T}$ is a *self-affine tiling*, if there exists an affine transformation on $\mathbf{R}^n$, such that the image of a tile is a union of some tiles in $\mathcal{T}$.

Self-affine tilings appear in several quite different contexts. Their roots can be found in the work on the construction of Markov partitions [1, 2, 9]. They also serve as a models for real quasicrystals [11, 6]. The theory of wavelets is another new field which has people to become interested in self-affine tilings [4, 8].

Thurston [12] shows a construction for non-periodic self-similar tilings which we call the *solitaire tilings*. Figure 1 shows an example of the solitaire tilings generated by the polynomial $x^3 - x^2 - x - 1$. This polynomial has a root $\alpha = -0.41964337760708\ldots - 0.6062907292071993\ldots i$. For each word w over $\{0, 1\}$, the tile $T(w)$ is defined by

$$T(w) = \left\{ \sum_{n=-l}^{\infty} a_n \alpha^n : a_n \in \{0,1\},\ a_n \times a_{n+1} \times a_{n+2} = 0,\ a_{-l} \ldots a_{-1} = w \right\}.$$

w determines the principal parts of the serieses which constitute the tile. As can be seen in this example, many self-affine tilings have tiles which have fractal boundaries. Strichartz and Wang [10] showed a method to compute the Hausdorff dimension of the boundaries of periodic self-affine tilings. Their method requires

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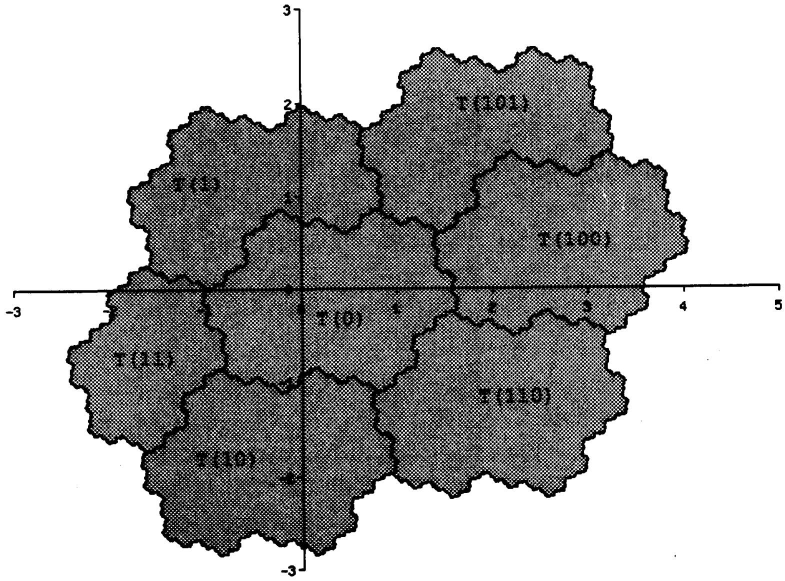

Figure 1 Pisot tiling generated by roots of $x^3 - x^2 - x - 1$

an algorithm to decide the adjacency of tiles and they used directed graphs (or automata). In this paper, we modify and extend their algorithm to apply to the solitaire tilings.

We consider the problem of coloring the solitaire tilings, which motivated our study. Since we can not use so many colors on the monochrome papers, wise color assingment, such that two adjacent tiles are assigned distinct colors, is required to draw the pictures of the tilings. We present an algorithm to construct a coloring function of a given solitaire tiling $\mathcal{T}$. By using this algorithm, for example, the tiling in the Figure 9.6 in [12] can be 3-colored as is in Figure 2.

Our algorithm is divided into the following three steps:

1. Embed all tiles into $\mathbf{Z}^r$ using a certain injection ϕ called the address map [6] so that the distance between the images of two adjacent tiles are bounded.
2. Determine a subset of $\mathbf{Z}^r$, $D = \{\phi(T) - \phi T' : T, T' \in \mathcal{T}, T$ and T' are adjacent$\}$.
3. Find an integer N and a $\mathbf{Z}$-linear map $l : \mathbf{Z}^r \to \mathbf{Z}/N\mathbf{Z}$, such that $l(d) \neq 0$ for all $d \in D$.

Then the map $c := l \circ \phi : \mathcal{T} \to \mathbf{Z}/N\mathbf{Z}$ will be a N-coloring function, that is, if T and $T'$are adjacent then $c(T) \neq c(T')$.

To execute step 2, we need informations on the adjacency. We show an algorithm to decide the intersections of two distinct tiles in Theorem 1, which will give the adjacency of two tiles.

Another aim of this paper is to show explicit computations, in which three-

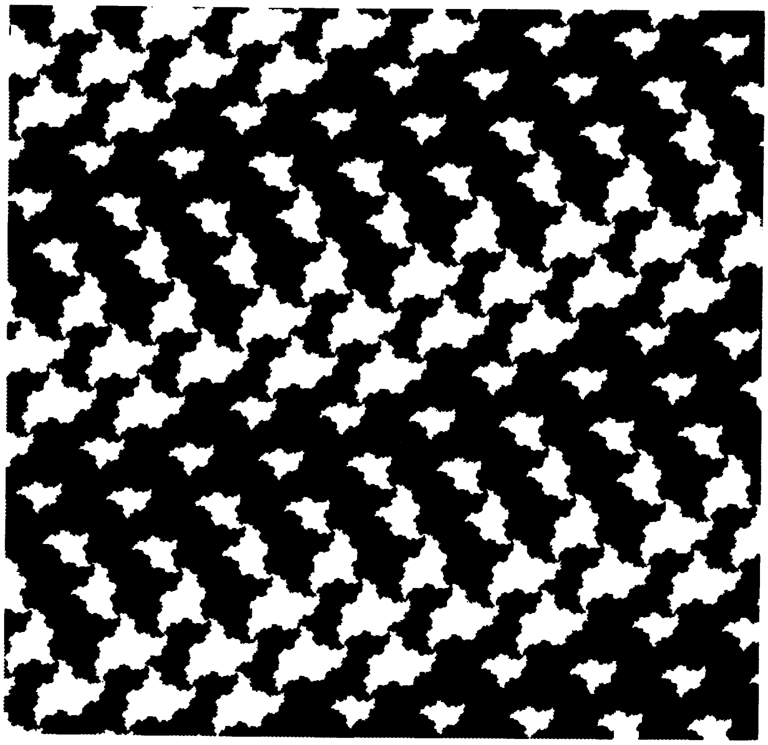

Figure 2

colorable and four-colorable examples are presented. We begin with the definition of the solitaire tilings.

