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ON DISTANCES OF POSETS WITH THE SAME UPPER BOUND GRAPHS

By

Kenjiro Ogawa

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Abstract. In this paper, we consider transformations for posets with the same upper bound graph. Those posets can be transformed into each other by a finite sequence of two kinds of transformations, called *addition* and *deletion* of an order relation. This result induces a characterization on unique upper bound graphs. We deal with the distance of those posets that have the same upper bound graph.

Introduction

In this paper, we consider finite undirected simple graphs. Let $P = (X, \leq)$ be a poset. The *upper bound graph* (*UB-graph*) of P is the graph $UB(P)$ over X obatined by joining a pair of distinct elements u and v in X whenever there exists $m \in X$ such that $u, v \leq m$. We say that a graph G is a *UB-graph* if there exists a poset whose upper bound graph is isomorphic to G. These concepts were introduced by F.R. McMorris and T. Zaslavsky [2].

The total ordered set with n elements is not isomorphic to the height-one poset with a unique maximal element and $n-1$ minimal elements if $n \geq 3$, but their UB-graphs are isomorphic. Thus a natural question arises; how are those two posets that have the same UB-graphs related. In this paper, we shall answer this quetion, introducing two kinds of transformations of such posets.

1. Upper bound graphs

In this section, we introduce two kinds of transformations for those posets that have the same UB-graph.

A characterization of upper bound graphs can be found in [2] as follows: A *clique* in a graph G is a maximal set of vertices which induces a complete subgraph in G. A family $\mathcal{C}$ of complete subgraphs is said to *edge-cover* G if for each edge $uv \in E(G)$, there exists $C \in \mathcal{C}$ such that $u, v \in C$.

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THEOREM 1. (F.R. McMorris and T. Zaslavsky [2]) *A graph G is a UB-graph if and only if there exists a family $\mathcal{C} = \{C_1, ..., C_n\}$ of complete subgraphs of G such that*

(i) *$\mathcal{C}$ edge-covers G, and*

(ii) *for each C_i, there is a vertex $v_i \in C_i - (\bigcup_{j \neq i} C_j)$.*

Furthermore, such a family $\mathcal{C}$ must consist of cliques of G and is the only such family if G has no isolated vertices. ■

For a UB-graph G and a clique edge cover $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ satisfying the conditions of Theorem 1, a *kernel* $K_{UB}(G)$ of G is a vertex subset $\{v_1, v_2, \ldots, v_n\}$ such that $v_i \in C_i - (\bigcup_{j \neq i} C_j)$ for each $i = 1, \ldots, n$. In this paper, we consider a fixed labeled UB-graph G with a fixed kernel $K_{UB}(G)$.

Let $P = (X, \leq)$ be a poset and $x \in X$ an element of P. We put $L_P(x) = \{y \in X \ ; \ y < x\}$ and $U_P(x) = \{y \in X \ ; \ y > x\}$, and denote by $\mathrm{Max}(P)$ the set of all maximal elements of P. For a UB-graph G, $\mathcal{P}_{UB}(G) = \{P \ ; \ UB(P) = G, \mathrm{Max}(P) = K_{UB}(G)\}$. Each poset P in $\mathcal{P}_{UB}(G)$ is identified with the set of relations in P and hence $\mathcal{P}_{UB}(G)$ can be regared as a poset by set inclusion. For a poset $P = (X, \leq_P)$, the *canonical poset* of P is the height-one poset $\mathrm{can}(P) = (X, \leq_{\mathrm{can}(P)})$, where $x \leq_{\mathrm{can}(P)} y$ if and only if (1) $x \notin \mathrm{Max}(P), y \in \mathrm{Max}(P)$ and $x \leq_P y$, or (2) $x = y$.

