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EVERY 5-CONNECTED TRIANGULATIONS OF THE KLEIN BOTTLE IS HAMILTONIAN

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Abstract. In this paper, we shall prove that every 5-connected triangulation of the Klein bottle has a Hamilton cycle, which is contractible.

Introduction

Let G be a graph. A cycle C in G is said to be a *Hamilton cycle* if C contains all vertices of G. When G has a Hamilton cycle, G is said to be *Hamiltonian.* Many people have studied sufficient conditions for graphs to be Hamiltonian. In this paper, we also consider it with the embeddability of graphs in closed surfaces.

Tutte proved a cerebrated theorem, which states that every 4-connected planar graph is Hamiltonian [7]. This is an extention of Whitney's theorem [10] for 4-connected maximal planar graphs. Moreover, Thomassen [6] showed in 1983 that every 4-connected planar graph is Hamiltonian-connected, that is, any two vertices are linked by a path containing all the vertices of the graph.

Similar problems also have been considered for surfaces other than the plane. Thomas and Yu have proved that every 4-connected projective-planar graph is Hamiltonian-connected [8]. It is a famous conjecture posed by Nash-Williams that every 4-connected toroidal (or Klein bottlal) graph is Hamiltonian. The conjecture is still open now, but weaker versions of the conjecture have been established. Richter and the first author [2] have proved that 5-connected triangulations of the torus are Hamiltonian. (A *triangulation* of a closed surface is an embedding of a simple graph such that each face is bounded by a cycle of length 3.) Thomas and Yu recently showed that 5-connected toroidal graphs are Hamiltonian [9]. Also, Yu [11] has showed that a 5-connected triangulation on a closed surface of Euler genus g has a Hamilton cycle if the triangulation has representativity at least $96(2^g-1)$. The 5-connectedness is necessary since there are 4-connected non-Hamiltonian graphs on closed surface with negative Euler

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characteristics. (The *representativity* of an embedding G on a closed surafce F^2 is the minimum number of intersecting points of G with any non-contractible closed curve on F^2 [5]. Note that the representativity of a triangulation G coincides with the length of the shortest non-contractible cycle of G and is at least 3.)

In this paper, we establish the theorem for triangulations of the Klein bottle. By Yu's theorem, if such a triangulation has representativity at least 288, then it is Hamiltonian. However, we claim that no restriction for representativity is needed. The following is our main theorem:

THEOREM 1. *Every* 5*-connected triangulation of the Klein bottle is Hamiltonian.*

In our proof of this theorem, we shall find a Hamilton cycle which is contractible. Is there a non-contractible Hamilton cycle in a 5-connected triangulation of the Klein bottle?

1. Proof of the theorem

A simple closed curve in the Klein bottle is called a *meridian* if it is orientation-preserving and is non-separating. Any meridian cuts open the Klein bottle into an annulus or a cylinder. A cycle C in a graph G embedded on the Klein bottle is called a *meridian cycle* if it is a meridian as a simple closed curve on the Klein bottle.

The following theorem presents one of the key facts for our proof of Theorem 1 and has been proved by Lawrencenko and Negami [4] as a consequence of their classification of the *irreducible triangulations* of the Klein bottle.

THEOREM 2. (Lawrencenko and Negami [4]) *Every* 4*-connected triangulation of the Klein bottle has two disjoint meridian cycles.*

Let G be a graph embedded on a surface F^2. A subgraph K in G is called a *cylindrical subgraph* if there are two disjoint cycles C_1 and C_2 that bound together a cylinder in F^2 containing the whole of K and those cycles are called its *boundary cycles*. For example, if G is embedded on the Klein bottle, and if C_1 and C_2 are two disjoint meridian cycles in G, then $C_1 \cup C_2$ bounds a cylindrical subgraph at each side.

LEMMA 3. *Let G be a triangulation of the Klein bottle. If G contains two disjoint meridian cycles, then G has a* 2*-connected spanning cylindrical subgraph.*

Proof. Let $K_{\{C_1,C_2\}}$ denote one of the two cylindrical subgraphs bounded by a pair of disjoint cycles $\{C_1, C_2\}$. It is easy to see that $K_{\{C_1,C_2\}}$ is 2-connected. Otherwise, there would be a simple closed curve ℓ which meet $K_{\{C_1,C_2\}}$ only at a cut vertex v. Since ℓ has to cross C_i at an even number of vertices, ℓ meet G only at v, which is contrary to either G being 3-connected or 3-representative.

