Yokohama Mathematical
Journal Vol. 47, 1999
Special Issue

FINITE-TYPE INVARIANTS OF EMBEDDINGS OF A THETA-CURVE UP TO TYPE 4

By

Atsushi Koike

(Received February 16, 1999; Revised September 16, 1999)

Abstract. We study finite-type invariants of embeddings of a theta-curve up to type 4. We give a basis of type 4 invariants by using chord diagrams, and apply it to consider symmetry of some embeddings of a theta-curve.

Introduction

The purpose of this paper is to study finite-type invariants of embeddings of a theta-curve into $\boldsymbol{R}^3$. A *theta-curve* is a graph of the shape of the Greek letter θ. We assume that three edges of the graph are oriented and distinguished, and we denote the graph by θ. Finite-type invariants were originaly defined by Vassiliev [7] as knot invariants. Birman and Lin re-defined them in a combinatorical way [2], and by generalizing it, Stanford defined finite-type invariants for embedded graphs [6]. The Stanford's work motivated the author to begin this work.

We assume that embeddings have loose or topological vertices, while Stanford discuss on rigid vertices case. However, loose vertices case is not different from rigid vertices case when G is a trivalent graph, e.g. the graph θ.

In [3], Kanenobu gives a basis for the space of finite-type invariants of type 3 of embeddings of a theta-curve. He describes the basis in terms of finite-type invariants of the assosiated 3-component link of theta-curve. Here we give a basis of finite-type invariants of type 4 in terms of chord diagrams (Theorem 1). We can use this basis to detect symmetry of some embeddings of a theta-curve (Corollary 2, 3).

The author wishes to thank Professor J. Murakami and M. Wakui for much encouragement and helpful advice.

1991 Mathematics Subject Classification: 57M15,57M25
Key words and phrases: theta-curve, finite-type invariant, chord diagram

1. Finite-type invariants and chord diagrams

By the term graph we mean an abstract graph in this paper. A finite graph is a graph which consists of a finite number of vertices and edges. An oriented graph is a graph each edge of which is oriented. In this paper, we denote a finite and oriented graph by G, and simply call it a graph. The valency of a vertex of a graph is the number of edges attached to it. Provided that the edge is loop, we count it twice.

We call an oriented graph illustrated in Fig. 1 a theta-curve. Here P_1 and P_2 are vertices, and three edges E_1, E_2, E_3 are oriented as Fig. 1. We denote by θ the oriented graph. The graph θ is a *trivalent graph*, that is, a graph each of whose vertices has valency 3. We study the class of ambient isotopy invariants of embeddings of θ into $\boldsymbol{R}^3$ which are called *finite-type invariants.*

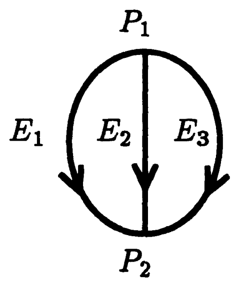

Figure 1. Theta-curve

A *regular projection* of an embedding f of G is defined similarly to one of link. It is also required that $f(P)$ is not a double point of ψ for a vertex P of G. Here by a *regular diagram* of an embedding f of G, we mean a pair of a composite map $\psi \circ f$ and information of all crossings that either of two edges that consists of them is upper, where ψ is a regular projection of the embedding f of G.

In this paper, we call simply an embedding of a graph into $\boldsymbol{R}^3$ an *embedding.* Two embeddings of G are said to be *ambient isotopic* if there exists a deformation between their regular diagrams by a finite sequence of five generalized (for graph embeddings) Reidemeister's moves R1-R5 shown in [4, Fig. 1] and planar isotopies. Note that we consider ambient isotopy classes of embeddings of a graph G into $\boldsymbol{R}^3$, but not that of the images of embeddings. For example, the two embeddings of θ in Fig. 2 are not ambient isotopic though their images are ambient isotopic.

To study finite-type invariants of embeddings of a graph G, we treat immersions of G all singularities of which are not vertices of G and are transversally crossing double points. We call them *singular embeddings.* The *degree* of a singular embedding f is defined as the number of the double points of f. We regard

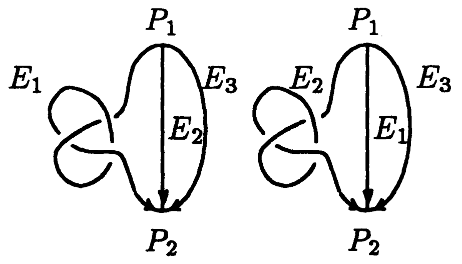

Figure 2.

embeddings as singular embeddings of degree zero. The lefthand figure of Fig. 3 is an example of a singular embedding of θ. In this paper, we represent a double point by a thick dot in order to distinguish one from a vertex of a graph.

