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DIAGONAL FLIPS IN PSEUDO-TRIANGULATIONS ON CLOSED SURFACES WITHOUT LOOPS

By

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Abstract. A pseudo-triangulation on a closed surface without loops is a graph embedded on the surface so that each face is triangular and may have multiple edges, but no loops. We shall establish a theory of diagonal flips in those pseudo-triangulations. Our theory will work in parallel to that for simple triangulations basically, but it will present more concrete theorems than the latter.

Introduction

A *triangulation* on a closed surface is a simple graph embedded on the surface so that each face is triangular and that any two faces share at most one edge. A *diagonal flip* of an edge ac in such a triangulation is to replace the diagonal ac with bd in the quadrilateral $abcd$ consisting of the two faces sharing ac. We do not perform a diagonal flip if it results in a nonsimple graph.

After Negami [13] proved the following theorem, many studies have appeared to establish a theory on diagonal flips in triangulations; [2], [4], [5], [9], [10], [14] and so on.

THEOREM 1. *For any closed surface F^2, there exists a natural number $N = N(F^2)$ such that two triangulations G_1 and G_2 on F^2 can be transformed into each other, up to homeomorhpism, by a sequence of diagonal flips if $|V(G_1)| = |V(G_2)| \geq N$.*

Let $N(F^2)$ denote its minimum value which makes the theorem valid. For example, the results given by Wanger [18], Dewdney [3], Negami and Watanabe [11] imply that $N(S^2) = 4$, $N(T^2) = 7$, $N(P^2) = 6$ and $N(K^2) = 8$ for the sphere S^2, the projective plane P^2, the torus T^2 and the Klein bottle K^2 in order. These values coincide with the minimum number of vertices of triangulations on these surfaces, but it does not hold in general. It is so difficult to determine the precise

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value of $N(F^2)$ for a given closed surface F^2. Also Negami [15] has already shown that

$$N(F^2) \le 19\, V_{\mathrm{irr}}(F^2) - 18\,\chi(F^2)$$

where $\chi(F^2)$ denotes the Euler characteristic of F^2. However, this bound includes an unknown quantity $V_{\mathrm{irr}}(F^2)$, which is the maximum order of irreducible triangulations of F^2. We have $V_{\mathrm{irr}}(S^2) = 4$, $V_{\mathrm{irr}}(P^2) = 7$, $V_{\mathrm{irr}}(T^2) = 10$ and $V_{\mathrm{irr}}(K^2) = 11$ (see [17], [1], [6] and [7], for irreducible triangulations of these surface in order) but it has been known only $|V(F^2)| \le 171(2 - \chi(F^2)) - 72$ for other surfaces [8], which implies the above upper bound for $N(F^2)$ is of linear order with respect to the genus of F^2.

One of points in the difficulty is that we have to keep the simpleness of graphs during flipping edges in triangulations. What happens if we neglect the simpleness of graphs? For example, Negami [15] has already given an answer to this question, which we shall present as Theorem 10 in Section 3, and has shown the previous upper bound for $N(F^2)$, as an application of his answer. We shall show another answer in this paper, establishing a theory which is more concrete than that for simple triangulations.

A *pseudo-triangulation* on a closed surface F^2 is a triangular embedding of a graph on F^2 which may have loops and multiple edges, according to Negami's definition in [15]. However, we shall exclude the loops and show the following theorem in the same style as Theorem 1:

THEOREM 2. *Given a closed surface F^2, there exists a natural number $n(F^2)$ such that two pseudo-triangulations G_1 and G_2 on F^2 without loops can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips through those pseudo-triangulations if $|V(G_1)| = |V(G_2)| \ge n(F^2)$.*

Let $n(F^2)$ denote its minimum value hereafter, as well as $N(F^2)$. We shall give the following upper bound for $n(F^2)$, which does not include any unknown quantity.

THEOREM 3. *If a closed surface F^2 is one of the sphere, the projective plane, the torus and the Klein bottle, then $n(F^2) = 3$. Otherwise, we have:*

$$4 \le n(F^2) \le 18 - 5\chi(F^2)$$

For convenience, we say that two pseudo-triangulations without loops are *equivalent under diagonal flips* if they can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips through those pseudo-triangulations without loops, and often call a pseudo-triangulation without loops simply a *pseudo-triangulation* hereafter, omitting "without loops".