Observe that if G is a UB-graph, then all posets in $\mathcal{P}_{UB}(G)$ have the same canonical poset by Theorem 1. For a UB-graph G, the *canonical poset* of G is the poset $\mathrm{can}(G) = (V(G), \leq_{\mathrm{can}(G)})$, where $x \leq_{\mathrm{can}(G)} y$ if and only if (1) $y \in K_{UB}(G)$ and $xy \in E(G)$, or (2) $x = y$. Then for a UB-graph G and any poset $P \in \mathcal{P}_{UB}(G)$, $\mathrm{can}(G) = \mathrm{can}(P)$, and $\mathrm{can}(G)$ is the minimum poset of $\mathcal{P}_{UB}(G)$.

Let x and y be two distinct elements of a poset P. Suppose that $y \notin \mathrm{Max}(P)$ and $x < y$. Then, another poset $P^-_{x<y}$ is obtained from P by subtracting the relation $x \leq y$ from P. We call this transformation the *deletion* of $x < y$ (*$x < y$-deleteion*). Now let x and y be an incomparable pair of elements in P such that $y \notin \mathrm{Max}(P)$, $U_P(y) \subseteq U_P(x)$ and $L_P(y) \supseteq L_P(x)$. Then, a poset $P^+_{x<y}$ is obtained from P by adding the relation $x \leq y$ to P. We call this transformation the *addition* of $x < y$ (*$x < y$-addition*).

We easily obtain the following facts on these transformations. Any poset P and $P^-_{x<y}$ have the same UB-graph, and P and $P^+_{x<y}$ also have the same UB-graph. Moreover the $x < y$-addition and the $x < y$-deletion are inverse transformations to each other.

The following lemma follows immediately from the above facts:

LEMMA 2. *For a UB-graph G, every poset in $\mathcal{P}_{UB}(G)$ is obtained from* $\mathrm{can}(G)$ *by additions of order relations only.* ■

The following is one of our main theorem:

THEOREM 3. *Let G be a UB-graph and let P and Q be two posets in $\mathcal{P}_{UB}(G)$. Then P can be transformed into Q by a sequence of deletions and additions of order relations.*

Proof. Both P and Q have the same canonical poset $\mathrm{can}(P) = \mathrm{can}(Q) = \mathrm{can}(G)$. Thus, it is easy to see that they can be transformed intoe each other via $\mathrm{can}(G)$. First, delete order relations among $P - \mathrm{Max}(P)$ in order untill $\mathrm{can}(G)$ is obtaiend. Next, add order relations to $\mathrm{can}(G)$ to get Q. ■

Lemma 2 implies the following result, which is Theorem 2 in F.R. McMorris and G.T. Myers [1]. For a UB-graph G with a kernel $K_{UB}(G)$, G is a *unique UB-graph* if it has only one realizing poset.

COROLLARY 4. (F.R. McMorris and G.T. Myers [1]) *Let G be a UB-graph with a kernel $K_{UB}(G)$. Then the followings are equivalent:*

(1) *G is a unique UB-graph.*

(2) $\mathrm{can}(G)$ *is only one poset whose UB-graph is G.*

(3) *$\{\mathrm{Ma}(v) \ ; \ v \in V(G) - K_{UB}(G)\}$ is an antichain with respect to set inclusions, where* $\mathrm{Ma}(v) = \{m_i \in K_{UB}(G) \ ; \ vm_i \in E(G)\}$. ■

2. The distance of posets

In this section, we introduce the distance of those posets that have the same upper bound graphs. Given a UB-graph G with a kernel $K_{UB}(G)$, the *distance* between two posets P and Q in $\mathcal{P}_{UB}(G)$, denoted by $d_{UB}(P,Q)$, is the shortest length of a sequence of deletions and additions of order relations from P to Q. Let X be a subset in $\mathcal{P}_{UB}(G)$. The *diameter* $d(X)$ is defined as $\max\{d_{UB}(P,Q) \ ; \ P,Q \in X\}$. The distance $d_{UB}(P,Q)$ satisfies the axioms of a distance. We shall consider the following two problems:

PROBLEM 1. *Let G be a UB-graph with a kernel $K_{UB}(G)$, and $P_{\max}$ be a maximal poset in $\mathcal{P}_{UB}(G)$. Estimate the value $d([\mathrm{can}(G), P_{\max}])$, where $[\mathrm{can}(G), P_{\max}] = \{Q \ ; \ \mathrm{can}(G) \leq_{\mathcal{P}_{UB}(G)} Q \leq_{\mathcal{P}_{UB}(G)} P_{\max}\}$.*

