YOKOHAMA MATHEMATICAL
JOURNAL VOL. 47, 1999
SPECIAL ISSUE

TRIANGURATIONS OF CLOSED SURFACES WITH ACHROMATIC NUMBER 3

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(Received December 24, 1998)

Abstract. The achromatic number of a graph G is the maximum number k such that G has a k-coloring each pair of whose colors appear at the ends of an edge. We shall show that a triangulation G of a closed surface has achromatic number 3 if and only if G is isomorphic to $K_{n,n,n}$ for some n.

Introduction

We consider only finite, simple, undirected graphs in this paper. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. A *k-coloring* of G is a map $c: V(G) \to \{0, 1, \ldots, k-1\}$ such that adjacent vertices of G get different colors. A *complete k-coloring* of G is a k-coloring such that each pair of colors appears on at least one edge. If G has a k-coloring (resp., a complete k-coloring), then G is said to be *k-colorable* (resp., *complete k-colorable*). The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number k such that G is k-colorable. On the other hand, the *achromatic number* of G, denoted by $\psi(G)$, is the maximum number k such that G is complete k-colorable. They are introduced by Harary et al [1].

By definition, we have $\chi(G) \leq \psi(G)$. Moreover, a $\chi(G)$-coloring of G is always a complete coloring of G. (For otherwise, there is a pair of colors, say i and j, which are not adjacent in G. Here we can give the same color to the vertices colored i and j. Now we could decrease the number of colors, contrary to G being colored by $\chi(G)$ colors.)

A complete k-coloring of a graph G can be regarded as a subjective homomorphism $h: G \to K_k$. That is, h satisfies that:

(i) if $xy \in E(G)$, then $h(x) \neq h(y)$, and

(ii) for any $x'y' \in E(K_k)$, there exists $xy \in E(G)$ such that $x' = h(x)$ and $y' = h(y)$.

1991 Mathematics Subject Classification: 05C15, 05C10
Key words and phrases: achromatic number, triangulations

We call such h simply a *homomorphism.*

Harary, Hedetniemi and Prins [2] have studied homomorphisms from given graphs to complete graphs, and showed that if a graph G admits two homomorphisms to K_t and $K_{t'}$ $(t \leq t')$, then G also has a homomorphism to K_s for any s with $t \leq s \leq t'$. Equivalently, G has a complete k-coloring for any k with $\chi(G) \leq k \leq \psi(G)$.

It is clear that $\psi(G) = 1$ if and only if G has no edges. It is however so difficult to characterize those graphs that have a given achromatic number $k \geq 2$ in general. So, we restrict graphs to be triangulations on closed surfaces, which are typical objects in topological graph theory, and shall show the following theorem. (A simple graph G is called a *triangulation* of a closed surface if G is embedded in the surface so that each face is triangular. The complete graph K_3 embedded on the sphere is usually excluded from the triangulations, but we regard it as a triangulation.)

THEOREM 1. *Let G be a triangulation of a closed surface. Then $\psi(G) = 3$ if and only if G is isomorphic to $K_{n,n,n}$ for some $n \geq 1$.*

In Section 1, we shall determine the achromatic number of complete multipartite graphs and give a characterization for a connected graph to have achromatic number 2. In Section 2, we shall prove our main theorem.

1. The achromatic number of complete multipartite graphs

First, we consider the achromatic numer of complete k-partite graphs.

THEOREM 2. *The achromatic number of a complete k-partite graph is equal to k for any positive integer k.*

Proof. Let G be a complete k-partite graph with partite sets $A_1, \ldots, A_k$ and consider any complete coloring $c : V(G) \to \{1, 2, \ldots\}$ of G. Since any two vertices in different partite sets are adjacent, $c^{-1}(i)$ must be contained in only one of $A_1, \ldots, A_k$ for any color i. If $c^{-1}(i)$ and $c^{-1}(j)$ were contained together in some A_h, there would be no edge between two colors i and j. Thus, we may assume that $c^{-1}(i) = A_i$. This implies that any complete coloring of G uses precisely k colors and hence $\psi(G) = k$. ■

Applying the above theorem, we can prove the following theorem:

THEOREM 3. *Let G be a connected graph. Then $\psi(G) = 2$ if and only if G is isomorphic to a complete bipartite graph.*

Proof. By Theorem 2, the sufficiency is obvious. We shall prove the necessity. Since $\psi(G) = 2$, there is a bipartition $V(G) = A_1 \cup A_2$ such that $A_1 \cap A_2 = \emptyset$ and each of A_1 and A_2 is independent set of G. We may suppose that for $i = 1, 2$, A_i is colored by i in the complete 2-coloring of G. Suppose that there are $u \in A_1$ and $v \in A_2$ such that $uv \notin E(G)$. Then color $\{u, v\}$ by 3. Since G is connected, u is adjacent with vertices in A_2 and v is adjacent with vertices in A_1. Therefore, the resulting 3-coloring of G is complete, contrary to $\psi(G) = 2$. Thus, for any $u \in A_1$ and $v \in A_2$, $uv \in E(G)$, and hence G is a complete bipartite graph. ■

Note that if we omit the connectedness of G in Theorem 3, the theorem does not hold. Consider the disjoint union of several isolated vertices and a complete bipartite graph. Clearly, the achromatic number of this is 2.