In the next section, we shall define the notions of *minimal*, *pseudo-minimal* and *frozen pseudo-triangulations* to carry out the same arguments as for simple triangulations developed in [13] and [16]. Distinguishing these notions is important in the theory for simple triangulations, but they are the same for pseudo-triangulations without loops, which enables us to establish the above concrete bound for $n(F^2)$.

1. Minimal pseudo-triangulations

A pseudo-triangulation on a closed surface without loops is said to be *minimal* if it has the fewest vertices among those. Since it has no loop, the three corners of each face consist of three distinct vertices. Thus, it is clear that any minimal pseudo-triangulation without loops has at least three vertices and also it is easy to construct pseudo-triangulations with precisely three vertices under the following conditions.

LEMMA 4. *Let G be a minimal pseudo-triangulation on a closed surface F^2 with V vertices, E edges and F faces and without loops. Then we have*

$$V = 3, \quad E = 9 - 3\chi(F^2), \quad F = 6 - 2\chi(F^2)$$

and G is an F-regular graph such that all faces are incident to each vertex. Thus, G can be obtained from a wheel W_{2n} by identifying the vertices and edges along its rim of length $2n$ suitably.

Proof. It is easy to show that

$$E = 3(V - \chi(F^2)), \quad F = 2(V - \chi(F^2))$$

for a pseudo-triangulation G on a closed surfcae F^2 in general, using Euler's formula. Since we can construct a minimal pseudo-triangulation with precisely three vertices actually, we obtain the three equalities in the lemma, assigning 3 to V in the above.

Let $V(G) = \{u, v, w\}$. Then each face of G has to have these three vertices u, v and w at its corners. This implies all of F faces are incident to v (and also to u and w) and they form a wheel with v at its center. The rim of this wheel W_{2n} is a closed walk of length $F = 2n$ representing the link of v, denoted by $\mathrm{lk}(v)$. and includes only u and w. So we need to identify the vertices which come from the same vertex, u or w, to obtain the actual form of G. ■

LEMMA 5. *A minimal pseudo-triangulation without loops is unique for each of the sphere, the projective plane, the torus and the Klein bottle.*

Proof. By Lemma 4, it is clear that the only minimal pseudo-triangulation of the sphere is K_3, the cycle of length 3, which has two faces. Also, the unique minimal pseudo-triangulation of the projective plane can be obtained from the wheel W_4 by identifying each pair of antipodal points on its boundary.

Those of the torus and the Klein bottle can be obtained from W_6 by suitable identification along its boundary. For the torus, the identification is clear; each parallel pair of edges should be identified. To represent it we give each egde a label so that two edges which should be identified have the same label. In this case, we have $xyzxyz$. Since the vertiecs has been labeled with u and w, the labeling on edges determines the identification uniquely.

On the other hand, we need a slight argument on the identification of W_6 for the Klein bottle. To obtain a nonorientable surface, we have to identify at least one pair of edges so that the surface includes a Möbius band. To do this, the identification should be represent with labeling "$x \bullet x \bullet \bullet \bullet$" or its cyclic shift, where each "$\bullet$" stands for one label. It is not difficult to determine the unknown labels and it will be $xyxzyz$ uniquely up to symmetry. Otherwise, the resulting pseudo-triangulation would have more than three vertices. ∎

LEMMA 6. *Any closed surface F^2 with $\chi(F^2) < 0$ admits two or more minimal pseudo-triangulations without loops.*

Proof. First, consider minimal pseudo-triangulations on the orientable closed surface of genus $g \geq 2$. By Lemma 4, they can be constructed from W_F with $F = 4g + 2$ by identifying vertices and edges on its rim. For example, the two identification with labeling

$$x_1x_2\cdots x_Fx_1x_2\cdots x_F; \quad x_1x_2\cdots x_{F-2}x_{F-1}x_Fx_1x_{F-1}x_Fx_2\cdots x_{F-2}$$

yield two pseduo-triangulations with three vertices. They are not homeomorphic to each other since their duals are not isomorphic as abstract 3-regular graphs.