2. The solitaire tilings

Let G be a (directed labeled) graph. We denote by $\mathcal{V}(G)$ the vertex set of G and by $\mathcal{E}(G)$ the edge set of G. Every edge $e \in \mathcal{E}(G)$ has the *starting point* $s(e) \in \mathcal{V}(G)$ and the *end point* $t(e) \in \mathcal{V}(G)$, and carry a label $l(e) \in \Sigma(G)$ where $\Sigma(G)$ is a finite set called the *alphabet* of G. A sequence of edges $e_1 \dots e_l$ is called a *path*of G if $t(e_i) = s(e_{i+1})$. In this paper we assume all labeled graph has a vertex i_G called the *initial state* of G. Σ^* denotes the set of all of the words over an alphabet Σ. A word $w = a_1 \dots a_l \in \Sigma^*$ is *accepted* by G if there exists a path $p = e_1 \dots e_l$ starting from i_G such that $l(e_1) \dots l(e_l) = w$. A *language* over Σ is a subset of Σ^*. We denote by $L(G)$ the set of all of the words accepted by G. And we say a language L is *accepted* by G if $L = L(G)$. An infinite word $(a_i)_{i \geq 0}$ over Σ is accepted by G if $a_0 \dots a_h \in L(G)$ for all $h \geq 0$. We denote by $X(G)$ the set of all of the infinite words accepted by G.

Let β be a real algebraic integer; β is a real root of a polynomial $f(x) \in \mathbf{Z}[x]$

which is irreducible over $\mathbf{Q}$. β is called to be a *Pisot number* if $\beta > 1$ and all of the roots of $f(x)$ other than β have modulus smaller than 1. A *Pisot unit* is a Pisot number which is an algebraic unit, that is, the constant term of $f(x)$ is ± 1. Let β be a Pisot unit of degree d, and let

$$\left\{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r)}, \overline{\alpha^{(r+1)}}, \ldots, \alpha^{(r+c)}, \overline{\alpha^{(r+c)}}\right\}$$

be all Galois conjugates of β other than β where $\alpha^{(i)} \in \mathbf{R}$ for $1 \le i \le r$, $\overline{\alpha}$ denotes the complex conjugate of α, and so $r + 2c = d - 1$.

Then we introduce the β-expansions of numbers. For further detail of β-expansion, see [3]. Let β be a Pisot unit and let $a \ge 0$. The *β-expansion* of a is computed as follows. We denote by $[y]$ and $\{y\}$ the integer part and the fractional part of a real number y respectively. There exists $k \in \mathbf{Z}$ such that $\beta^k \le a < \beta^{k+1}$. Let $a_k = [a/\beta^k]$, and $r_k = \{a/\beta^k\}$. For $k > i > -\infty$, put $a_i = [\beta r_{i+1}]$, and $r_i = \{\beta r_{i+1}\}$. Then we get an expansion $a = a_k\beta^k + a_{k-1}\beta^{k-1} + \cdots$ called the β-expansion of a. Let $\Sigma_\beta = \{0, 1, \cdots, [\beta]\}$ and let L_β consist of all of the words $a_h a_{h-1} \cdots a_0$ over Σ_β such that $a_h\beta^h + a_{h-1}\beta^{h-1} + \cdots + a_0$ is a β-expansion of a number.

There exists a graph G such that $L(G) = L_\beta$. We illustrate a construction for G. First we compute a sequence of Σ_β, $(c_n)_{n\ge 0}$, which we call the *carry sequence*of β and denote by *carry*(β). Put $x_0 = 1$, and for $n > 0$, let c_n be the largest integer less than βx_{n-1} and let $x_n = \beta x_{n-1} - c_n$. It is known that *carry*(β) is periodic if β is a Pisot number [12]. So there exist nonnegative integers p, q such that $c_{k+p} = c_k$ for $k > q$. We denote it by $c_1 \ldots c_q(c_{q+1} \cdots c_{q+p})^\infty$. Then the graph G will be constructed as follows. G has the vertices, $\mathcal{V}(G) = \{0, 1, \cdots, p+q-1\}$ with the initial state $i_G = 0$. G has the following edges. For $0 \le n < p+q-1$ there exists an edge from n to $n+1$ labeled c_{n+1} and an edge from $p+q-1$ to q labeled c_{p+q}. And for each $l \in \Sigma_\beta$ and $n \in \mathcal{V}(G)$, there exists an edge from n to 0 labeled l if $l < c_{n+1}$.

We denote by L^R the *reverse* of a language L, that is, $L^R = \{a_0 a_1 \cdots a_l : a_l a_{l-1} \cdots a_0 \in L\}$. It is well known that there exists a graph which accepts the reverse of a language accepted by a graph [5].

EXAMPLE 1. Let β be the Pisot number whose minimal polynomial is $x^d - x^{d-1} - \cdots - x - 1$. Then *carry*$(\beta) = (\overbrace{11\cdots 10}^{d-1\,\text{times}})^\infty$ and so the graph shown in Figure 3 accepts L_β. In this case

$$\begin{aligned} L_\beta &= L_\beta^R \\ &= \{a_0 a_1 \cdots a_l : a_i \in \{0,1\},\ a_i \times a_{i+1} \times \cdots \times a_{i+d-1} = 0\ 0 \le i \le l-d+1\}. \end{aligned}$$

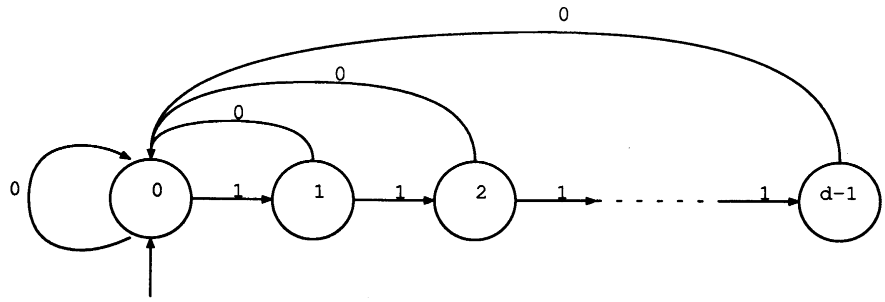

Figure 3

Let β be a Pisot unit, let Σ be a finite set of integers and let

$$F_\Sigma = \left\{ \sum_{n=l}^{\infty} a_n x^n : a_n \in \Sigma,\ l \in \mathbf{Z} \right\}.$$

Then we define the *projection map* π_β by

$$\pi_\beta = F_\Sigma \ni f(x) \mapsto \big(f(\alpha^{(1)}), \dots, f(\alpha^{(r+c)})\big) \in \mathbf{R}^r \times \mathbf{C}^c \simeq \mathbf{R}^{d-1}.$$

Let G be a labeled graph whose alphabet Σ is a finite set of integers and let $w \in L(G)$ be a word over Σ. Let

$$F(w, G) = \left\{ \sum_{i=-l}^{\infty} a_i x^i : (a_i)_{i \geq 0} \in X(G), a_{-l} \cdots a_{-1} = w \right\},$$

and

$$F(\epsilon, G) = \left\{ \sum_{i=0}^{\infty} a_i x^i : (a_i)_{i \geq 0} \in X(G) \right\},$$

where ϵ denotes the empty word. Let G_β be a graph which accepts L_β^R. Then for each word $w \in L_\beta^R$, we define a *tile* $T(w)$by

$$T(w) = \pi_\beta\big(F(w, G_\beta)\big).$$

If $w \in L_\beta$ then $w00\cdots0 \in L_\beta$ and so $T(00\cdots0v) = T(v)$ for any $v \in L_\beta^R$. We call $\mathcal{T}_\beta := \big\{T(w) : w \in L(G_\beta)\big\}$ the *tiling* by β.