Choose such a pair $\{C_1, C_2\}$ so that $K_{\{C_1,C_2\}}$ includes faces of G as many as possible, and let A be the annulus bounded by $C_1 \cup C_2$ which does not include $K_{\{C_1,C_2\}}$. Suppose that $K_{\{C_1,C_2\}}$ is not spanning, that is, there is a vertex v of G inside A. Since G is 3-connected, there are three inner disjoint paths from v to $C_1 \cup C_2$ and we may assume that at least two of them, say P_1 and P_2, terminate at vertices v_1 and v_2 on C_1. Then there is a meridian cycle C_1' passing through $C_1 \cup P_1 \cup P_2$ and the cylindrical subgraph $K_{\{C_1',C_2\}}$ includes faces of G more than $K_{\{C_1,C_2\}}$. This is contrary to the maximality of $K_{\{C_1,C_2\}}$. Therefore, $K_{\{C_1,C_2\}}$ must be 2-connected and spanning. ■

A cylindrical subgraph K with boundary cycles C_1 and C_2 is said to be *internally k-connected* if $K - X$ does not contain a component disjoint from $C_1 \cup C_2$ for any subset $X \subset V(K)$ with $|X| < k$. It is clear that if K is a cyclindrical subgraph in a k-connected graph G, then K is internally k-connected.

LEMMA 4. *Every 5-connected triangulation G of the Klein bottle has a 3-connected, internally 5-connected, spaning cylindrical subgraph bounded by two disjoint meridian cycles.*

Proof. Let $K = K_{\{C_1,C_2\}}$ be the cylindrical subgraph as in the proof of Lemma 3, which exists actually by Theorem 2. Then it is 2-connected, internally 5-connected and spanning. If K is not 3-connected, then there is a *cut pair* of vertices $\{u, v\}$ for K, that is, $K - \{u, v\}$ is disconnected. Since K is a subgraph in a triangulation, those two vertices u and v have to lie together on one of C_1 and C_2, and they are joined with an edge not lying on the boundary cycle. Thus, we assume that the number of cut pairs of vertices in K is the fewest among all of such cylindrical subgraphs.

Suppose that K has a cut pair $\{u, v\}$ lying on C_2 and let A' and A'' be the two segments of C_2 bounded by $\{u, v\}$ so that $A'' + uv$ is contractible and $C_2' = A' + uv$ is a meridian cycle in G. Furthermore, we may assume that A'' does not contain any other cut pair of K. Let L and M be the two subgraphs of K such that $L \cup M = K$ and $L \cap M = \{u, v\}$ and that L includes A''.

We now construct C_1' as follows. Let $u_1, \dots, u_s$ and $v_1, \dots, v_t$ be the neighbours of u and v in L respectively in their cyclic order such that $u_1, v_t \in V(A'')$ and $u_s = v_1$. Since G is a triangulation, each of the edges u_iu_{i+1} and v_jv_{j+1} is in G for $i = 1, \dots, s-1$ and $j = 1, \dots, t-1$. Let $\bar{A}$ be the walk $u_1 \cdots u_s(=$

$v_1)v_2 \cdots v_t$. Note that $\bar{A}$ is a path. For if $u_i = v_j$ for some $i \neq s$ and $j \neq 1$, then $\{u_i = v_j, u, v\}$ would be a 3-cut of G, contrary to G being 5-connected.

The edge uu_1 is in two triangles of G. One of them contains the vertices u, u_1 and u_2, and the other triangle contains u, u_1 and another vertex, say y. Clearly, $y \in V(C_1)$. Similarly, let z be the vertex of C_1 that is in a triangle together with v and v_t. Let A_0 be the path with ends y and z obtained by adding y, yu_1, v_tz, z to $\bar{A}$. Decompose C_1 into two paths A_1 and A_2 with ends y and z so that the cycle $A_0 \cup A_1$ is non-contractible. Clearly, A_2 has length at least 2; for otherwise $\{u, v, y = z\}$ or $\{u, v, y, z\}$ would be a 3- or 4-cut of G, contrary to the 5-connectedness of G. Let $C_1' = A_0 \cup A_1$. See Figure 1.

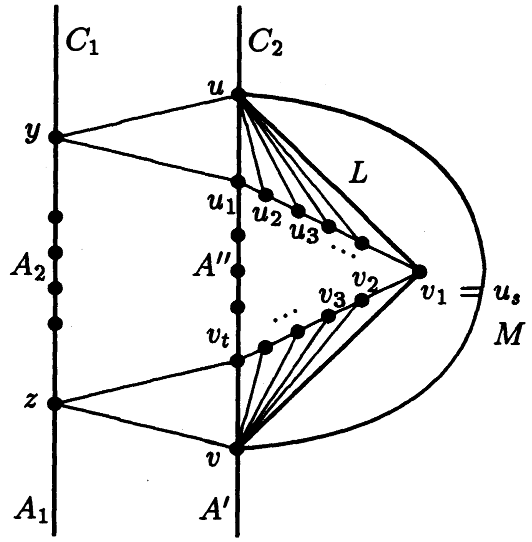

Figure 1. Structure around L

It is easy to see that the cylindrical subgraph $K' = K_{\{C_1', C_2'\}}$ with boundary cycles C_1' and C_2' is 2-connected and spanning. We shall show that K' has fewer cut pairs than K. First, $\{u, v\}$ is not a cut pair of K'. Let $\{w, x\}$ be any cut pair of K'. Since there is an edge bewteen w and x, $\{w, x\}$ is contained in either A_1 or A_0. In the fist case, $\{w, x\}$ is a cut pair of K, too. In the second case, $\{u, v, w, x\}$ would be a 4-cut of G, contrary to G being 5-connected. Thus, the cut pairs of K' correspond to those of K except $\{u, v\}$ injectively, but this is contrary to the assumption of K.