2. Triangulations with achromatic number 3

In this section, we shall prove our main theorem and discuss the existense of triangulations of closed surfaces with achromatic number 3.

Proof of Theorem 1. By Theorem 2, the sufficiency is obvious. We shall prove the necessity. Since G is a triangulation, we have $\chi(G) \geq 3$. On the other hand, since $\psi(G) = 3$, we have $\chi(G) \leq 3$, and hence $\chi(G) = 3$. Thus, we characterize triangulations G such that $\chi(G) = 3$.

It is easy to see that for a triangulation G on a closed surface, either $G \cong K_3$ or each vertex of G has degree at least 3. In the former case, we have $G \cong K_3 \cong K_{1,1,1}$, and hence the theorem follows.

Now we consider the latter case. Let v be any vertex of G and let $v_1, \dots, v_m$ be the neighbors of v lying around v in this cyclic order. Since G is a triangulation, v is surrounded by the cycle $v_1 \cdots v_m$. We call this cycle the *link* of G and denote it by $\mathrm{lk}(v)$.

CLAIM 1. For each vertex v of G, the link of v has an even length.

Proof. Suppose that G is colored by $\{1, 2, 3\}$, and a vertex v is colored by 1. Then the vertices on $\mathrm{lk}(v)$ must be colored by $\{2, 3\}$. Thus, the length of $\mathrm{lk}(v)$ must be even. ■

According to the 3-coloring of G, we decompose $V(G) = A_1 \cup A_2 \cup A_3$ such that for $i \neq j$, $A_i \cap A_j = \emptyset$ and for each i, A_i is independent.

CLAIM 2. For each $i \in \{1, 2, 3\}$, $|A_i| \geq 2$.

Proof. Observe that for $v \in A_1$, the length of $\mathrm{lk}(v)$ is even and at least 4, by Claim 1. Since the vertices colored by 2 and those colored by 3 appear on $\mathrm{lk}(v)$

alternately, and since $\mathrm{lk}(v)$ is a cycle, the vertices on $\mathrm{lk}(v)$ are all distinct. Thus, we have $|A_2| \geq 2$ and $|A_3| \geq 2$. Focusing on $u \in A_2$ and $\mathrm{lk}(u)$, we also have $|A_1| \geq 2$. ■

Now we show that G is a complete tripartite graph with partite sets A_1, A_2 and A_3. Suppose that there are $u \in A_1$ and $v \in A_2$ such that $uv \notin E(G)$. Consider the new partition $V(G) = A_1' \cup A_2' \cup A_3' \cup A_4'$ such that $A_4' = \{u, v\}$. $A_1' = A_1 - \{u\}$, $A_2' = A_2 - \{v\}$ and $A_3' = A_3$. Clearly, this satisfies that for $i \neq j$, $A_i' \cap A_j' = \emptyset$ and that A_i' is independent for each i. We define

$$e(X,Y) = \big|\{xy \in E(G) : x \in X, y \in Y\}\big|$$

for two disjoint sets $X, Y \subset V(G)$.

Since vertices in A_2 and those in A_3 appear alternately on the link $\mathrm{lk}(u)$ of u, we have $e(A_2', A_4') \geq 2$ and $e(A_3', A_4') \geq 2$, by Claim 2. Similarly, focusing on v, we have $e(A_1', A_4') \geq 2$, and focusing on $w \in A_3$, we have $e(A_1, A_3) \geq 2$ and $e(A_2, A_3) \geq 2$, and hence $e(A_1', A_3') \geq 1$ and $e(A_2', A_3') \geq 1$. It is clear that $G - A_3$ is a 2-cell embedding each of whose faces contains a vertex in A_3 at its center, and hence it is connected. Thus, we have $e(A_1', A_2') \geq 1$. Therefore, the partition $V(G) = A_1' \cup A_2' \cup A_3' \cup A_4'$ gives a complete 4-coloring, a contradiction. Thus, G is a complete tripartite graph.

Now we show that $|A_1| = |A_2| = |A_3|$. For $u \in A_1$, all the vertices in A_2 and all the vertices in A_3 appear on $\mathrm{lk}(v)$ alternately. Thus, we have $|A_2| = |A_3|$. Similarly, focusing on $v \in A_2$, we have $|A_1| = |A_3|$. Hence, $|A_1| = |A_2| = |A_3|$. Therefore, G is isomorphic to $K_{n,n,n}$ with $n = |A_1| = |A_1| = |A_2|$. ■

The orientable genus of $K_{n,n,n}$ has been already determined as in the following theorem [3], [5], [4].

THEOREM 4. (A. White et al) *The complete tripartite graph $K_{n,n,n}$ has a triangular embedding in the orientable closed surface of genus g if and only if*

$$g = \frac{(n-1)(n-2)}{2}.$$

The following is an immediate consequence of Theorems 1 and 4.

COROLLARY 5. *For the orientable closed surface of genus g, there exists a triangulation G on it such that $\psi(G) = 3$ if and only if there exists a natural number n satisfying*

$$g = \frac{(n-1)(n-2)}{2}.$$

If we have a formula for the nonorientable genus of $K_{n,n,n}$, then we can obtain the fact corresponding to Corollary 5. We don't however know it now.

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