Similarly, we can give two identifications on the boundary of W_F with $F = 2q + 2$ for the nonorientable closed surface of genus $q \geq 3$:

$$x_1x_2\cdots x_Fx_1x_F\cdots x_2; \quad x_1x_2\cdots x_{F-1}x_Fx_1x_Fx_2\cdots x_{F-1}$$

They also yield non-homeomorphic pseudo-triangulations with three vertices whose duals are not isomorphic. ∎

Here, we shall show an easy way to construct a series of minimal pseudo-triangulations inductively. Let G_1 and G_2 be pseudo-triangulations on two disjoint closed surfaces F_1^2 and F_2^2, respectively. Choose one face of G_1 and of G_2, say A_1 and A_2. Paste F_1^2 and F_2^2 along A_1 and A_2, and remove the open 2-cell

$A_1 = A_2$. Then we obtain a pseudo-triangulation on the *connected sum* $F_1^2 \# F_2^2$ of the two surfaces F_1^2 and F_2^2. The resulting pseudo-triangulation also is called a *connected sum* of G_1 and G_2 and is denoted by $G_1 \# G_2$. If each of G_1 and G_2 has precisely three vertices, then $G_1 \# G_2$ also has precisely three vertices. By Lemma 4, $G_1 \# G_2$ is a minimal pseudo-triangulation of $F_1^2 \# F_2^2$.

For example, a series of minimal pseudo-triangulations of the orientable closed surfaces of genus $2, 3, 4, \dots$ can be constructed from many copies of the unique minimal pseudo-triangulation of the torus by joining them repeatedly in the above way. Each of their duals has a nontrivial 3-edge-cut, that is, a set of three edges whose removal disconnects it into nontrivial components. Thus, we cannot construct the first type given in the proof of Lemma 6 in this way since its dual does not have such a 3-edge-cut.

2. Pseudo-minimal pseudo-triangulations

Let G be a pseudo-triangulation on a closed surface F^2 without loops and ac an edge in G with two faces abc and adc incident to it. The *contraction* of ac is to shrink ac to a point and to replace the resulting two digonal faces with edges $ab = cb$ and $ad = cd$, respectively. We perform the contraction of an edge only when it results in another pseudo-triangulation on F^2 without loops, denoted by G/ac, and call such an edge a *contractible edge*.

A pseudo-triangulation is said to be *contractible* if it has a contractible edge and to be *irreducible* otherwise. For example, any minimal pseudo-triangulation is irreducible since an edge contraction decreases the number of vertices. A pseudo-triangulation is said to be *pseudo-minimal* if it cannot be transformed into any contractible pseudo-triangulation by diagonal flips. Any pseudo-triangulation equivalent to a pseudo-minimal one is pseudo-minimal.

Let G be a pseudo-triangulation on a closed surface F^2 without loops and let $\delta(G)$ denote the minimum degree of G. In general, we have $\delta(G) \geq 2$; otherwise, we could find a loop around a vertex of degree 1. Suppose that $\delta(G) = 2$ and let v be a vertex of degree 2 in G. Then v has two distinct neighbors u and w and there are multiple edges between u and w which bound a digonal region including the path uvw of length 2. Replace this digonal part with a single edge uw to obtain another pseudo-triangulation without loops. We call this deformation the *elimination* of a vertex v of degree 2. Note that each of the two edges incident to a vertex v of degree 2 is contractible and its contraction realizes the elimination of v.

LEMMA 7. *A pseudo-triangulations on a closed surface without loops, except K_3, is pseudo-minimal if and only if it is equivalent to no pseudo-triangulation with a vertex of degree 2 under diagonal flips.*

Proof. The necessity is clear since a pseduo-triangulation is contractible if it has a vertex of degree 2. To prove the sufficiency, it suffices to show that a contractible pseudo-triangulation is equivalent to one with a vertex of degree 2 under diagonal flips.

Let v be a vertex and $u_1, \ldots, u_n$ its neighbors lying on $\mathrm{lk}(v)$ around v in this cyclic order. Suppose that vu_n is a contractible edge in G. Since G/vu_n has no loops, each of $u_1, \ldots, u_{n-1}$ is distinct from u_n. Thus, we can flip vu_1 to $u_n u_2$, vu_2 to $u_n u_3$, ..., vu_{n-2} to $u_n u_{n-1}$. The vertex v will have degree 2 finally. ■

The next lemma follows from the above immediately:

LEMMA 8. *Any pseudo-triangulation on a closed surface without loops can be transformed into a pseudo-minimal one by a sequence of diagonal flips and elimination of vertices of degree* 2. ■

Negami [13] has defined the pseudo-minimal triangulations in a similar style, related to contraction of edges. They also play an important role to determine the value of $N(F^2)$. However, they are just theoretical objects and we know nothing about their concrete forms. (We can find several examples of pseudo-minimal triangulations in [16].) On the other hand, we can give a good characterization of the pseudo-minimal pseudo-triangulations, as follows, which suggests how to construct them.

