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NOTE ON FROZEN TRIANGULATIONS ON CLOSED SURFACES

By

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Abstract. In this note, we introduce the notions of frozen triangulations on closed surfaces, as ones to which any diagonal flip is not applicable and consider the relationship between those and other concepts for triangulations on closed surfaces. Those arguments will lead us to estimate a lower bound for the minimum number $N = N(F^2)$ such that two triangulations on a closed surface F^2 with the same number of vertices can be transformed into each other by a sequence of diagonal flips whenever they have at least N vertices.

Introduction

A *triangulation* on a closed surface is a simple graph embedded on the surface so that each face is triangular and any two faces share at most one edge. Two triangulations G_1 and G_2 on a closed surface F^2 are said to be *equivalent* or *homeomorphic* if there is a homeomorphism $h : F^2 \to F^2$ with $h(G_1) = G_2$. Such a homeomorphism induces a graph isomorphism between G_1 and G_2 which induces a bijection between their face sets.

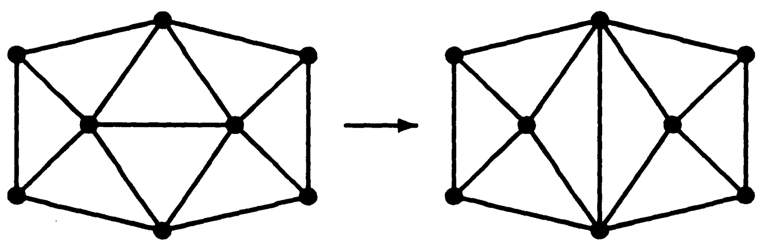

Figure 1. Diagonal flip

A *diagonal flip* is a local transformation in a triangulation which flip an edge in the quadrilateral with the edge as a diagonal, as shown in Figure 1. We have

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to keep the simpleness of triangulations and hence do not perform the diagonal flip if it yields multiple edges.

There have been many studies on the diagonal flips in triangulations on closed surfaces. The origin of this topic is the theorem proved by Wagner [33] which states that any two triangulations on the sphere with the same number of vertices can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips. Also Dewdney [10] proved the same fact for the torus while Negami and Watanabe [24] did it for the projective plane and the Klein bottle.

The same statement as their theorems does not hold in general, as we shall show later. However, the following theorem proved by Negami [26] shows that it also holds for other surfaces if we restrict the number of vertices of triangulations to exceed some constant. This theorem gives us a "breakthrough" toward a general theory of diagonal flips and is the starting point of a serise of recent studies on diagonal flips in triangulations; [9], [15], [16], [21], [22], [25], [26], [28], [29] and so on.

THEOREM 1. (Negami [26]) *For any closed surface F^2, there exists a natural number $N = N(F^2)$ such that two triangulations G_1 and G_2 can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips if* $|V(G_1)| = |V(G_2)| \geq N$.

An *edge contraction* is to shrink an edge to a point and replace each pair of the resulting multiple edges with a single edge to make another triangulation on the same surface, as shown in Figure 2. An edge is said to be *contractible* if its contraction does not yield multiple edges. We perform only the contraction of contractible edges.

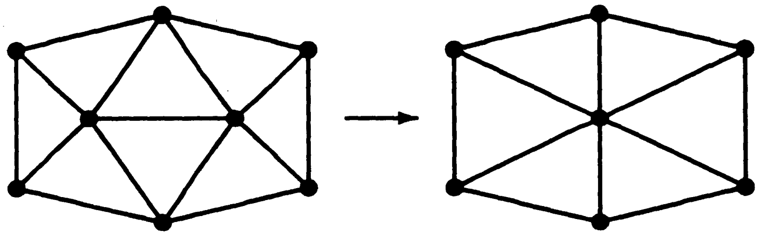

Figure 2. Edge contraction

One of the key ideas in Negami's proof of the above theorem is to connect the notion of diagonal flips to the edge contraction. (See [26] for the details.) This connection leads us to some notions on triangulations to give the theoretical value of $N(F^2)$. The "pseudo-minimal triangulation" is one of those notions and will be defined in the next section. Negami has asked in [26] whether or not every pseud-minimal triangulation might be minimal, but we shall show the negative answer to his question.

