

## ONE-LOOSELY TIGHT TRIANGULATIONS ON CLOSED SURFACES

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**Abstract.** A triangulation  $G$  on a closed surface is called *k-loosely tight* if any color assignment to vertices with  $k + 3$  colors yields a face whose corners are assigned three distinct colors. We shall show that a triangulation  $G$  of the sphere, the projective plane, the torus or the Klein bottle is 1-loosely tight if and only if both the independence number and the diameter of  $G$  do not exceed 2. Using this result, we shall classify all 1-loosely tight triangulations of the projective plane.

### Introduction

Let  $G$  be a triangulation of a closed surface, that is, a simple graph embedded in the surface so that any face of  $G$  is bounded by a cycle of length 3 and any pair of faces share at most one edge in common. (The latter condition is added only to exclude  $K_3$  embedded in the sphere.) We denote the sets of vertices, edges and faces of  $G$  by  $V(G)$ ,  $E(G)$  and  $F(G)$  respectively, and identify each face with the set of three vertices  $\{x, y, z\}$  which lie on its boundary. A face  $\{x, y, z\}$  of  $G$  is said to be *heterochromatic* for a color assignment  $f : V(G) \rightarrow \{1, 2, 3, \dots\}$  if the three vertices  $x$ ,  $y$  and  $z$  have distinct colors,  $f(x)$ ,  $f(y)$  and  $f(z)$ . A triangulation  $G$  is said to be *tight* if there is a heterochromatic face for any surjective color assignment  $f : V(G) \rightarrow \{1, 2, 3\}$ . A color assignment  $f$  is called *hetero-free* if there is no heterochromatic face for  $f$ .

Suppose that  $G$  has two vertices  $u$  and  $v$  which are not adjacent. Then we can define a hetero-free assignment  $f : V(G) \rightarrow \{1, 2, 3\}$  by  $f(u) = 1$ ,  $f(v) = 2$  and  $f(x) = 3$  for any other vertex  $x$  of  $G$ . This implies that  $G$  is a complete graph if it is tight. Thus, the notion of tightness works only for triangulations on closed surfaces with complete graphs.

For example, Arocha, Bracho and Neumann-Lara [1], [2] have discussed those and given a method to construct a series of untight triangulations with complete graphs and a series of tight ones. In particular, they have shown that any untight triangulation with a complete graph has at least 16 vertices and that there are

three non-isomorphic triangulations of the nonorientable closed surface of genus 26 with  $K_{16}$ , one of which is tight and the other two are untight. Lawrencenko, Negami and White [4] have found three non-isomorphic triangulations of the orientable closed surface of genus 20 with  $K_{19}$ , which are all tight.

Negami and Midorikawa [5] extended the concept of the tightness as follows in order to discuss the tightness of non-complete triangulations. A triangulation  $G$  is said to be *k-loosely tight* if there is a heterochromatic face for any surjective color assignment  $f : V(G) \rightarrow \{1, 2, \dots, 3 + k\}$ . It is obvious that all faces are heterochromatic for any  $f$  if  $|V(G)| = 3 + k$ . Hence any triangulation  $G$  of a closed surface is *k-loosely tight* for some  $k \leq |V(G)| - 3$ . The *looseness* of  $G$  is defined as the minimum value of  $k$  for which  $G$  is *k-loosely tight*, and it is denoted by  $\xi(G)$ .

A set of vertices of a graph  $G$  is said to be *independent* if any two vertices in it are not adjacent to each other. The *independence number* of  $G$ , denoted by  $\alpha(G)$ , is the maximum size of independent sets of vertices in  $G$ . The *distance* between two vertices  $u$  and  $v$  in  $G$ , denoted by  $d(u, v)$ , is defined as the length of a shortest path between  $u$  and  $v$  in  $G$ . The *diameter* of  $G$ , denoted by  $\text{diam}(G)$ , is defined as the maximum value of the distances taken over all pairs of vertices in  $G$ . Negami and Midorikawa [5] have shown that if  $G$  is a *k-loosely tight* triangulation, then  $\alpha(G) \leq k + 1$  and  $\text{diam}(G) \leq k + 1$ .