3. Description of the algorithm

First we show an algorithm to decide intersections of tiles by labeld graphs. The following lemma is fundamental.

LEMMA 1. *Let β be a Pisot unit and let Σ be a finite set of integers. Then there exists a labeled graph G which generates all of the sequences $(a_n)_{n\geq 0}$ of Σ such that*

$$\pi_\beta\left(\sum_{n=0}^{\infty} a_n x^n\right) = (0,0,\ldots,0), \tag{1}$$

that is, (1) *holds if and only if* $(a_n)_{n\geq 0} \in X(G)$.

Proof. We construct the graph as follows. Let $M = \max\{|m| : n \in \Sigma\}$. We denote by $x^{(i)}$ the image of $x \in \mathbf{Z}[\beta]$ by the conjugate map which transforms β to $\alpha^{(i)}$. Let G have the following vertices,

$$\mathcal{V}(G) = \left\{ f \in \mathbf{Z}[\beta];\ |f^{(i)}| \leq \frac{M}{1-|\alpha^{(i)}|} i \in \{1,2,\ldots,d-1\},\ |f| \leq \frac{M}{|\beta|-1} \right\},$$

which is a finite set. Let $\delta : \mathbf{Z}[\beta] \times \Sigma \to \mathbf{Z}[\beta]$ be defined by $\delta(f,d) = (f+d)\beta^{-1}$, (note that $\beta^{-1} \in \mathbf{Z}[\beta]$ since β is a unit), and $\delta^* : \mathbf{Z}[\beta] \times \Sigma^* \to \mathbf{Z}[\beta]$ by $\delta^*(f, d_1 d_2 \cdots d_k) = \delta(\delta^*(f, d_1 d_2 \cdots d_k) = (f+d_1)\beta^{-k} + d_2\beta^{-k+1} + \cdots + d_k\beta^{-1}$. Let G have the following edges. There is an edge labeled d from $f \in \mathcal{V}(G)$ to $\delta(f,d)$ if $\delta(f,d) \in \mathcal{V}(G)$. And let 0 be the initial state of G.

Then G has the desired property. Indeed, if (1) holds for a sequence of Σ, $(a_n)_{n\geq 0}$ then for any $N \geq 0$ and $i \in \{1,\ldots,r+c\}$,

$$(\alpha^{(i)})^{-N} \sum_{n=0}^{\infty} a_n (\alpha^{(i)})^n = 0.$$

So

$$\begin{aligned}\left|\delta^*(0, a_0 \cdots a_{N-1})^{(i)}\right| &= \left|(\alpha^{(i)})^{-N} \sum_{n=0}^{N-1} a_n (\alpha^{(i)})^n\right| \\ &= \left|\sum_{n=0}^{\infty} a_{n+N} (\alpha^{(i)})^n\right| \\ &\leq \frac{M}{1-|\alpha^{(i)}|}.\end{aligned}$$

whose both sides converge to zero when $N \to \infty$. □

For reasons which will be clear later,it is convenient to consider *essential* graph [7]. The graph G in the proof above contains vertices which do not appear on any infinite path starting from the initial state, and so these vertices have no meaning for $X(G)$. In the following we assume all of the graphs are essentialized, that is, all of the edges and vertices which are not on any infinite paths starting from the initial state are removed.

EXAMPLE 2. Let $\beta = \frac{1+\sqrt{5}}{2}$ and $\alpha = \frac{1-\sqrt{5}}{2}$, and let $\Sigma = \{1, 0, 1\}$. The graph G of Lemma 1 is constructed as follows.

$$\begin{aligned}\mathcal{V}(G) &= \left\{ m\beta + n \in \mathbf{Z}[\beta] : |m\beta + n| \leq \frac{1}{\beta - 1},\ |m\alpha + n| \leq \frac{1}{1 - |\alpha|} \right\} \\ &= \{0, \pm 1, \pm(\beta - 1), \pm(\beta - 2), \pm\beta\}\end{aligned}$$

Adding the edges following the rules above, we obtain the graph G shown in the Figure 4. And essentialization remover the vertices β and $-\beta$. Then a infinite-path on the resulting graph generates a power series of base α whichconverges to zero, for example,

$$1 + \alpha + \alpha^3 + \alpha^5 + \alpha^7 + \cdots = 0,$$
$$1 - \alpha^2 + \alpha^3 - \alpha^4 + \alpha^5 - \alpha^6 + \cdots = 0,$$

and so on.

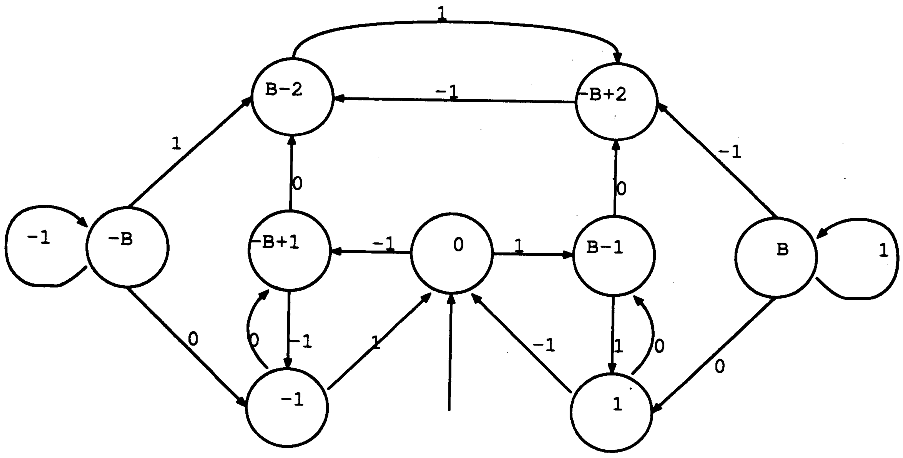

Figure 4

THEOREM 1. *For any positive integer k, there exists a labeled graph G^k_β with alphabet $\Sigma^k_\beta = \{(a_1, \ldots, a_k) : a_i \in \Sigma_\beta\}$ such that, $(a_{1,n}, a_{2,n}, \ldots, a_{k,n})_{n \geq -l} \in X(G^k_\beta)$ if and only if*

$$\pi_\beta\left(\sum_{n=-l}^{\infty} a_{1,n}x^n\right) = \cdots = \pi_\beta\left(\sum_{n=-l}^{\infty} a_{k,n}x^n\right), \tag{2}$$

and

$$(a_{i,n})_{n \geq -l} \in X(G_\beta) \quad i = 1, \ldots, k, \tag{3}$$

where G_β denotes the graph which accepts L_β^R. So if we are given k words $w_1 = (a_{1,n})_{-l\le n\le -1}, \dots, w_k = (a_{k,n})_{-l\le n\le -1} \in L_\beta^R$, G_β^k generates all of the points in $T(w_1)\cap T(w_2)\cap\cdots\cap T(w_k)$.