Let $N(F^2)$ denote its minimum value which makes Theorem 1 valid. We have $N(F^2) = 4$, 6, 7 and 8 if F^2 is the sphere, the projective plane, the torus and the Klein bottle, and these values coincide with the minimum number of vertices of triangulations for these surfaces. However, it does not hold in general and hence the bound $N(F^2)$ for the number of vertices is needed actually for the theorem. Negami [29] has already given an upper bound for $N(F^2)$ by a linear function of the genus of F^2. So, we would like to estimate a lower bound for $N(F^2)$ and to know when $N(F^2)$ does not coincide with the number of vertices of minimal triangulations.

To do it, we shall introduce several notions on triangulations and discuss the relationship among them in Section 1. The "frozen triangulation" in the title is one of them and the point in our arguments below is how to construct frozen triangulations with non-complete graphs. By the existence of such frozen triangulations and that of inequivalent triangulations with complete graphs, we shall specify some cases that $N(F^2)$ has a non-trivial lower bound, in Section 2.

1. Properties of triangulations

First, we shall define several properties of triangulations related to diagonal flips and edge contraction and give some comments for them. In particular, a *pseudo-minimal triangulation* has been defined in [26] to describe the value of $N(F^2)$ theoretically, while a *frozen triangulation* has been defined in [23] with an additional point related to the minimum degrees.

- **Complete triangulations:** Ones that are isomorphic to complete graphs as graphs. The complete graph K_n triangulates an orientable closed surface if and only if $n \equiv 0, 3, 4$ or $7 \mod 12$, while it triangulates a non-orientable one if and only if $n \equiv 0$ or $1 \mod 3$ and $n \neq 7$. These facts are well-known as consequences of the solution of "Map Color Theorem" [30].

- **Minimal triangulations:** Ones with fewest vertices among all triangulations on a closed surface. If the surface admits a complete triangulation, then the triangulation becomes necessarily minimal. The number of vertices of a minimal triangulation has been determined in [14] and [31] as follows:

$$V_{\min}(F^2) = \left\lceil \frac{7 + \sqrt{49 - 24\chi(F^2)}}{2} \right\rceil$$

where F^2 is neither the orientable closed surface of genus 2, the non-orientable one of genus 3, nor the Klein bottle. The values of $V_{\min}(F^2)$ are 10, 9 and 8 for these exceptional surfaces, respectively.

- **Frozen triangulations:** Ones to which no diagonal flip is applicable. That is, any diagonal flip in a frozen triangulation yields a pair of multiple edges. Thus, a triangulation is frozen if and only if the four vertices lying on the quadrilateral with e as its diagonal induces K_4 for any edge $e \in E(G)$. This implies that any complete triangulation is frozen. So, our question is whether or not there is a non-complete frozen triangulation. We shall show later the affirmative answer to this question.
- **Pseudo-minimal triangulations:** Ones from which no sequence of diagonal flips results in a triangulation including a contractible edge. Especially, any pseudo-minimal triangulation has no contractible edge and hence is irreducible. By Negami's arguments in [26], a triangulation is pseudo-minimal if and only if no sequence of diagonal flips results in a triangulation having a vertex of degree 3. We do not know any other reasonable characterization of pseudo-minimal triangulations.
- **Irreducible triangulations:** Ones which have no contractible edge. There exit only finitely many irreducible triangulations of each closed surface. Let $V_{\mathrm{irr}}(F^2)$ denote the maximum number of vertices of irreducible triangulations of a closed surface F^2. The following bound for $V_{\mathrm{irr}}(F^2)$ is the best one at present, and is given in [20]:

$$V_{\mathrm{irr}}(F^2) \leq 171\gamma(F^2) - 72$$

where $\gamma(F^2)$ stands for the *Euler genus* of F^2 and is defined by $\gamma(F^2) = 2 - \chi(F^2)$ with the Euler characteristic $\chi(F^2)$ of F^2. However, this is far from the truth. We have $V_{\mathrm{irr}}(F^2) = 4$, 7, 10 and 11 for the sphere [32], the projective plane [3], the torus [17] and the Klein bottle [18], by the classification of irreducible triangulations of these surfaces.