The converse of their result does not hold in general since the looseness depends on the embeddings. We shall however prove the following theorem.

**THEOREM 1.** *A triangulation  $G$  of the sphere, the projective plane, the torus or the Klein bottle is 1-loosely tight if and only if  $\alpha(G) \leq 2$  and  $\text{diam}(G) \leq 2$ .*

This characterization of 1-loosely tight triangulations of these surfaces will enable us to classify them. Negami and Midorikawa [5] have already classified the 1-loosely tight triangulations of the sphere; they are precisely eight in number, up to isomorphism. In this paper, we shall classify those of the projective plane.

**THEOREM 2.** *There are precisely twenty 1-loosely tight triangulations of the projective plane, up to isomorphism, as shown in Figure 4.*

In Section 1, we shall give some general observations about the looseness and prove Theorem 1. In Section 2, we shall show how to construct the complete list of 1-loosely tight triangulations of the projective plane and prove Theorem 2.

## 1. Characterization of 1-loosely tight triangulations

Negami and Midorikawa [5] have already given a characterization of *k-loosely tight* triangulations, in terms of cycles in  $G^*$ , as follows:

**THEOREM 3.** (Negami and Midorikawa [5]) *A triangulation  $G$  of a closed surface is  $k$ -loosely tight if and only if  $G^*$  does not contain a union of disjoint cycles which separate the surface into  $k + 3$  regions.*

Let  $G$  be a triangulation on a closed surface  $F^2$  and  $G^*$  its dual and let  $f : V(G) \rightarrow \{1, 2, \dots, 3 + k\}$  be a surjective color assignment. We define  $H^*$  as the subgraph in  $G^*$  induced by all the edges dual to the edges whose ends are assigned two different colors by  $f$ . It is clear that each vertex of  $H^*$  has degree 2 or 3 in  $H^*$  and that  $f$  is hetero-free if and only if  $H^*$  has no vertex of degree 3. In this case,  $H^*$  consists of a union of disjoint cycles in  $G^*$  and those cycles separate  $F^2$  into at least  $k + 3$  regions. Each of those regions contains only vertices of  $G$  with the same color. Conversely, if there is a union of cycles in  $G^*$  which separates  $F^2$  into  $k + 3$  regions, then we can define a hetero-free color assignment  $f : V(G) \rightarrow \{1, 2, \dots, 3 + k\}$  so that  $H^*$  can be obtained as above. This is the meaning of Theorem 3.

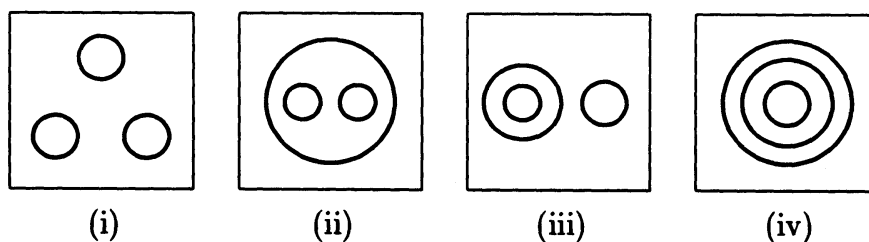
*Proof of Theorem 1.* The necessity of the theorem follows from Negami and Midorikawa's result in [5]. So we shall show only the sufficiency, using Theorem 3 with  $k = 1$ .