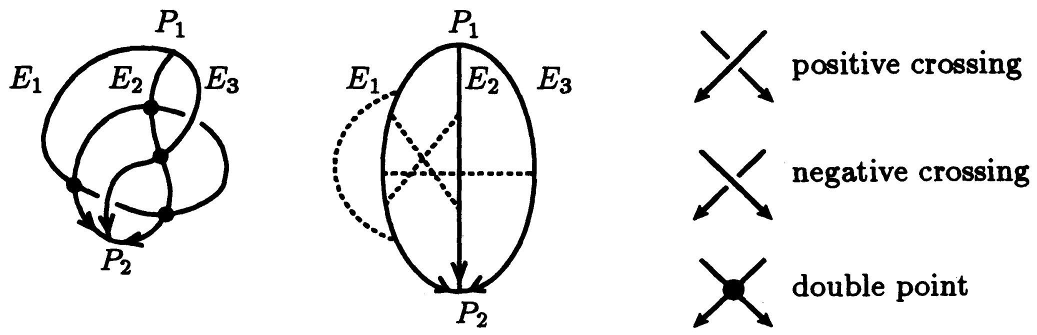

Figure 3.

We state the definition of finite-type invariant. Let v be a $\boldsymbol{C}$-valued ambient isotopy invariant of embeddings of G. It is possible to extend v to an invariant of singular embeddings inductively and uniquely by the Vassiliev skein relation:

$$v(f_x) = v(f_{x+}) - v(f_{x-})$$

where f_x is a singular embedding of G with a double point x and f_{x+}, f_{x-} are ones obtained from f_x by replacing x by a positive crossing and a negative crossing respectively. The extended v is called an invariant of type n if $v(f) = 0$ for an arbitrary singular embedding f that has more than n double points, where n is a natural number. If v is an invariant of type n for some natural number n, it is called a *finite-type invariant.*

For $m \leq n$, an invariant of type m is also an invariant of type n by definition. Finite type invariants satisfy 1-term relation [2, Fig. 7], 4-term relation [2, Fig. 14], vertex-edge relation [6, Fig. 10], and the relation in Fig. 4. They are correspond to Reidemeister moves R1, R3, R4, and R5 respectively.

Let $\mathcal{F}(G)$ be the complex vector space spanned by all embeddings of G. Since we consider only finite-type invariants in this paper, we regard a singular

v () $= 0$

Figure 4.

embedding as an element of $\mathcal{F}(G)$ by removing all singularities applying the relation $f_x = f_{x+} - f_{x-}$ inductively, where f_{x+}, f_{x-}, and f_x are as above. Then, we regard $\mathcal{F}(G)$ as the complex vector space spanned by all singular embeddings subject to this relation. We denote by $\mathcal{F}_i(G)$ the subspace of $\mathcal{F}(G)$ generated by all singular embeddings of degree i. Thus we obtain a filtration $\mathcal{F}(G) = \mathcal{F}_0(G) \supset \mathcal{F}_1(G) \supset \mathcal{F}_2(G) \cdots$. We denote by $V_i(G)$ the vector space of all invariants of type i of embeddings of G. In the same fashion as knot case, $V_i(G)/V_{i-1}(G)$ is isomorphic to $(\mathcal{F}_i(G)/\mathcal{F}_{i+1}(G))^*$ as a vector space.

We introduce a concept of chord diagrams with support G. They have information of the relative position of the inverse images of double points of singular embeddings.

A *chord diagram* with *support* G is a graph G together with finitely many dashed segments (not necessary straight) whose arbitrary endpoints are on $G -$ {the vertices of G} and are all distinct. The dashed segments are called *chords*. Arbitrary two chords of a chord diagram do not intersect even if they cross in figure. The *degree* of a chord diagram is defined as the number of chords of it. The middle figure of Fig. 3 is an example of chord diagram with support θ. We identify two chord diagrams D and D' if there exists a homeomorphism $h : D \to D'$ such that h maps chords to chords and G to G.