Proof. We construct the graph G_β^k as follows. Let G_β be a graph which accepts L_β^R, and let G be the graph in Lemma 1 by putting $\Sigma = \{0, \pm 1, \pm 2, \dots, \pm[\beta]\}$. Let $\mathcal{V}(G_\beta^k) = \mathcal{V}(G_\beta)^k \times \mathcal{V}(G)^{k-1}$. Let G_β^k have an edge labeled $(d_1, d_2, \cdots, d_k)$ from $(v_1, v_2, \dots, v_k, f_2, \dots, f_{k-1})$ to $(v_1', \dots, v_k', f_2', \dots, f_{k-1}')$, if there are edges in G_β from v_j to v_j' labled by d_j for $j = 2, \dots, k$ and edges in G from f_j to f_j' labeled $d_1 - d_j$ for $j = 2, \dots, k$. Let $i_{G_\beta^k} = (i_{G_\beta}, \dots, i_{G_\beta}, 0, \dots, 0)$. Then G_β^k has the desired property.

Indeed, if $(a_{1,n}, \dots, a_{k,n})_{n\ge -l} \in X(G_\beta^k)$, then it is clear that $(a_{j,n})_{n\ge -l} \in X(G_\beta)$ for $j = 1, 2, \dots, k$, and $(a_{1,n} - a_{j,n})_{n\ge -l} \in X(G)$ for $j = 2, \dots, k$. It follows from Lemma 1 that $\sum_{n=-l}^{\infty} a_{j,n}(\alpha^{(i)})^n = 0$ for $i = 1, \dots, r+c$ and hence $\sum_{n=-l}^{\infty} a_{1,n}(\alpha^{(i)})^n = \sum_{n=-l}^{\infty} a_{j,n}(\alpha^{(i)})^n$.

Conversely, assume (2) and (3) hold for a sequence $(a_{1,n}, \dots, a_{k,n})_{n\ge -l}$. If we are on the vertex $(v_1, \dots, v_k, f_2, \dots, f_k)$ after reading the first $m+l+1$ labels $(a_{1,n}, \dots, a_{k,n})_{m\ge n\ge -l}$ as an input sequence to G_β^k then, since (3) holds, there exists an edge from v_j to a vertex $v_j' \in \mathcal{V}(G_\beta)$ labeled $a_{j,m+1}$ for $j = 1, \dots, k$. And from (2) and Lemma 1, $(a_{1,n} - a_{j,n})_{n\ge -l} \in X(G)$ for $j = 2, \dots, k$. Therefore there exists an edge labeled $a_{1,m+1} - a_{j,m+1}$ from f_j to some vertex $f_j' \in \mathcal{V}(G)$. So there is an edge from $(v_1, \dots, v_k, f_2, \dots, f_k)$ to $(v_1', \dots, v_k', f_2', \dots, f_k')$ labeled $(a_{1,m+1}, \dots, a_{k,m+1})$. □

Roughly saying the first k columns of a vertex of G_β^k are used to check the validity of each input sequence $(a_{i,n})_{n\ge -l}$ and the last half are used to keep track of the differences between sequences.

EXAMPLE 3. Let β and α be the sameas Example 1. Then *carry* $(\beta) = (10)^\infty$ and (essential) G_β^2 is shown in the Figure 5. From this graph we can see that the intersection of distinct two tiles in this tiling consists of at most one point. For example we can see $T(\epsilon)(= T(0)) \cap T(1) = \{-1\}$ as follows. Start from the initial state and follow the path labeled `(0,1)` then we get to the state `(0,1,-B+1)` from which there is only one road labeled $(0,0)(1,0)((0,1)(1,0))^\infty$. In fact,

$$\alpha + \alpha^3 + \alpha^5 + \cdots = \frac{1}{\alpha} + \alpha^2 + \alpha^4 + \alpha^6 + \cdots = -1.$$

In the same manner we can see $T(\epsilon)\cap T(10) = \{-1/\alpha\}$, $T(\epsilon)\cap T(100) = \emptyset$, and so on.

We show the algorithm to construct the coloring function of the solitaire tilings.

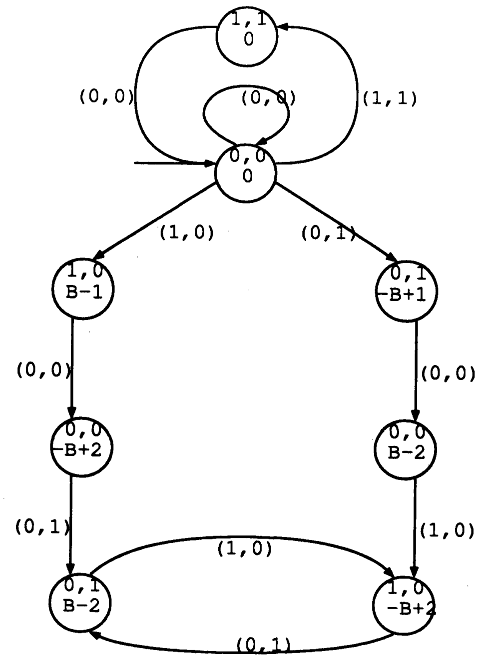

Figure 5

THEOREM 2. *There exists a finite algorithm to construct a coloring function of a given tiling $\mathcal{T}_\beta$ by using a finite number of colors.*

Proof. Using (essential) G_β^2, we can determine the distances between two intersecting tiles: Two tiles $T(a_{-l}a_{-l+1}\cdots a_{-1})$ and $T(b_{-l}b_{-l+1}\cdots b_{-1})$ intersect if and only if there exists a path in G_β^2 labeled $(a_n, b_n)_{-l\leq n\leq -1}$. So $(a_{-l}-b_{-l})\beta^{-l}+(a_{-l+1}-b_{-l+1})\beta^{-l+1}+\cdots+(a_0-b_0)\beta^{-1}$ is equal to one of the third column of a vertex of G_β^2. We denote by D_β the set of all of the third columns of $\mathcal{V}(G_\beta^2)$.

The address map $\phi : \mathcal{T}_\beta \to \mathbf{Z}^d$ is defined as follows. Let $w = a_{-l}a_{-l+1}\cdots a_{-1}$ be aword over Σ_β and $a_{-l}\beta^{-l}+a_{-l+1}\beta^{-l+1}+\cdots+a_1\beta^{-1} = b_0+b_1\beta+\cdots+b_{d-1}\beta^{d-1}$, where $b_0,\ldots,b_{d-1}\in\mathbf{Z}$. Then we define ϕ by

$$\phi(T(w)) = (b_0, b_1, \ldots, b_{d-1}).$$

And the coloring function $c : \mathcal{T}_\beta \to \mathbf{Z}/N\mathbf{Z}$ for some integer N is defined as follows. Find $(c_1,\ldots,c_d)\in(\mathbf{Z}/N\mathbf{Z})^d$, such that $c_1d_1+\cdots+c_{d-1}d_{d-1}\neq 0\in\mathbf{Z}/N\mathbf{Z}$ for all $(d_1,\ldots,d_d)\in D_\beta\backslash\{(0,\ldots,0)\}$. Since D_β is a finite set,we can obtain such $c_1,\ldots,c_d$ if we increse N enough. Then the map $c : \mathcal{T}_\beta \ni T \mapsto \phi(T)^t(c_1,\ldots,c_{d-1})\in\mathbf{Z}/N\mathbf{Z}$ has the desired property. □

EXAMPLE 4. Let $\beta=\frac{1\pm\sqrt{5}}{2}$. From Example 3 we can see $D_\beta=\{0,\pm(\beta-1),\pm(\beta-2)\}=\{0,\pm(\beta^{-1}),\pm(1-\beta^{-1})\}$ and $\phi(D_\beta)=\{(0,0),\pm(1,-1),\pm(-2,1)\}$. So we can take $(c_0,c_1)=(0,1)$.