Recall that we must not flip an edge in a pseudo-triangulation without loops if it yields a loop. A pseudo-triangulation is said to be *frozen* if any diagonal flip is not applicable to it. That is, any frozen pseudo-triangulation is not equivalent to any other pseudo-triangulation under diagonal flips.

LEMMA 9. *For a pseudo-triangulation G on a closed surface without loops, the following four are equivalent to one another:*

(i) *G is frozen.*

(ii) *G is pseudo-minimal.*

(iii) *G is minimal.*

(iv) *G has precisely three vertices.*

Proof. By Lemma 4, the equivalence between (iii) and (iv) is obvious. So we shall show the equivalence among (i), (ii) and (iv) below.

(i) implies (ii): Suppose that there is a vertex v of degree 2. Then it has two distinct neighbors u and w and they are joined by multiple edges. Each of the multiple edges between u and w is flippable in G. Thus, any frozen pseudo-triangulatioin has minimum degree at least 3. Since it is not equivalent to any other pseudo-triangulation, it is pseudo-minimal by Lemma 7.

(ii) implies (iv): Let G be a pseudo-minimal pseudo-triangulation. We may suppose that $\delta(G)$ is the smallest among those pseudo-triangulations that are equivalent to G under diagonal flips. Let v be a vertex of G with $\deg v = \delta(G) \geq 3$. By the minimality of $\delta(G)$, each edge incident to v is not flippable; otherwise, flipping it would decrease $\deg v$ by one. This implies that $\deg v$ is an even number ≥ 4 and that two distinct vertices u and w lie alternately along $\mathrm{lk}(v)$. Each face incident to v consists of the three vertices $\{u, v, w\}$.

Consider the link $\mathrm{lk}(w)$ of w $(= w_1)$ and suppose that there is a fourth vertex x on $\mathrm{lk}(w)$, different from u, v and w. Then we can find a segment $xuvu$ along $\mathrm{lk}(w)$. To distinguish two u's in the segment, we donte it by xu_1vu_2 and let $u_1w_1u_2w_2\cdots$ be the walk along $\mathrm{lk}(v)$ starting at u_1. Flip w_1u_1 to vx, vw_1 to u_2x and vu_2 to w_2x. This sequence of diagonal flips decreases $\deg v$ finally by one, contrary to the minimality of $\delta(G)$. Therefore, $\mathrm{lk}(w)$ consists of only v and u, and $\mathrm{lk}(u)$ also consists of only v and w, similarly. This implies $\{u, v, w\}$ induces a connected component of G. Since G is connected, G has only these three vertices u, v and w.

(iv) implies (i): If $V(G) = \{u, v, w\}$, then flippying any edge, say uv, yields a loop at w. Thus, no diagonal flip is applicable to G. ■

Any pseudo-minimal pseudo-triangulation is irreducible. However, we can make those irreducible pseudo-triangulations that are not pseudo-minimal, for each closed surface F^2 except the sphere and the projective plane, as follows.

Prepare the wheel W_{4g} which subdivides a $4g$-gonal disk, for the orientable closed surface of genus $g \geq 1$ and identify the boundary of the disk to obtain the surface so that all of the $4g$ vertices of W_{4g} except its center v become a single vertex, say u. The resulting graph has two vertices and $2g$ loops, which come from edges on the rim of W_{4g}, and the $4g$ spokes form multiple edges between v and u. Subdivide each loop into a pair of multiple edges with its middle point as a vertex and join the new vertex to the center v with an edge.

Now we obtain a pseudo-triangulation without loops which has precisely $2g+2$ vertices, and hence it is not minimal or equivalently not pseudo-minimal by Lemmas 4 and 9. Each of its edges lies on a cycle of length 2 and hence it is irreducible. Similarly, we can construct those with $q + 2$ vertices from W_{2q} for the nonorientable closed surface of genus $q \geq 2$. It is not difficult to see that the irreducible pseudo-triangulations of the sphere and of the projective plane are the unique minimal ones given in Lemma 5.