Let  $G$  be a triangulation on a closed surface  $F^2$  which is homeomorphic to one of the sphere, the projective plane, the torus and the Klein bottle. Suppose that  $G$  is not 1-loosely tight. By Theorem 3, its dual  $G^*$  contains a union of disjoint cycles  $\mathcal{C} = \{C_1, C_2, C_3, \dots\}$  which separates  $F^2$  into four regions. We may assume that each cycle  $C_i$  is a common boundary cycle of two distinct regions, and hence it is 2-sided, that is,  $C_i$  has a neighborhood in  $F^2$  homeomorphic to an annulus. We shall discuss the configurations of such cycles on  $F^2$  and find either an independent set of three vertices or a pair of vertices with distance 3.

First, consider the projective plane as  $F^2$ . Any 2-sided simple closed curve in the projective plane is trivial and bounds a 2-cell region there. Thus, it is easy to see that the configuration of  $\mathcal{C}$  is homeomorphic to one of those given as (i) to (iv) in Figure 1. To obtain their real form, each square in the figure should be identified along its the boundary so that each pair of antipodal points becomes a single point in the projective plane. We can choose three vertices of  $G$  from three non-adjacent regions in cases of (i) and (ii), so that they form an independent set. On the other hand, we can find a pair of vertices with distance 3 in cases of (iii) and (iv).

When  $F^2$  is homeomorphic to the sphere, both (i) and (ii) represent the same configuration on the sphere, and so (iii) and (iv) do. We also obtain the same conclusion as in the previous case.

Now we shall consider the torus and the Klein bottle. To do it, we should explain about essential simple closed curves on these surfaces. Each of them

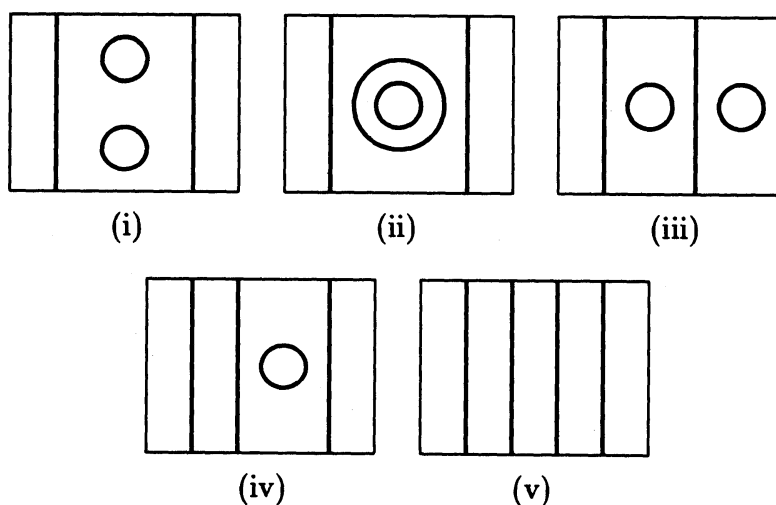


**Figure 1.** Disjoint cycles on a surface I

includes a 2-sided simple closed curve which cuts open it into an annulus. We call such a curve a *meridian*. Furthermore, there is another simple closed curve which cross a given meridian at a point transversely. This is called a *longitude*. A longitude on the torus is 2-sided while that on the Klein bottle is 1-sided, that is, it runs along the center line of a Möbius band.

To present a configuration on the torus or the Klein bottle, we cut the surface along a fixed pair of a meridian and a longitude into a rectangle and the vertical pair of its sides corresponds to the meridian and the horizontal one to the longitude in our figures. There is the third type of an essential simple closed curve on the Klein bottle, called an *equator*. An equator cross a meridian twice and splits into two horizontal segments in a rectangle presenting the Klein bottle.

If  $\mathcal{C}$  consists of only trivial cycles, then the same pictures in Figure 1 present all the cases and we obtain the same conclusion. Thus, we may assume that  $\mathcal{C}$  contains at least one essential cycle. It must be a meridian in case of the torus while the essential cycles in  $\mathcal{C}$  are either all meridians or all equators in case of the Klein bottle.