PROBLEM 2. *Let G be a UB-graph with a kernel $K_{UB}(G)$. Estimate the value $d(\mathcal{P}_{UB}(G))$.*

The poset $\mathrm{can}(G)$ has two types of binary relations. One is the set of reflexive relations. The number of reflexive relations of $\mathrm{can}(G)$ is $|V(G)|$. Tthe other is the set of relations between maximal elements and minimal elements. Note that $\mathrm{Max}(\mathrm{can}(G)) = K_{UB}(G)$. Let $N_G(v)$ be the neighborhood of a vertex v in G, that is, the set of vertices adjacent to v in G. A maximal element $m \in K_{UB}(G)$ covers all elements of $L_{\mathrm{can}(G)}(m)$, and $N_G(m) = L_{\mathrm{can}(G)}(m) - \{m\}$ for $m \in K_{UB}(G)$. Thus we have the following result:

LEMMA 5. *Let G be a UB-graph with a kernel $K_{UB}(G)$. The number of binary relations in* $\mathrm{can}(G)$ *is equal to* $|\mathrm{can}(G)| = \sum_{m \in K_{UB}(G)} |N_G(m)| + |V(G)|$. ■

For a UB-graph G with a kernel $K_{UB}(G) = \{m_1, \ldots, m_n\}$, we have

$$|\mathcal{P}_{UB}(G)| \leq \prod_{i=1}^{n} 3^{\binom{|N_G(m_i)|}{2}}.$$

We also have results on maximal posets in $\mathcal{P}_{UB}(G)$.

THEOREM 6. *Let G be a UB-graph with a kernel $K_{UB}(G)$, and P a maximal poset in $\mathcal{P}_{UB}(G)$. Then for all $x, y \in V(G) - K_{UB}(G)$ such that* $\mathrm{Ma}(x) \neq \mathrm{Ma}(y)$, *$x \leq_P y$ if and only if* $\mathrm{Ma}(x) \supset \mathrm{Ma}(y)$, *where* $\mathrm{Ma}(x) = \{m_i \in K_{UB}(G) = \mathrm{Max}(P)\ ;\ x \leq_P m_i\}$.

Proof. The necessity is obvious. To prove the sufficiency, we assume that there exist incomparable elements $x, y \in V(G) - K_{UB}(G)$ such that $\mathrm{Ma}(x) \supseteq \mathrm{Ma}(y)$. We fix x and select a maximal element satisfying the conditions as y. That is, every element $z \in U_P(y)$ is comparable with x. If $z \leq_P x$, then $y \leq_P z \leq_P x$, which is a contradiction. Hence $x \leq_P z$ for every element $z \in U_P(y)$, and $U_P(x) \supseteq U_P(y)$. Furthermore $y \notin \mathrm{Max}(P) = K_{UB}(G)$. Next we choose a minimal element x for this y under the conditions and y is maximal. Thus every element $w \in L_P(x)$ is comparable with y. If $y \leq_P w$, then $y \leq_P w \leq_P x$, which is a contradiction. So $w \leq_P y$ for every element $w \in L_P(x)$, and $L_P(x) \subseteq L_P(y)$. Therefore we can perform the $x < y$-addition for these x and y. This contradicts the maximality of P. ■

By Theorem 6, we obtain the following result immediately:

COROLLARY 7. *Let G be a UB-graph with a kernel $K_{UB}(G)$, and P be a maximal poset on $\mathcal{P}_{UB}(G)$. Then for all $x, y \in V(G) - K_{UB}(G)$, if* $\mathrm{Ma}(x) = \mathrm{Ma}(y)$, *then $x \leq_P y$ or $y \leq_P x$.* ■