4. Computational examples

In this section we show explicit computational examples of planar tilings. For calculations to be simple, we redefine the address map ϕ to be a map from $\mathcal{T}_\beta$ to $\mathbf{Z}^{d-1}$ as follows. Since β is a unit, $\mathbf{Z}[\beta] = \mathbf{Z}[\beta^{-1}]$. Let $w = a_{-l}a_{-l+1}\cdots a_{-1}$ be a word over Σ_β. Then $w(\beta) := a_{-l}\beta^{-l} + \cdots + a_{-1}\beta^{-1} \in \mathbf{Z}[\beta]$ has a unique representation, $w(\beta) = n_0 + n_1\beta^{-1} + \cdots + n_{d-1}\beta^{-d+1}$ where $n_0, n_1, \ldots, n_{d-1} \in \mathbf{Z}$, and we define $\phi(T(w)) = (n_1, \ldots, n_{d-1}) \in \mathbf{Z}^{d-1}$. ϕ is a surjective homomorphism with kernel $\mathbf{Z}$. So $\mathbf{Z}[\beta]/\mathbf{Z} \simeq \mathbf{Z}^{d-1}$. Each tile is represented by a word $w = a_{-l}\cdots a_{-1}$ such that $a_{-1}\beta^{-1} + a_{-2}\beta^{-2} + \cdots a_{-l}\beta^{-l}$ is a β-expansion of a number a, and so $a \in \mathbf{Z}[\beta] \cap [0,1)$.

For $w = a_{-l}a_{-l+1}\cdots a_{-1} \in L_\beta^R$, we also denote $T(w)$ by $T_{n_1,n_2,\ldots,n_{d-1}}$ if $i(a_{-l}\beta^{-l} + \cdots + a_{-1}\beta^{-1}) = (n_1, n_2, \ldots, n_{d-1})$.

There is a heuristic reasoning for our choice of this ϕ. As can be seen in the following examples, this ϕ seems to make $\phi(D_\beta)$ inside the smaller diskincluding the origin. The bound of $\phi(D_\beta)$ yields the upper-bound of the number of colors required: If $|x|, |y| \le 1$ for all $(x,y) \in \phi(D_\beta)$, then we can take $N \le 4$.

In the following we say two distinct tiles are *adjacent* if the intersection of tiles consists of infinitely many points. And we say a map $c : \mathcal{T}_\beta \mapsto \mathbf{Z}/N\mathbf{Z}$ is a coloring function if $c(T) \ne c(T')$ for all two adjacent tiles T and T'. The map c in the proof above clearly satisfies this condition.

4.1 three-colorable example

Let β be the Pisot number whose minimalpolynomial is $x^3 - x^2 - x - 1$. Then β has two Galois conjugates, $\alpha = -0.41964337760708\cdots + 0.6062907292071993\cdots i$ and $\overline{\alpha}$. *carry*$(\beta) = 0.(110)^\infty$. Figure 1 shows the tiles $T(0) = T(\epsilon)$, $T(1)$, $T(10)$, $T(11)$, $T(100)$, $T(101)$, $T(110)$. G_β^2 is shown in the Figure 6 where `[m,n,l]` denotes $m\beta^2 + n\beta + l$. From this graph we can see

$$\begin{aligned} D_\beta &= \{0, \pm(\beta^2 - \beta - 1), \pm(\beta^2 - \beta - 2), \pm(\beta^2 - 2\beta), \pm(\beta^2 - 2\beta - 1), \\ &\qquad \pm(\beta - 1), \pm(\beta - 2)\} \\ &= \{0, \pm\beta^{-1}, \pm(-1 + \beta^{-1}), \pm\beta^{-2}, \pm(1 + \beta^{-2}), \pm(\beta^{-1} + \beta^{-2}), \\ &\qquad \pm(-1 + \beta^{-1} + \beta^{-2})\}. \end{aligned}$$

Taking the coefficients of β^{-1} and β^{-2},

$$\phi(D_\beta) = \{(\pm 1, 0), (0, \pm 1), \pm(1,1)\}.$$

Hence, this tiling is 3-colorable, for we can take the coloring function c as

$$c : \mathcal{T}_\beta \ni T_{n,m} \mapsto (n + m) \bmod 3 \in \mathbf{Z}/3\mathbf{Z},$$