Let D be a chord diagram with support G. A singular embedding f_D of G is called an embedding of D if the following condition is satisfied: $f_D(p) = f_D(p')$ if and only if p and p' are the two endpoints of a chord in D or $p = p'$. The degree of a chord diagram and that of its embedding are same by definition. the singular embedding in Fig. 3 is an embedding of the chord diagram in Fig. 3. For any singular embedding f, there exists a chord diagram D_f such that f is an embedding of D_f, and D_f is determined uniquely under the above identification of chord diagrams.

We denote by $\mathcal{D}(G)$ the $\boldsymbol{C}$-linear space spanned by chord diagrams with support G. $\mathcal{D}(G)$ is naturally graded by the number of chords. We denote by $\mathcal{D}_i(G)$ the subspace of $\mathcal{D}(G)$ that is spanned by chord diagrams of degree i. We define the following four kinds of relations in $\mathcal{D}(G)$ and in $\mathcal{D}_i(G)$, framing independence relations (FI), 4-term relations (4T), vertex-edge relations (VE), and the relations induced from the Reidemeister move 5 (R5c), as shown in Fig. 5. The signature of each term in the VE relation is determined by the orientation of the edge which is attached to the vertex in figure and has the endpoint of the chord in figure. It is plus if the edge is oriented away from the vertex, and it is minus if it is oriented toward the vertex. Each term in the four kinds of relations FI, 4T, VE, and R5c is a local part of a chord diagram whose embedding is a singular embedding in the previous four relations respectively (and the other parts are identical). These relations are homogenious. We denote the quotient space $\mathcal{D}_i(G)/($ FI, 4T, VE, and R5c $)$ by $\mathcal{A}_i(G)$.

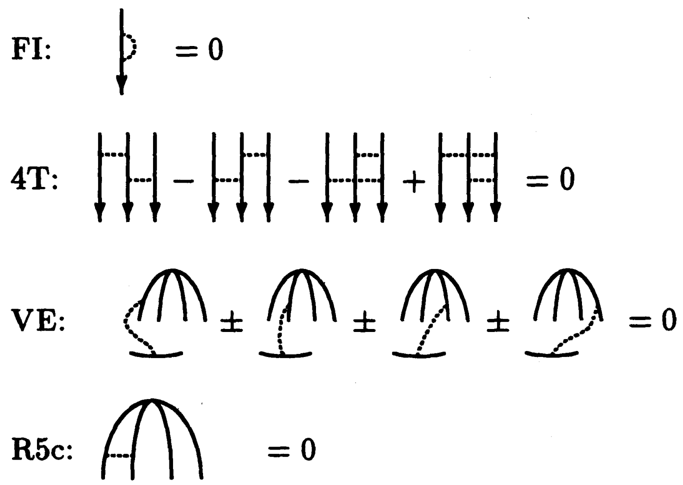

Figure 5.

The following is due to Stanford [6, Proposition 1.1]. Two singular embeddings f, f' are two embeddings of a chord diagram if and only if f' can be obtained from f by a finite sequence of crossing changes and ambient isotopy in $\boldsymbol{R}^3$. On account of it, a mapping corresponding each chord diagram of degree k with support G to its arbitrary embedding induces a surjective linear map $\varphi : \mathcal{A}_k(G) \to \mathcal{F}_k(G)/\mathcal{F}_{k+1}(G)$.

Let G be a trivalent graph. Then there exists the linear map $\hat{Z} : \mathcal{F}(G) \to \mathcal{D}(G)$ which is called *Kontsevich invariant* [4]. The map $\hat{Z}$ induces a linear map $[\hat{Z}] : \mathcal{F}_k(G)/\mathcal{F}_{k+1}(G) \to \mathcal{A}_k(G)$, and $[\hat{Z}]$ is the inverse map of φ (see [2] (knot case)). Therefore, φ is an isomorphism. So studying the space of $\mathcal{A}_i(G)$ for $i \leq n$ is studying the space of invariants of type n.

For each of elements of a basis of $\mathcal{A}_i(G)$ for $i \leq n$, we take an arbitrary chord diagram which is a representative element of it respectively. Corresponding with each of the representative elements, we assign an arbitrary singular embeddings which is an embedding of the chord diagram respectively. Corresponding with each of the singular embeddings, we assign a value of invariant on the singular embedding respectively. We call a triplet of the representative elements, singular embeddings and the value of invariant an *initial data*. The fact that φ is an isomorphism implies that an initial data determines an invariant of type n, and conversely that an arbitrary invariant of type n is obtained by giving an initial data.