**Figure 2.** Disjoint cycles on a surface II

Assume that  $\mathcal{C}$  includes at least one meridian, which is the case when  $F^2$  is the torus. Its configuration is homeomorphic to one of (i) to (v) in Figure 2. It is easy to find an independent set of three vertices in case of (i) and a pair of vertices with distance 3 in (ii) and (iii). We need more arguments on (iv) and (v).

Let  $S_1, S_2, S_3$  and  $S_4$  be the four regions into which  $\mathcal{C}$  divides  $F^2$  and let  $H_i$  be the subgraph in  $G$  induced by the vertices inside  $S_i$ . If  $S_i$  is not a disk, then  $H_i$  should include a cycle and should have at least three vertices.

Consider Case (iv) and assume that  $S_1$  and  $S_2$  are the two annuli while  $S_4$  is the disk. Since  $S_1 \cup S_2$  contains at least six vertices of  $G$ , there is a pair of non-adjacent vertices among them, say  $u$  and  $v$ ; otherwise, the annulus would include  $K_6$ , which is not planar. Then,  $\{u, v, w\}$  will be an independent set for any vertex  $w$  inside  $S_4$ .

Consider Case (v). In this case, all  $S_i$ 's are annuli. We may assume that they are labeled cyclically modulo 4 with  $S_4 = S_0$  and that  $S_i$  and  $S_{i+1}$  have a common boundary cycle. If  $H_i$  includes a pair of non-adjacent vertices, say  $u$  and  $v$ , then  $\{u, v, w\}$  will be an independent set for any vertex  $w$  inside  $S_{i+2}$ . Thus, may we assume that each  $H_i$  is a complete graph.

If there is a vertex  $v \in V(H_2)$  which is not adjacent to some vertex  $u \in V(H_1)$  and is not adjacent to some vertex  $w \in V(H_3)$ , then  $\{u, v, w\}$  will be an independent set. Otherwise, each vertex  $v \in V(H_2)$  is adjacent to all vertices of  $H_1$  or to all of vertices of  $H_3$ . Since  $V(H_2)$  consists of at least three vertices, we may assume that at least two vertices  $v_1, v_2 \in V(H_2)$  are adjacent to all vertices of  $H_1$ . Then  $V(H_1) \cup \{v_1, v_2\}$  induces a complete subgraph in  $G$ . This subgraph is embedded in the annulus  $S_1 \cup S_2$ , but it would be non-planar since it has at least five vertices. Thus, this is not the case.

Assume that  $\mathcal{C}$  includes an equator. Then its configuration is homeomorphic to one of (i) to (vi) in Figure 3. We can find an independent set of three vertices in cases of (ii) and (v) and a pair of vertices with distance 3 in the other cases.

Now we have discussed all of the possible cases and the theorem follows. ■

Note that Theorem 1 does not hold for other surface in general. For example, we can construct a triangulation  $G$  of a suitable closed surface such that  $\alpha(G) \leq 2$  and  $\text{diam}(G) \leq 2$ , but  $\xi(G) \geq 2$ , as follows.

Let  $G_1$  and  $G_2$  be two untight triangulations with complete graphs embedded separately on two closed surfaces. Since they are not tight, they have hetero-free color assignments with three colors, say  $f_1 : V(G_1) \rightarrow \{1, 2, 3\}$  and  $f_2 : V(G_2) \rightarrow \{1, 2, 4\}$ . We may assume that each  $G_i$  has a face  $\{x_i, y_i, z_i\}$  with  $f(x_i) = 1$  and  $f(y_i) = f(z_i) = 2$ . Identify  $x_1$  with  $x_2$ ,  $y_1$  with  $y_2$  and  $z_1$  with  $z_2$ , and omit the face  $\{x_1, y_1, z_1\} = \{x_2, y_2, z_2\}$  to form a triangulation  $G$  on the connected sum of two surfaces where  $G_1$  and  $G_2$  are embedded. This construction induces a