Theorem 6 means that relations on a maximal poset in $\mathcal{P}_{UB}(G)$ are determined by the set inclusions on $\mathrm{Ma}(x)$. Let G be a UB-graph with a kernel $K_{UB}(G)$ and put

$$N_G[S] = \bigcap_{m \in S} N_G(m) - \bigcup_{m \in K_{UB}(G) - S} N_G(m)$$

for $S \subseteq K_{UB}(G)$. We decompose $V(G)$ as follows:

$$V(G) = K_{UB}(G) \ \cup \bigcup_{\emptyset \neq S \subseteq K_{UB}(G)} N_G[S].$$

In addition, for a maximal poset P in $\mathcal{P}_{UB}(G)$, $x \in N_G[S]$ and $y \in N_G[T]$, $x \leq_P y$ if and only if $S \supset T$. The elements in $N_G[S]$ form a total order in P, and $K_{UB}(G) = \mathrm{Max}(P)$. Therefore we have the following result, which is Theorem 2.2.12 in G.T. Myers [3]:

THEOREM 8. (G.T. Myers [3]) *Let G be a UB-graph with a kernel $K_{UB}(G)$. Then two maximal posets on $\mathcal{P}_{UB}(G)$ are isomorphic. Furthermore, the differences of two maximal posets on $\mathcal{P}_{UB}(G)$ are only total orderings of the elements in $N_G[S]$ for each nonempty set $S \subseteq K_{UB}(G)$.* ■

From Theorem 8 and the descomposition of $V(G)$ on $\mathrm{Ma}(x)$, we have the following result:

LEMMA 9. *Let G be a UB-graph with a kernel $K_{UB}(G)$, and $P_{\max}$ be a maximal poset in $\mathcal{P}_{UB}(G)$. The number of binary relations in $P_{\max}$ can be obtainde by:*

$$|P_{\max}| = \sum_{\emptyset \neq S \subseteq K_{UB}(G)} \binom{|N_G[S]|}{2} + \sum_{\emptyset \neq S \subset T \subseteq K_{UB}(G)} |N_G[S]| \times |N_G[T]| + |\mathrm{can}(G)|$$

■

Let Σ_1 and Σ_2 denote the following:

$$\Sigma_1 = \sum_{\emptyset \neq S \subseteq K_{UB}(G)} \binom{|N_G[S]|}{2}, \quad \Sigma_2 = \sum_{\emptyset \neq S \subset T \subseteq K_{UB}(G)} |N_G[S]| \times |N_G[T]|$$

For two posets $Q, R \in [\mathrm{can}(G), P_{\max}]$, R is obtained from Q by adding relations in $R \setminus Q$ and deleting relations in $Q \setminus R$. Thus we obtain the following result by above results.

THEOREM 10. *Let G be a UB-graph with a kernel $K_{UB}(G)$, and $P_{\max}$ be a maximal poset in $\mathcal{P}_{UB}(G)$. Then $d([\mathrm{can}(G), P_{\max}]) = \sum_1 + \sum_2$.*

Proof. For a poset $Q \in [\mathrm{can}(G), P_{\max}]$, $d_{UB}(Q, P_{\max}) = |P_{\max}| - |Q|$, and $d_{UB}(\mathrm{can}(G), Q) = |Q| - |\mathrm{can}(G)|$, where $|Q|$ is the number of binary relations on Q. Thus,

$$d_{UB}(\mathrm{can}(G), Q) + d_{UB}(Q, P_{\max}) = |P_{\max}| - |\mathrm{can}(G)| = d_{UB}(\mathrm{can}(G), P_{\max}).$$

For any two posets $Q, R \in [\mathrm{can}(G), P_{\max}]$, we have

$$\begin{aligned} d_{UB}(Q,R) &\le \min\{d_{UB}(Q,\mathrm{can}(G)) + d_{UB}(\mathrm{can}(G),R), \\ &\qquad d_{UB}(Q,P_{\max}) + d_{UB}(P_{\max},R)\} \\ &\le d_{UB}(\mathrm{can}(G),P_{\max}), \end{aligned}$$

since

$$\begin{aligned} d_{UB}(\mathrm{can}(G),P_{\max}) &= d_{UB}(\mathrm{can}(G),Q) + d_{UB}(Q,P_{\max}) \\ &= d_{UB}(\mathrm{can}(G),R) + d_{UB}(R,P_{\max}). \end{aligned}$$