2. The results

We obtain the following result about the vector space $\mathcal{A}_i(\theta)$. The edges of chord diagrams in figures are labeled E_1, E_2, E_3 from lefthand to righthand like a chord diagrm in Fig. 3.

THEOREM 1. *For $i \le 4$, the chord diagrams in Fig. 6 give a basis of $\mathcal{A}_i(\theta)$.*

$\mathcal{A}_0(\theta) = \langle\theta\rangle$
$\mathcal{A}_1(\theta) = 0$
$\mathcal{A}_2(\theta) = \langle\alpha_1, \alpha_2, \alpha_3\rangle$
$\mathcal{A}_3(\theta) = \langle\beta_1, \beta_2, \beta_3, \gamma\rangle$
$\mathcal{A}_4(\theta) = \langle\delta_1, \delta_2, \delta_3, \epsilon_1, \epsilon_2, \epsilon_3, \zeta_{12}, \zeta_{23}, \zeta_{13}, \eta\rangle$

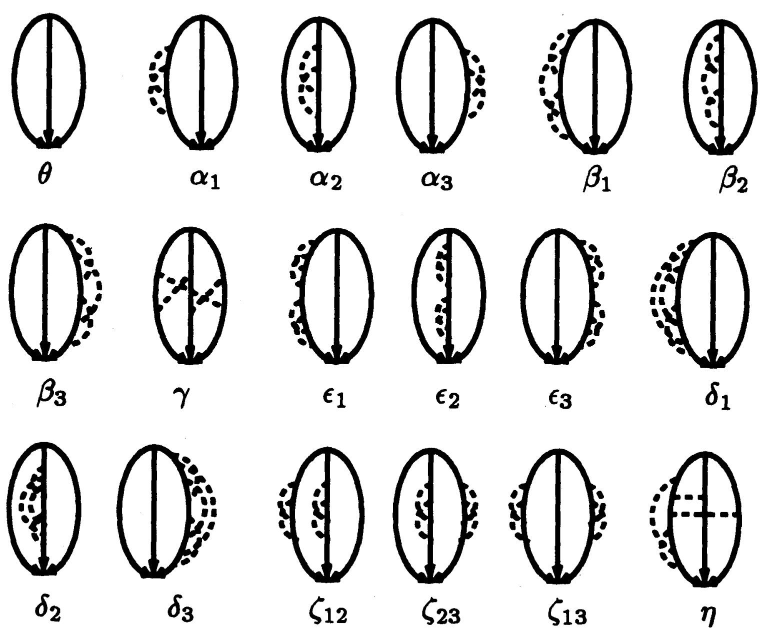

Figure 6.

Proof. Here is outline of the proof. It is easy to check that whether a chord diagram vanishes by FI or R5c relations. Such a chord diagram is 0 in $\mathcal{A}_i(\theta)$. We solve the system of equation of all 4T and VE relations which consists of chord diagrams of respective degree. ■

We are ready to exhibit a table of the coefficients of the chord diagrams in $\mathcal{A}_i(\theta)$ ($i \le 4$) when we express them by the linear combination of the basis, and the system of equations we solved.

Let W_i be the vector space of coordinates vector relative to the basis of $\mathcal{A}_i(\theta)$ for $i \le 4$. We make automorphisms of the graph θ act on chord diagrams with support θ. The actions move only the support θ of chord diagrams. An automorphism of θ induces a linear map $\mathcal{D}(\theta) \to \mathcal{D}(\theta)$, a linear map $\mathcal{A}_i(\theta) \to \mathcal{A}_i(\theta)$, and a linear map $W_i \to W_i$. We denote by $\mathrm{Aut}_+(\theta)$ the group of all orientation preserving automorphisms of the graph θ. Let ϕ be in $\mathrm{Aut}_+(\theta)$. We denote the linear map $W_i \to W_i$ induced from ϕ by $\tilde{\phi}_i$. The relations in Fig. 7 hold in $\mathcal{A}_3(\theta)$ and $\mathcal{A}_4(\theta)$. From the shape of the chord diagrams which form the basis and from the relations in Fig. 7, the following Corollary holds immediately.