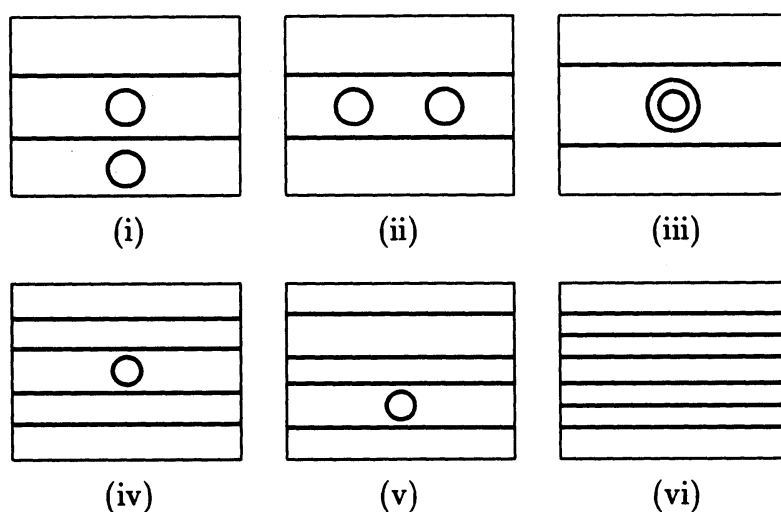


Figure 3. Disjoint cycles on a surface III

hetero-free color assignment  $f : V(G) \rightarrow \{1, 2, 3, 4\}$  and hence  $G$  is not 1-loosely tight. It is clear that  $\alpha(G) = \text{diam}(G) = 2$ .

## 2. The 1-loosely tight triangulations on the projective plane

Let  $G$  be a triangulation of a closed surface and  $U$  a set of vertices of  $G$ . The *closed neighborhood* of  $U$  in  $G$  is the set which consists of  $U$  and the neighbors of vertices in  $U$ .

$$N[U] = U \cup \{v \in V(G) : \exists u \in U, uv \in E(G)\}$$

The following lemma has been given in [5]:

**LEMMA 4.** *Let  $G$  be a  $k$ -loosely tight triangulation of a closed surface. If  $G$  has an independence set  $U = \{v_1, \dots, v_k\}$ , then the following two conditions hold.*

- (i)  $G - N[U]$  is a complete graph.
- (ii) Any vertex  $v \in V(G - N[U])$  is adjacent to some vertex  $u \in N[U] - U$ .

Let  $G$  be a 1-loosely tight triangulation on a closed surface and  $v$  any vertex of  $G$ . Then, the neighbors of  $v$  form a cycle, each of whose edges, say  $uw$ , shares a face  $uvw$  with  $v$ . This cycle is called the *link* of  $v$  and is denoted by  $\text{lk}(v)$ . The *wheel neighborhood* is often defined as the subgraph which consists of  $\text{lk}(v) \cup \{v\}$  with all edges incident to  $v$  and is denoted by  $W(v)$  in this paper. The subgraph  $\langle N[v] \rangle$  induced by the closed neighborhood  $N[v]$  of  $v$  consists of  $W(v)$  and possibly some edges joining vertices on  $\text{lk}(v)$ . By Lemma 4,  $G - N[v]$  is a complete graph.

**LEMMA 5.** *Let  $G$  be a 1-loosely tight triangulation of the projective plane and  $v$  any vertex of  $G$ . Then one of  $\langle N[v] \rangle$  and  $G - N[v]$  includes an essential cycle of length 3 and the other does not.*

*Proof.* Since any two essential closed curves in the projective plane must intersect each other, it does not happen that both  $\langle N[v] \rangle$  and  $G - N[v]$  include essential cycles. So, we suppose that none of them does. Then they are contained in two disjoint disks separately. Shrinking each of these disks to a point, we obtain an embedding of a graph with only two vertices such that each face is bounded by a pair of multiple edges. If there are  $X$  edges between these two vertices, then there are  $X$  faces and we have  $2 - X + X = 2$ . This is contrary to Euler's formula for the projective plane. Thus, either  $\langle N[v] \rangle$  or  $G - N[v]$  includes an essential cycle.