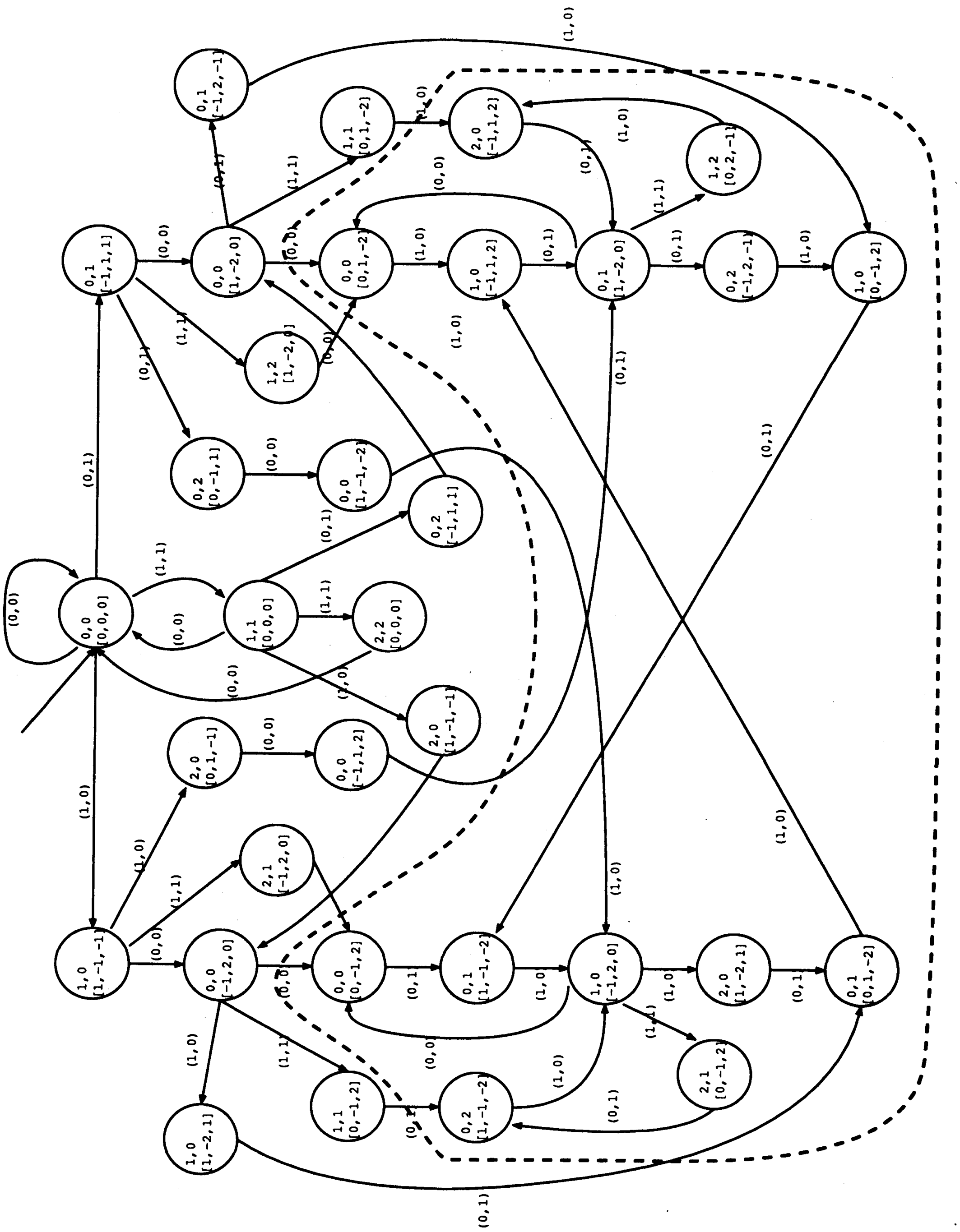

Figure 6

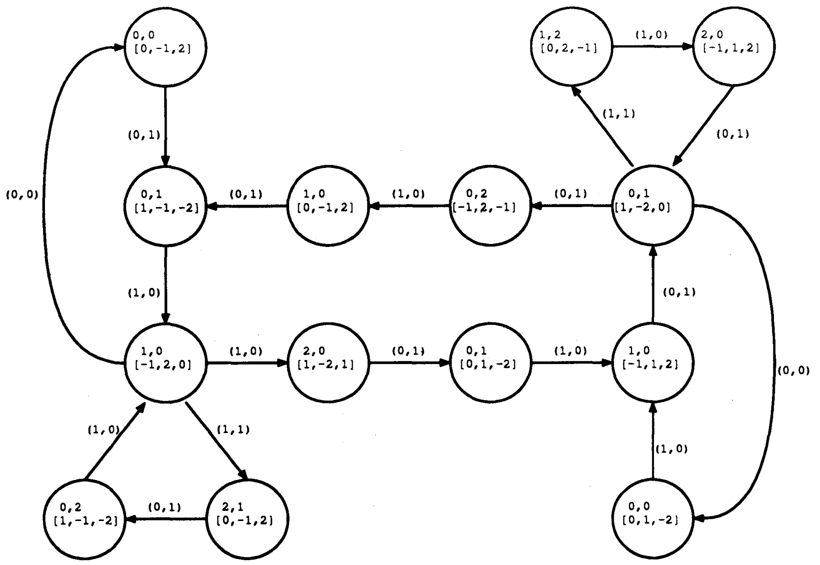

Figure 7

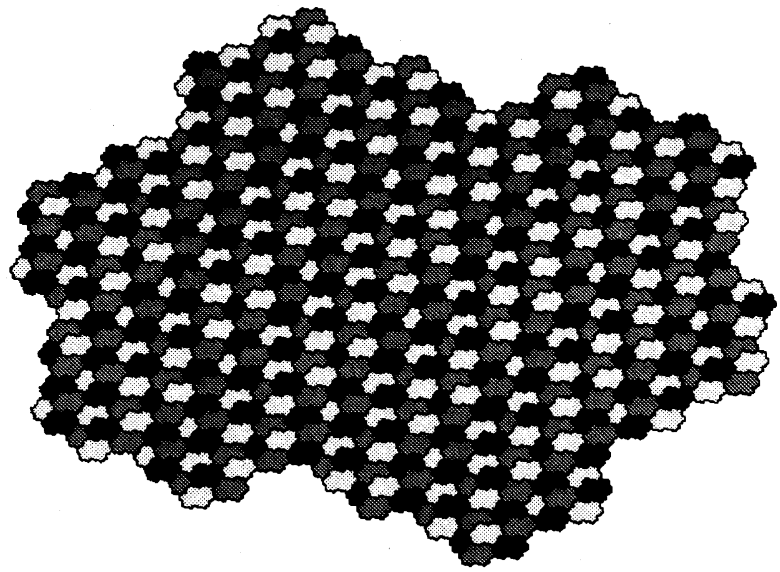

Figure 8

(see Figure 8).

This graph has two strongly connected component containing loops. One of them is the trivial one including the initial state and another is shown in Figure 7. The non-trivial component generates infinitely many points on the intersections of two distinct tiles. So every intersection of two distinct tiles consists of infinitely many points.

G_β^3 has just 171 vertices, from which we can determine the 'vertices' of $T(0)$:

$T(0) \cap T(1) \cap T(11) = \{-1\}$.

$T(0) \cap T(11) \cap T(10) = \{-\frac{1}{\alpha+1}\}$.

$T(0) \cap T(10) \cap T(110) = \{-\frac{1}{\alpha}\}$.

$T(0) \cap T(110) \cap T(100) = \{-\frac{1}{1-\alpha^3}\}$.

$T(0) \cap T(100) \cap T(101) = \{-\frac{1}{1-\alpha} + 1\}$.

$T(0) \cap T(101) \cap T(1) = \{-\frac{1}{\alpha+1}\}$.

In fact, for example

$$\sum_{n=0}^{\infty} \alpha^{3n}(\alpha+\alpha^2) = \alpha^{-1}+\alpha+\sum_{n=1}^{\infty} \alpha^{3n}(1+\alpha) = \alpha^{-2}+\alpha^{-1}+\sum_{n=0}^{\infty} \alpha^{3n}(\alpha^2+\alpha^3) = 1\,.$$

The first representation of -1 is by $T(0)$, the second is by $T(1)$, the last is by $T(11)$ and no other admissible representation exists.