Therefore, K has no cut pair and is 3-connected. The lemma follows. ■

Let G be a graph and C a cycle in G. A *C-bridge* is the subgraph B induced by a component of $G - C$ and all edges incident to the component and the vertices of B lying on C are called its vertices of attachment. An edge uv not on C also is called a C-bridge if u and v lie on C, but it is said to be *trivial*.

Let $F \subset E(G)$ be a set of edges. A cycle C in G is said to be *F-admssible* if every C-bridge in G has at most three vertices of attachment and if every C-bridge in G containing an edge of F has at most two vertices of attachment. The following lemma shows another key fact to prove the theorem and has been established by Thomas and Yu [8] to prove that every 4-connected projective planar graph is Hamiltonian.

LEMMA 5. (Thomas and Yu [8]) *Let G be a 2-connected cylindrical graph with boundary cycles C_1 and C_2 and let $F = E(C_1) \cup E(C_2)$. Given an edge $e \in E(C_1)$, there exists an F-admissible cycle C such that $e \in E(C)$ and that no C-bridge contains edges of both C_1 and C_2.*

Now we have prepared what we need to prove Theorem 1.

Proof of Theorem 1. Let G be a 5-connected triangulation of the Klein bottle. By Lemma 4, G has a 3-connected internally 5-connected spanning cylindrical subgraph K bounded by two disjoint cycles C_1 and C_2. By Lemma 5, K contains a $(E(C_1) \cup E(C_2))$-admissible cycle C through any edge of C_1. Let B be a nontrivial C-bridge. If B is disjoint from $C_1 \cup C_2$ of if B meets $C_1 \cup C_2$ only in its attachment, then B has at most three vertices of attachment. But this is contrary to the internal 5-connectedness of K. If B contains a vertex v of $C_1 \cup C_2$ which is not of attachment, then the edges of the C_i incident to v must also be contained in B and so B must have at most two vertices of attachment. But. this is a contradiction with the 3-connectedness of K. Therefore, all C-bridge are trivial, and hence C is a Hamilton cycle of G. ■

We can find two disjoint non-contractible cycles in any irreduicible triangulation listed in [3]. This implies that every triangulation of the torus also contains those. Using this fact instead of Theorem 1, we can give a brief proof of the same theorem for the torus, that is, the theorem that every 5-connected triangulation of the torus is Hamiltonian, already proved in [2]. All of our arguments on cylindrical subgraphs in this paper, except Theorem 2, will work well for the torus.

References

[1] R. Brunet, M.N. Ellingham, Z. Gao, A. Metzlar, R.B. Richter, Spanning planar subgraphs of graphs in the torus and Klein bottle, *J. Combin. Theory, Ser. B*, **65** (1995), 7–22.

[2] R. Brunet and R.B. Richter, Hamiltonicity of 5-connected toroidal triangulations, *J. Graph Theory*, **20** (1995), 267–286.

[3] S. Lawrencenko, The irreducible triangulations of the torus, *Ukrain. Geom. Sb.*, **30** (1987), 52–62. [In Russian; MR 89c:57002; English translation: *J. Soviet Math.*, **51**, No.5 (1990), 2537–2543.]

[4] S. Lawrencenko and S. Negami, Irreducible triangulations of the Klein bottle, *J. Combin. Theory, Ser. B*, **70** (1997), 265–291.

[5] N. Robertson and R. Vitray, Representativity of surface embeddings, "Paths, Flows, and VLSI-Layouts" (B. Korte, L. Lovász, H.J.Prömel and A. Schrijver, Eds), Springer-Verlag, Berlin (1990), 293–328.

[6] C. Thomassen, A theorem on paths in planar graphs, *J. Graph Theory*, **7** (1983), 169–176.

[7] W.T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.*, **82** (1956), 99–116.

[8] R. Thomas and X. Yu, Every 4-connected projective planar graphs are Hamiltonian, *J. Combin. Theory, Ser. B*, **62** (1994), 114–132.

[9] R. Thomas and X. Yu, Five-connected toroidal graphs are Hamiltonian, *J. Combin. Theory, Ser. B*, **69** (1997), 79–96.

[10] H. Whitney, A theorem on graphs, *Ann. of Math.*, **32** (1931), 378–390.

[11] X. Yu, Disjoint paths, planarizing cycles, and spanning walks, *Trans. Amer. Math. Soc.*, **349** (1997), 1333–1358.

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