If  $\langle N[v] \rangle$  includes an essential cycle, then such a cycle must pass through an edge which joins two vertices  $u$  and  $w$  on  $\text{lk}(v)$  and which does not lie in  $W(v)$ . It is obvious that the cycle  $uvw$  is essential, too.

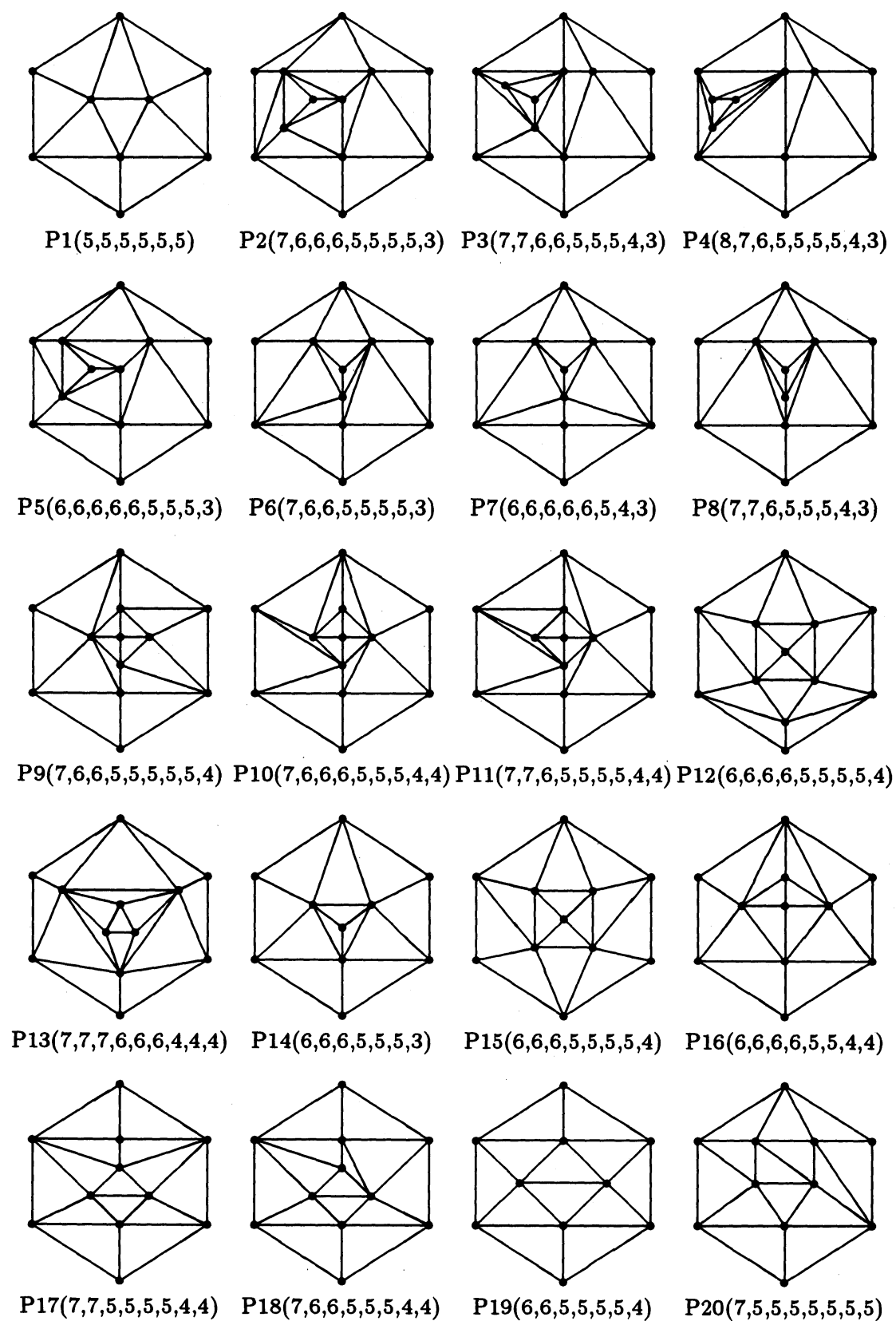
Now suppose that  $G - N[v]$  includes an essential cycle. If there is a cycle of length 3 in  $G - N[v]$  which bounds no face of  $G$ , then it must be essential; otherwise, Condition (ii) in Lemma 5 would not hold. Thus, this cycle is the required one. If there is no such cycle, then all cycles of length 3 in  $G - N[v]$  bound faces. From this and Condition (ii), it follows that the complete graph  $G - N[v]$  consists of at most three vertices, but any essential cycle could not exist in  $G - N[v]$ .