4.2 not three-colorable example

Let β be the real root of x^3-x^2-1. Then $\beta = 1.46557123187676802 66\cdots$ has two complex conjugates $\alpha = -0.23278561593838401\cdots - 0.79255199251544784 83\cdots i$ and $\overline{\alpha}$. *carry*$(\beta) = 0.(100)^\infty$. In this case the tiling $\mathcal{T}_\beta$ is not three-colorable since the central tile $T(\epsilon)$ and the seven tiles surrounding $T(\epsilon)$ need at least four colors to be colored. G_β^2 obtained by our algorithmhas 31 vertices and

$$\begin{aligned} D_\beta &= \{0, \pm(\beta^2-\beta), \pm(\beta-1), \pm(\beta^2-\beta-1), \pm(\beta^2-2), \pm(\beta^2-2\beta), \pm(\beta-2)\} \\ &= \{0, \pm(\beta^{-1}), \pm(\beta^{-2}), \pm(1+\beta^{-1}), \pm(-1-\beta^{-1}+\beta^{-2}), \\ &\quad \pm(-1+\beta^{-1}+\beta^{-2}), \pm(-1+\beta^{-2})\} \end{aligned}$$

and so

$$\phi(D_\beta) = \{\pm(1,0), \pm(0,1), \pm(1,1), \pm(1,-1)\}\,.$$

Hence we can take a four-coloring function c as

$$c : \mathcal{T}_\beta \ni T_{m,n} \mapsto (m+2n) \bmod 4 \in \mathbf{Z}/4\mathbf{Z}\,.$$

(see Figure 9).

4.3 another three-colorable example

Let β be the real root of $x^3 - x - 1$. Then $\beta = 1.324717957244746\cdots$ has conjugates $\alpha = -0.662358978622373\cdots - 0.562279512062301\cdots i$ and $\overline{\alpha}$.

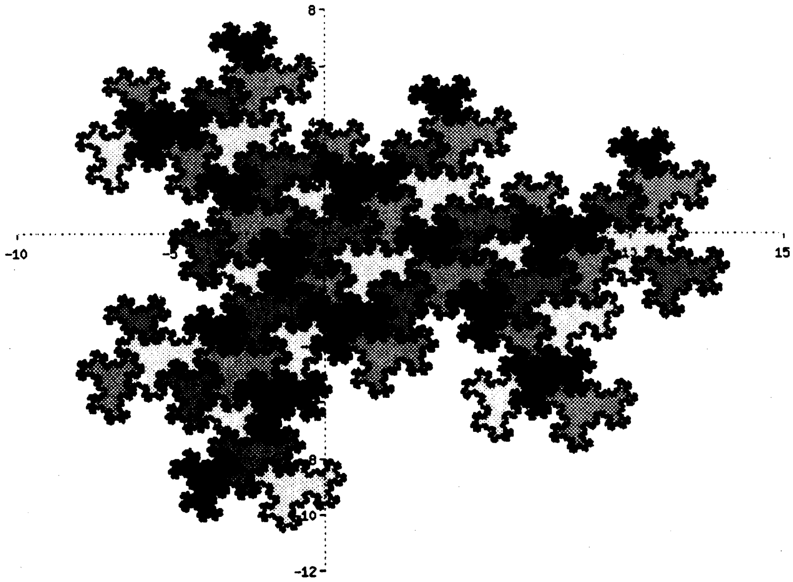

Figure 9

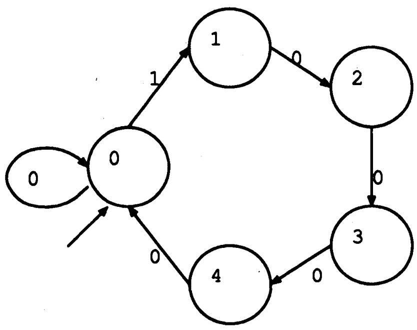

Figure 10

carry(β) = $(10000)^\infty$, and Figure 10 shows a graph which accepts L_3^R. G_β^2 is shown in Figure 11. The digits 0, a, b and c in Figure 11 represent input alphabets $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ respectively. An edge labeled by $l_1 l_2 \cdots l_n$ is the abbreviation for the path,

$$\bigcirc \xrightarrow{l_1} \bigcirc \xrightarrow{l_2} \bigcirc \to \cdots \to \bigcirc \xrightarrow{l_n} .$$

G_β^2 has 77 vertices. In this tiling there are tiles which intersect at only one point. For example, from this G_β^2 we can see $T(\epsilon)(= T(0000000)) \cap T(1000000) = \{-\alpha^{-3}\}$ as follows. Read in the first 7 input $(1,0)$, $(0,0)$, $(0,0)$, $(0,0)$, $(0,0)$, $(0,0)$, $(0,0)$ and folow the path starting from the initial state labeled a000000. Then we are at thevertex S from which there is only one road, which is labeled