3. Proof of theorems

Negami [15] has shown the following theorem for pseudo-triangulations possibly with loops. In such pseudo-triangulations, there is no restriction to flip edges. His proof of this theorem suggests an algorithm to transform G_1 into G_2, which is greedy in a sense, and gives an upper bound for the length of a sequence of diagonal flips from G_1 to G_2. The quantity $\mathrm{cr}_\nabla(G_1, G_2)$ is called the *crossing number* of G_1 and G_2 *under vertex coincidence* and is the minimum number of crossing points in $G_1 \cup G_2$ when we embed G_1 and G_2 together on the same surface F^2 with $V(G_1) = V(G_2)$.

THEOREM 10. *Let G_1 and G_2 be two labeled pseudo-triangulations on a closed surface F^2 with the same number of vertices. Then they can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips of length at most* $\mathrm{cr}_\nabla(G_1, G_2)$.

As an application of this theorem, we shall prove Theorems 2 and 3 for pseudo-triangulations without loops, as follows.

Let G be a pseudo-triangulation on a closed surface F^2 without loops and uv an edge in G. Replace uv with a pair of multiple edges between u and v bounding a digonal region which includes a path uxv of length 2. Then we obtain another pseudo-triangulation G' with a new vertex x of degree 2. We call this local deformation of G into G' the *insertion* of a vertex x of degree 2 along an edge uv and denote G' by $G + \Theta_1$. Furthermore, let $G + \Theta_m$ denote a pseudo-triangulation without loops obtained from G by inserting m vertices of degree 2 along edges in order. The insertion of a vertex of degree 2 is the inverse operation of the elimination of a vertex of degree 2.

Let uvw be a face of a pseudo-triangulation G and insert a vertex x of degree 2 along an edge uv with multiple edges e_1 and e_2 so that e_1 lies in the face uvw. Flip e_1 to xw and xu to vw. The resulting pseudo-triangulation can be regarded as the one obtained from G by inserting x along wv. Repeating this deformation, we can move a vertex of degree 2 freely to anywhere. This fact implies that any two pseudo-triangulations with the same notation $G+\Theta_1$ are equivalent to each other under diagonal flips and hence it is the same for $G + \Theta_m$ with any natural number m.

The following theorem will give an essense of our proof of Theorems 2 and 3:

THEOREM 11. *Let G_1 and G_2 be two pseudo-triangulations on a closed surface F^2 without loops which have the same number of vertices. Then $G_1 + \Theta_m$ can be transformed into $G_2 + \Theta_m$, up to homeomorphism, by a sequence of diagonal flips through pseudo-triangulations without loops if* $m \geq 5(|V(G)| - \chi(F^2))$.

Proof. By Theorem 10, G_1 can be transformed into G_2 by a sequence of diagonal flips, but this sequence $T_0, T_1, \ldots, T_n$ might include pseudo-triangulations with many loops although $G_1 = T_0$ and $G_2 = T_n$ have no loops. We shall translate this sequence into that from $G_1 + \Theta_m$ to $G_2 + \Theta_m$, as follows.

Consider the barycentric subdivision G_1' of G_1. That is, G_1' can be obtained from G_1 by putting a new vertex at the middle point of each edge and adding the barycenter of each face as a vertex adjacent to all of six vertiecs along its boundary. The number of vertices added to G_1, say m_0, is equal to $|E(G)| + |F(G)| = 5(|V(G)| - \chi(F^2))$. Flipping edges in faces of G_1, we can make the additional vertices have degree 2. This implies that G_1' is equivalent to $G_1 + \Theta_{m_0}$ under diagonal flips.