Therefore, we have found an essential cycle in either  $\langle N[v] \rangle$  or  $G - N[v]$ , and the lemma follows. ■

*Proof of Theorem 2.* Let  $G$  be a 1-loosely tight triangulation on the projective plane and choose a vertex  $v$  so that it attains the minimum degree of  $G$ . Since any graph embedded in the projective plane has a vertex of degree at most 5 by Euler's formula, we have  $\deg v = 3, 4$  or  $5$ .

By Lemma 5, either  $\langle N[v] \rangle$  or  $G - N[v]$  includes an essential cycle of length 3. By Lemma 4,  $G - N[v]$  is isomorphic to  $K_n$  with  $n \leq 5$  and all of its vertices lie together along the boundary of a face of  $G - N[v]$ . Under these conditions, we can construct and classify those triangulations systematically; if it includes either an independent set of three vertices or a pair of vertices with distance 3, we must omit it by Theorem 1. Finally we obtain the twenty triangulations P1 to P20 given in Figure 4, which can be distinguished by their degree sequences added to their names. Table 2 presents the pair of  $\deg v$  and  $K_n$  for each of the triangulations.

Note that P1 is isomorphic to  $K_6$  and  $\xi(P1) = 0$  while  $\xi(G) = 1$  for the others. For P2 to P16,  $G - N[v]$  includes an essential cycle of length 3, which



**Figure 4.** The 1-loosely tight triangulation of the projective plane



	0	$K_1$	$K_2$	$K_3$	$K_4$	$K_5$
$\deg v = 3$	-	-	-	P14	P6, P7, P8	P2, P3, P4, P5
$\deg v = 4$	-	-	P19	P15, P16 P17, P18	P9, P10, P11 P12, P13	-
$\deg v = 5$	P1	-	P20	-	-	-

corresponds to the boundary of each hexagon, and  $v$  lies inside the hexagon in each figure. For P17 to P20,  $\langle N[v] \rangle$  includes an essential cycle of length 3 and the two vertices at the top and bottom of each hexagon correspond to  $v$ . ■

### References

- [1] J.L. Arocha, J. Bracho and V. Neumann-Lara, On the minimum size of tight hypergraphs, *J. Graph Theory*, **16** (1992), 319–326.
- [2] J.L. Arocha, J. Bracho and V. Neumann-Lara, Tight and untight triangulated surfaces, *J. Combin. Theory, Ser. B*, **63** (1995), 185–199.
- [3] D. Barnette, Generating the triangulations of the projective plane, *J. Combin. Theory Ser. B*, **33** (1982), 222–230.
- [4] S. Lawrencenko, S. Negami and A.T. White, Three nonisomorphic triangulations of an orientable surface with the same complete graph, *Discrete Math.*, **135** (1994), 367–369.
- [5] S. Negami and T. Midorikawa, Loosely-tightness of triangulations of closed surfaces, *Sci. Rep. Yokohama Nat. Univ., Sec. I*, **43** (1996), 25–41.

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