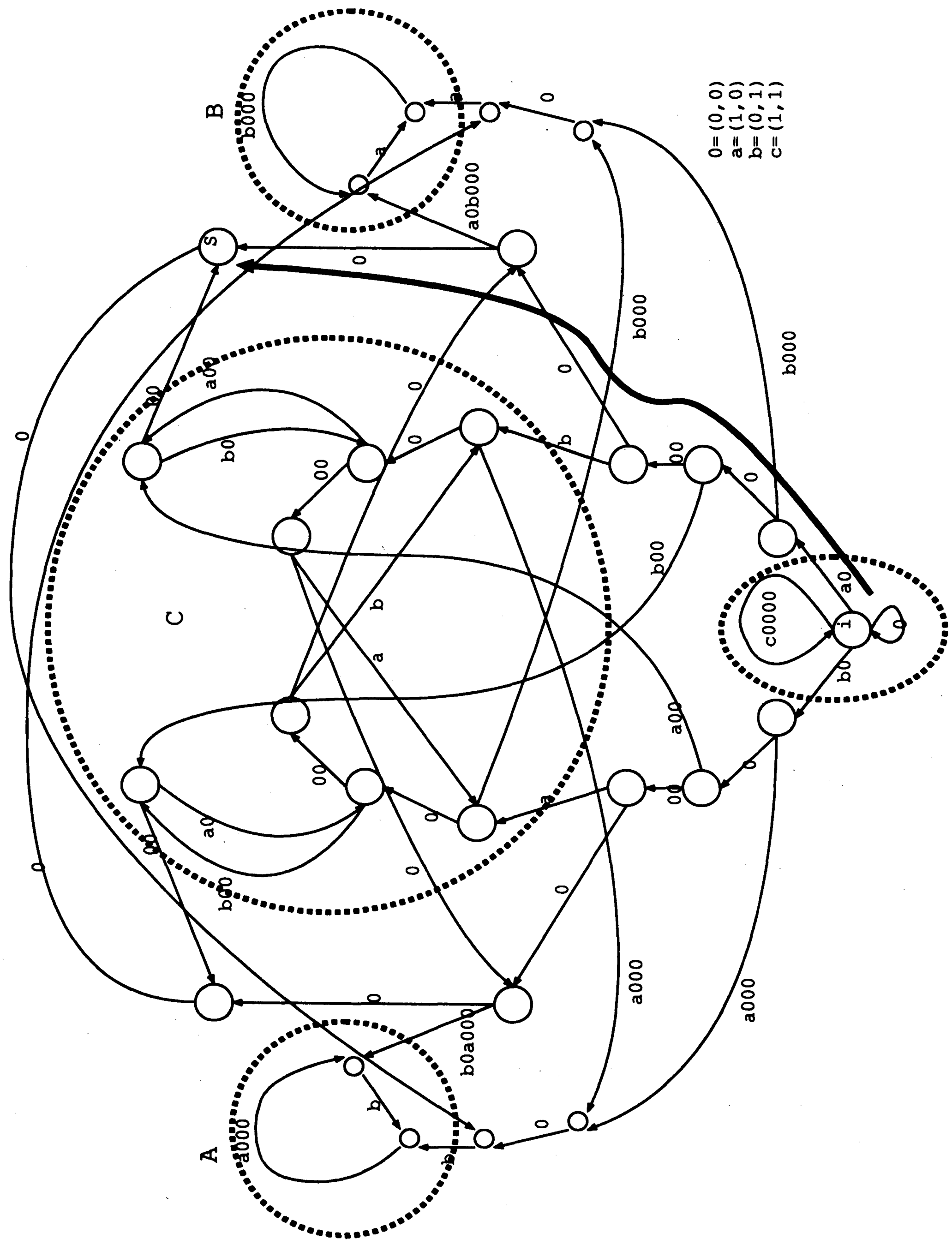

Figure 11

`0b(a000b)`$^\infty$. From this we can obtain

$$\alpha^{-7} + \alpha^2 + \alpha^7 + \alpha^{12} + \cdots = \alpha + \alpha^6 + \alpha^{11} + \cdots = -\alpha^{-3}.$$

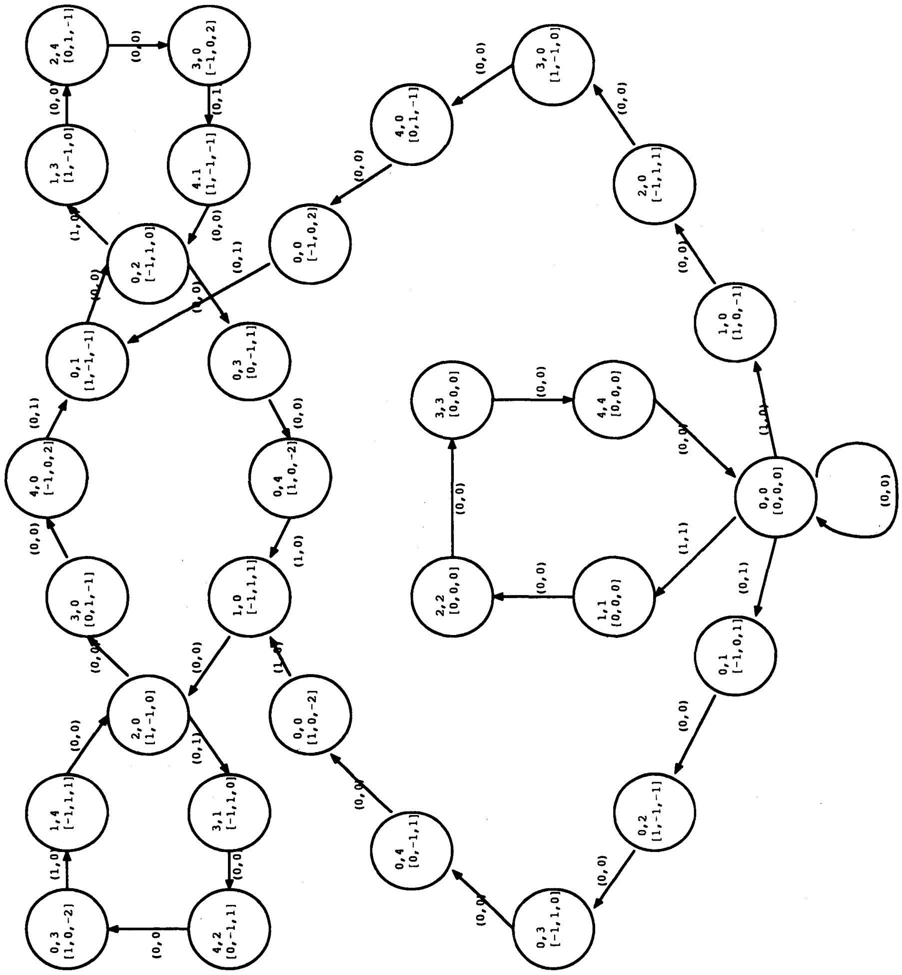

Figure 12

There are four strongly connected components containing loops, which are surrounded by dashed circles in Figure 11. Two of them, `A`, `B` consists of simple loops. They are sinks, that is, there is no out going edge from the components to outside on them So, two tiles $T(a_{-l}a_{-l+1}\cdots a_{-1})$ and $T(b_{-l}b_{-l+1}\cdots b_{-1})$ are adjacent if and only if, there exists a path starting from the initial state labeled $(a_n, b_n)_{-l\le n\le 1}$ which can be extended to a path termnating at a vertex in the component `C`. So, for the purpose to determine the adjacent tiles, we can reduce the graph by removing the vertices from which there is no path to `C`. Then we get a new graph shown in Figure 12. From this new graph, we obtain

$$D_\beta = \{\pm(\beta^2 - 1), \pm(\beta^2 - \beta - 1), \pm(\beta^2 - \beta), \pm(\beta - 1), \pm(\beta^2 - 2)\}$$

$$= \{\pm\beta^{-1}, \pm\beta^{-2}, \pm(-1+\beta^{-2}), \pm(-1+\beta^{-1}+\beta^{-2}), \pm(-1+\beta^{-1})\}.$$

So

$$\phi(D_\beta) = \{\pm(1,0), \pm(0,1), \pm(1,1)\}.$$

And hence we can define three-coloring function by

$$c(T_{m,n}) = (m+n) \bmod 3.$$

(see Figure 13).

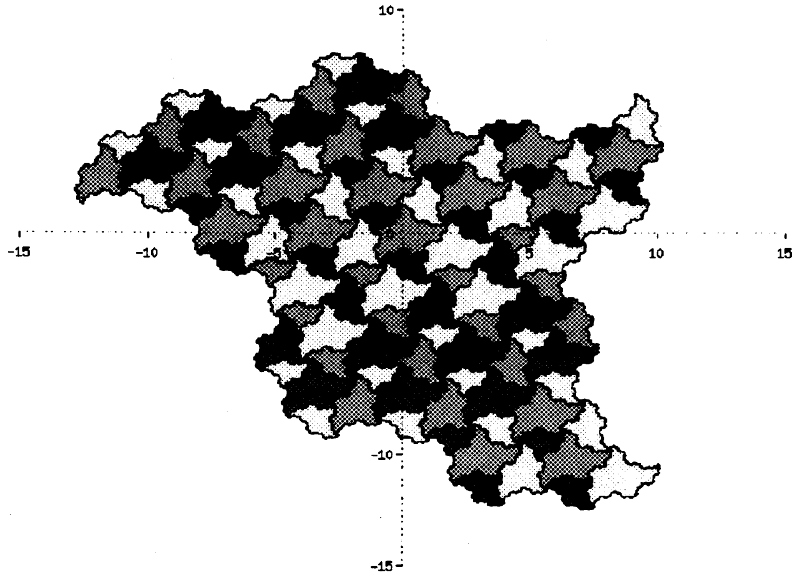

Figure 13

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