We would like to investigate the implications among these notions. To denote the statement for each of them, we shall use the ordered pair of initial letters of two notions. For example, "CM" means that if a triangulation is complete, then it is minimal. From the above comments, we can conclude easily CM, CF, FP, MP and PI, which the thick arrows present in Figure 3. We shall show below that the other implications do not hold in general.

MC does not hold: As is shown in the comment on minimal triangulations, if a closed surface admits a complete triangulation, then the triangulation is minimal. Furthermore, it is well-known that a closed surface F^2, except the

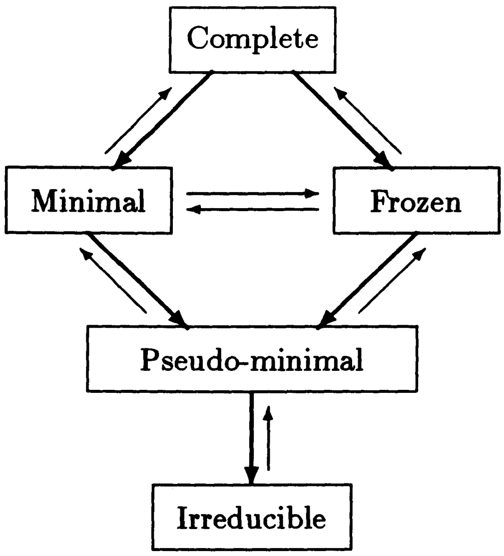

Figure 3. Properties of triangulations

Klein bottle, admits a complete triangulation if and only if $\frac{7+\sqrt{49-24\chi(F^2)}}{2}$ is an integer, which becomes the number of vertices in the complete triangulation. Thus, any minimal triangulation is not complete for each closed surface which does not satisfy this condition. The Klein bottle also does not admit any complete triangulation and has precisely six minimal triangulations with 8 vertices. ■

FC does not hold: Let $K_{n(m)}$ denote the complete n-partitle graph with partite sets of the same size m, that is, $K_{m,\ldots,m}$ with n m's. The following theorem is one of corollaries of Theorem 7.5 in [1], and is useful to construct a frozen triangulation which is not complete.

THEOREM 2. (Archdeacon [1]) *If the complete graph K_n triangulates a closed surface F^2 and if each prime factor of m is at least $n-1$ except the case of $n=4$, $m=3$, then $K_{n(m)}$ triangulates another surface.*

Following Archdeacon's method in [1], we can construct the triangulation with $K_{n(m)}$ on a closed surface $\tilde{F}^2$ as a *wrapped covering* of K_n embedded on F^2. That is, there is an m^2-fold branched covering $p : \tilde{F}^2 \to F^2$, only branched over $V(K_n)$, such that the neighborhood of each vertex $v \in V(K_{n(m)})$ wraps that of $p(v) \in V(K_n)$ cyclically. (See [1] and [13] for the precise definition of a wrapped covering.) If F^2 is non-orientable, then $\tilde{F}^2$ is non-orientable, too, since any feasible m in the theorem is odd for $n \geq 4$.

This structure guarantees that if $abcd$ is a quadrilateral with ac its diagonal in $K_{n(m)}$, then b and d belong to two different partite sets of $K_{n(m)}$ for $n \geq 4$. Thus, b and d are joined with an edge and hence ac cannot be flipped. This implies that $K_{n(m)}$ is a frozen triangulation of $\tilde{F}^2$.