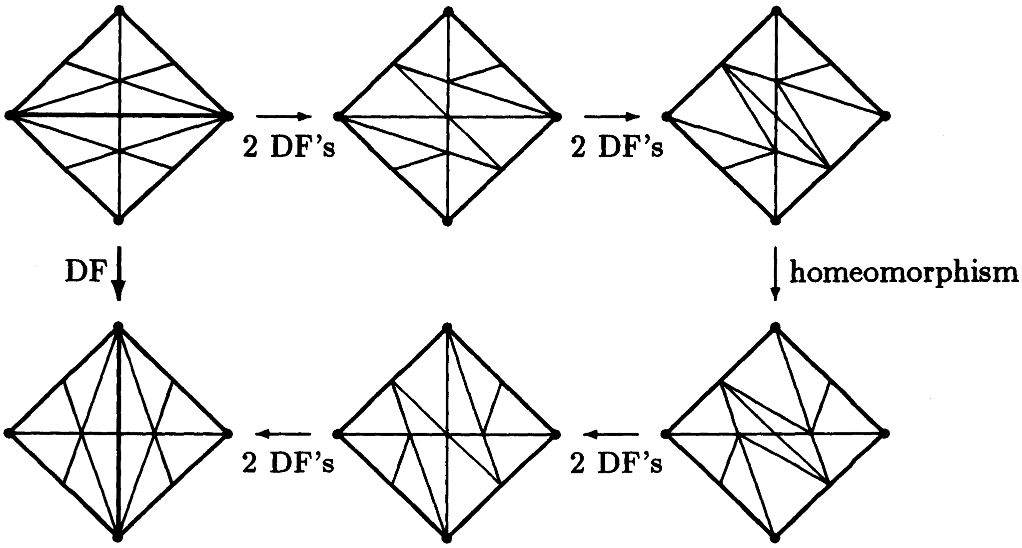

Figure 1. Diagonal flips in barycentric subdivisions

Similarly, consider the barycentric subdivision T_i' of each T_i in the sequence. Then the diagonal flip from T_i to T_{i+1} can be translated into a sequence of eight diagonal flips, as shown in Figure 1. Thus, $G_1 + \Theta_{m_0}$ is equivalent to the barycentric subdivision G_2' of G_2, which is equivalent to $G_2 + \Theta_{m_0}$ under diagonal flips. Since any vertex of degree 2 can be moved to anywhere, it is easy to see that $G_1 + \Theta_m$ is equivalent to $G_2 + \Theta_m$ with $m \geq m_0$ under diagonal flips; move a vertex of degree 2 far away if it disturbs a diagonal flip. ■

Proof of Theorems 2 and 3. Let G_1 be a pseudo-triangulation on F^2 without loops. If G_1 is not pseudo-minimal, then G_1 can be transformed into a pseudo-minimal one, say Q_1, by a sequence of diagonal flips and elimination of vertices of degree 2, by Lemma 8, and hence G_1 is equivalent to $Q_1 + \Theta_m$ under diagonal flips, where $m = |V(G_1)| - |V(Q_1)|$.

Similarly, let G_2 be a pseudo-triangulation on F^2 with the same number of vertices as G_1 and let Q_2 be the pseudo-minimal one such that G_2 is equivalent to $Q_2 + \Theta_m$ under diagonal flips. By Lemma 9, both Q_1 and Q_2 has precisely three vertices. By Theorem 11, if $m \geq 5(3-\chi(F^2)) = 15-5\chi(F^2)$, then $Q_1+\Theta_m$ and $Q_2+\Theta_m$ are equivalent under diagonal flips. This implies that G_1 is equivalent to G_2 via $Q_1+\Theta_m$ and $Q_2+\Theta_m$ under diagonal flips if $|V(G_1)| = |V(G_2)| \geq 18-5\chi(F^2)$. That is, $n(F^2) \leq 18-5\chi(F^2)$.

By Lemma 5, if F^2 is one of the sphere, the projective plane, the torus and the Klein bottle, then Q_1 and Q_2 are identical. Thus, there is no restriction on the number of vertices to transform two pseudo-triangulations into each other and hence $n(F^2) = 3$. Otherwise, there are two or more frozen pseudo-triangulations on F^2 which have precisely three vertices and no two of which are equivalent under diagonal flips. Thus, we have $n(F^2) \geq 4$. ■

It is not difficult to see that there are precisely two minimal pseudo-triangulations of the orientable closed surface S_2 of genus 2, up to homeomorphism. They are the ones obtained in the proof of Lemma 6, denoted by T_1 and T_2 here. We have already observed that $T_1+\Theta_1$ and $T_2+\Theta_1$ are equivalent under diagonal flips, which implies that $n(S_2) = 4$. We conjecture that $n(F^2) = 4$ for any closed surface F^2 with $\chi(F^2) < 0$, orientable or nonorientable.

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