Thus

$$d([\mathrm{can}(G),P_{\max}]) = d_{UB}(\mathrm{can}(G),P_{\max}) = \Sigma_1 + \Sigma_2.$$

■

Next we consider the diameter of $\mathcal{P}_{UB}(G)$. For two posets P, Q in $\mathcal{P}_{UB}(G)$, P can be obtained from Q by the following operations: (1) delete a relation from Q, (2) add a relation to Q, (3) delete a relation from Q and add its reverse relation to Q. We define two special posets related to a maximal poset P in $\mathcal{P}_{UB}(G)$. A maximal poset Q in $\mathcal{P}_{UB}(G)$ is a *quasi-dual maximal poset* of P, when $x \le_Q y$ if and only if:

(1) $x \le_P y$ if $\mathrm{Ma}(x) \neq \mathrm{Ma}(y)$, or

(2) $y \le_P x$ if $\mathrm{Ma}(x) = \mathrm{Ma}(y)$.

A poset R in $\mathcal{P}_{UB}(G)$ is a *submaximal poset* of P, when $x \le_R y$ if and only if:

(1) $y \in K_{UB}(G)$ and $x \le_P y$, or

(2) $x = y$, or

(3) $x, y \in V(G) - K_{UB}(G)$, $\mathrm{Ma}(x) = \mathrm{Ma}(y)$ and $x \le_P y$.

LEMMA 11. *Let G be a UB-graph with a kernel $K_{UB}(G)$, P a maximal poset in $\mathcal{P}_{UB}(G)$, and Q a quasi-dual maximal poset of P. Then $d_{UB}(P,Q) = 2\Sigma_1$.* ■

LEMMA 12. *Let G be a UB-graph with a kernel $K_{UB}(G)$, P a maximal poset in $\mathcal{P}_{UB}(G)$, and R a submaximal poset of P. Then $d_{UB}(P,R) = \Sigma_2$.* ■

LEMMA 13. *Let G be a UB-graph with a kernel $K_{UB}(G)$, P a maximal poset in $\mathcal{P}_{UB}(G)$, Q a quasi-dual maximal poset of P, and R a submaximal poset of P. Then $d_{UB}(Q,R) = 2\Sigma_1 + \Sigma_2$.* ■

By these lemmas we have the following result.

THEOREM 14. *For a UB-graph G with a kernel $K_{UB}(G)$, we have*

$$d(\mathcal{P}_{UB}(G)) = 2\Sigma_1 + \Sigma_2.$$

Proof. For two posets $A, B \in \mathcal{P}_{UB}(G)$, the number of relations in $A \cap B$ is greater than or equal to $|\mathrm{can}(G)|$. For $x, y \in V(G)$, if $x \leq_A y$ and $y \leq_B x$, then $\mathrm{Ma}(x) = \mathrm{Ma}(y)$. Thus,

$$\begin{aligned} d_{UB}(A, B) &\leq 2\,|\{\{x, y\}; \mathrm{Ma}(x) = \mathrm{Ma}(y)\}| \\ &\quad + |\{(x, y); \mathrm{Ma}(x) \subset \mathrm{Ma}(y), \text{ and} \\ &\qquad ((x \parallel y \text{ in } A \text{ and } x \leq_B y) \text{ or } (x \leq_A y \text{ and } x \parallel y \text{ in } B))\}| \\ &\leq 2\Sigma_1 + \Sigma_2, \end{aligned}$$

where $x \parallel y$ means that x and y are incompatible. By lemma13, this bound is sharp. Therefore $d(\mathcal{P}_{UB}(G)) = 2\Sigma_1 + \Sigma_2$. ■

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Department of Mathematical Sciences,
Tokai University,
1117 Kitakaname,
Hiratsuka 259-1292,
JAPAN
E-mail: kenjiro@ss.u-tokai.ac.jp