For example, since K_4 triangulates the sphere, $K_{4(m)}$ triangulates an orientable closed surface with any odd integer $m \neq 3$ and each of the triangulations with $K_{4(m)}$ is frozen. To construct a non-complete frozen triangulations on non-orientable closed surfaces, we can use $K_{6(m)}$ with m not divisible by 2 and 3 since K_6 triangulates the projective plane. (We can find other constructions of triangulations with $K_{n(m)}$ in [5], [6] and [7].) ■

PM and FM do not hold: The triangulation with $K_{4(m)}$ in the previous is pseudo-minimal since it is frozen. We can show that this is not minimal, comparing the number of its vertices with that of minimal ones. Suppose that $K_{4(m)}$ triangulates the orientable closed surface F^2 of genus $g > 0$ and let V and E denote the number of vertices and of edges in it. Then we have $V = 4m$, $E = 6m^2$ and $E = 3(V-(2-2g))$ by Euler's formula. From these, we obtain the quadratic equation $\frac{3}{8}V^2 = 3(V+2g-2)$ for V. Solving this, we have $V = 4+4\sqrt{g}$, which is strictly larger than $V_{\min}(F^2) = \left\lceil \frac{7+\sqrt{1+48g}}{2} \right\rceil$.

Similarly, suppose that $K_{6(m)}$ triangulates the non-orientable closed surface F^2 of genus $q > 1$. Then the number of its vertices is equal to $18 + 6\sqrt{5q-1}$, which is greater than $V_{\min}(F^2) = \left\lceil \frac{7+\sqrt{1+24q}}{2} \right\rceil$. Thus, $K_{6(m)}$ is pseudo-minimal, but not minimal. ■

PF and MF do not hold: The six minimal triangulations of the Klein bottle, which are pseudo-minimal, can be joined to one another by diagonal flips. Thus, they are not frozen.

For other surfaces, we can use a triangular embedding of $K_n - K_2$, which is the complete graph K_n with an edge deleted. For example, if $n \equiv 5 \bmod 12$, then $K_n - K_2$ admits a triangular embedding of a closed surface, orientable or non-orientable. Such a triangulation must be minimal and hence pseudo-minimal.

Let G be any triangulation on a closed surface F^2 with $K_n - K_2$ and let x and y be the unique pair of non-adjacent vertices in G. That is, the edge deleted from K_n is xy. It is clear that an edge ac is flippable in G if and only if the two faces incident to ac are acx and acy, and that flipping ac results in another triangulation G' with $K_n - K_2$, which is obtained from K_n by deleting the edge ac.

According to Ringel's construction of "Orientable Case 5" in [30], the rotations of some vertices, say a, in $K_n - K_2$ contain a segment $\cdots xcy \cdots$ for some vertex c. This implies that such a rotation scheme exhibits a minimal triangulation with $K_n - K_2$ which is not frozen and has a flippable edge ac. (On the other hand, Ringel's solution of "Non-Orientable Case 5" presents a frozen minimal triangulation with $K_n - K_2$. The surface admitting such a triangulation may be orientable or non-orientable.) ■

IP does not hold: One of the two irreducible triangulations of the projective plane, not isomotphic to K_6, is isomorphic to $K_4 + \overline{K_3}$ and is not pseudo-minimal. The 20 irreducible triangulations of the torus, other than K_7, are not pseudo-minimal. The 19 irreducible triangulations of the Klein bottle, other than the six minimal ones, are not pseudo-minimal. Many examples for other closed surfaces can be constructed easily by pasting these irreducible triangulations along their faces. ■

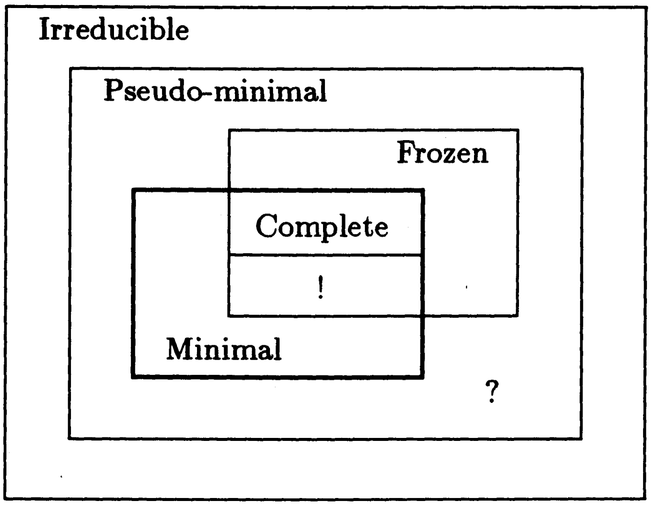

Figure 4. Classes of triangulations

Each pseudo-minimal triangulation in the above examples is either frozen or minimal. Does there exist a pseudo-minimal triangulation on a closed surface which is not minimal and is not frozen? Also, does there exist a frozen minimal triangulation which is not complete? Figure 4 will make the meaning of these questions clearer. The answer to the second question is affirmative and we have already shown an example for such a triangulation with $K_n - K_2$, corresponding to the region with "!". Thus, we want examples for the region with "?" in the figure. Examples corresponding to the other regions have been presented in the previous arguments. Is the unknown region empty or not?