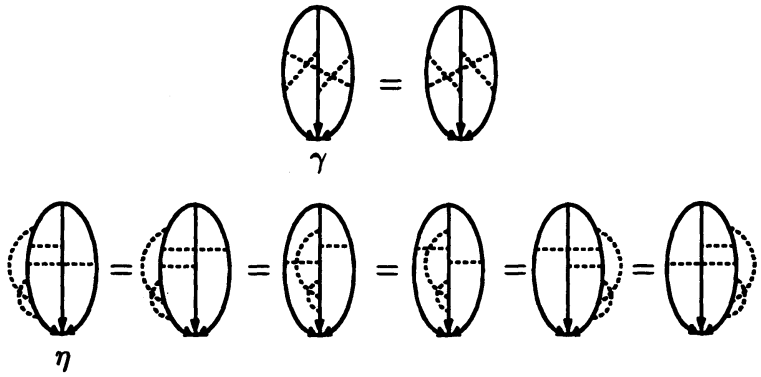

Figure 7.

COROLLARY 2. *The linear map $\tilde{\phi}_i$ induces a permutation of the basis of $\mathcal{A}_i(\theta)$ for $i \leq 4$. The permutation only exchanges subscript indices of the basis.*

Let $e_1, \ldots, e_l$ denote the singular embeddings which corresponding to the basis of $\oplus_{j=0}^{i} \mathcal{A}_j(\theta)$, where $l = \sum_{j=0}^{i} \dim \mathcal{A}_j(\theta)$. For an embedding f of θ there exist scalars $c_1, c_2, \cdots, c_l$ in $\boldsymbol{C}$ such that $v(f) = \sum_{k=1}^{l} c_k v(e_k)$. We denote the vector $(c_1, \ldots, c_l)$ by $\tilde{f}$. By correspondence of a singular embedding with the chord diagram, we regard $\tilde{f}$ as an element of $W_0 \oplus W_1 \oplus \cdots \oplus W_i$. For $\phi \in \mathrm{Aut}_+(\theta)$, we denote the map $W_0 \oplus W_1 \oplus \cdots \oplus W_i \to W_0 \oplus W_1 \oplus \cdots \oplus W_i$ induced from ϕ by $\tilde{\phi}_{\leq i}$. For an embedding f of θ and for $\phi \in \mathrm{Aut}_+(\theta)$, if $\tilde{f} \neq \tilde{\phi}_{\leq i}(\tilde{f})$ for $i \leq 4$, then $f \circ \phi$ is not ambient isotopic to f. By using Corollary 2, we can compute $\tilde{\phi}_{\leq 4}(\tilde{f})$ actually.

We denote by $\bar{\phi}$ the automorphism of the graph $\bar{\theta}$ which maps P_1 to P_2 and maps E_i to E_i $(i = 1, 2, 3)$ when we forget the orientations of the edges. On the process of obtaining the basis, we observe that the following corollary also holds.

COROLLARY 3. *For $i \leq 4$, and for an arbitrary chord diagram $D \in \mathcal{D}(\theta)$, D and $\bar{\phi}(D)$ are equivalent in $\mathcal{A}_i(\theta)$.*

Besides, D and $\bar{\phi}(D)$ are also equivalent in $\mathcal{D}_i(\theta)/($ FI, 4T, and R5c $)$ for $i \leq 4$. On account of Corollary 3, invariant of type 4 is invalid to discriminate two embeddings f and $(f \circ \bar{\phi})$.

References

[1] D. Bar-Natan, On the Vassiliev knot invariants, *Topology*, **34** (1995), 423–472.

[2] J. S. Birman and X.-S. Lin, Knot polynomials and Vassiliev's invariants, *Invent. Math.*, **111** (1993), 225–270.

[3] T. Kanenobu, Vassiliev-type invariants of a theta-curve, *J. Knot Theory & Its Ramifications*, **6** (1997), 455–477.

[4] L. H. Kauffman, Invariants of graphs in three-space, *Trans. Amer. Math. Soc.*, **311** (1989), 697–710.

[5] J. Murakami and T. Ohtsuki, Topological Quantum Field Theory for the Universal Quantum Invariant, *Commun. Math. Phys.*, **188** (1997), 501–520.

[6] T. Stanford, Finite-type invariants of knots, links, and graphs, *Topology*, **35** (1996), 1027-1050.

[7] V. A. Vassiliev, Cohomology of knot spaces, in "Theory of singularities and its applications" (V.I. Arnold, Ed.), *Advanced in Soviet Math.*, **1**, AMS Providence, RI (1990) 23–69.

Department of Mathematics,
Graduate School of Science,
Osaka University,
Toyonaka,
Osaka 560-0043,
JAPAN.
E-mail: `smv077ka@ex.ecip.osaka-u.ac.jp`