We can define another interesting class of triangulations on closed surfaces formally, as follows. A triangulation is said to be *isolated* if any diagonal flip in it results in itself, up to homeomorphsim. Thus, an isolated triangulation cannot be transformed into any other triangulation by a sequence of diagonal flips. For example, a frozen triangulation is isolated while an isolated triangulation is pseudo-minimal. Does there exist an isolated triangulation which is not frozen? The minimal triangulations with $K_n - K_2$ in the previous are candidates for this.

A triangulation is said to be *self-flippable* if it has an edge e such that flipping e does not change the homeomorphism type of the triangulation. A non-frozen isolated triangulation is self-flippable.

It is easy to construct a self-flippable triangulation on the sphere, as follows. Prepare two wheels W_n and W_{n+1} which triangulate two disks with rims $u_0u_1 \cdots u_{n-1}$ and $v_0v_1 \cdots v_n$ respectively along their boundary cycles. Join these rims with a cylinder (or an annulus) to make the sphere and place edges u_iv_i $(i = 0, 1, \ldots, n-1)$, u_jv_{j-1} $(j = 1, \ldots, n-1)$, u_0v_{n-1} and u_0v_n across the cylinder. Then the resulting triangulation on the sphere is self-flippable. Flipping u_0v_0 to u_1v_n (or u_0v_{n-1} to $u_{n-1}v_n$) results in the same triangulation.

Replacing the wheel parts with suitable objects, we can make another self-flippable triangulation on another closed surface, but those objects have to have cyclic symmetry of order n and of order $n+1$, which will restrict the genus of the whole surface. Characterize or classify those self-flippable triangulations on each closed surface.

2. Lower bounds for $N(F^2)$

Now we shall discuss when $N(F^2)$ does not coincide with $V_{\min}(F^2)$, applying the previous arguments on frozen triangulations. The next two theorems will follow from the existense of frozen triangulations which are not minimal.

THEOREM 3. *Let F^2 be the orientable closed surface of genus g. If g is an even square number more than 4, then we have:*

$$V_{\min}(F^2) < 4 + 4\sqrt{g} < N(F^2)$$

Proof. If g is an even square number, then we can put $g = 4x^2$ with a positive integer $x \geq 2$ and $m = 1 + \sqrt{g} = 1 + 2x$ is an odd integer, different from 3. By Theorem 2, we can construct a frozen triangulation with $K_{4(m)}$ for this m. This triangulation is not minimal and there is a triangulation, obtained from a minimal one by subdividing it suitably, which cannot be transformed into $K_{4(m)}$ by diagonal flips. Thus, $N(F^2)$ must be larger than $4m$, the number of vertices of $K_{4(m)}$. ∎

THEOREM 4. *Let F^2 be the non-orientable closed surface of genus q. If $5q-1$ is an even square number and if $q \not\equiv 2 \bmod 3$, then we have:*

$$V_{\min}(F^2) < 18 + 6\sqrt{5q-1} < N(F^2)$$

Proof. If $5q-1$ is an even square number, then $m = 3 + \sqrt{5q-1}$ is an odd integer. If $m \equiv 0 \bmod 3$, then $5q-1 \equiv -q-1 \equiv 0 \bmod 3$. This implies that $q \equiv 2 \bmod 3$, which is excluded by our assumption in the lemma. Thus, this m is divisible by neither 2 nor 3, and we can construct a frozen triangulation with $K_{6(m)}$ by Theorem 2. ■

Since any complete triangulation is minimal and frozen, if there are two or more inequivalent complete triangulations on a closed surface F^2, then they cannot be transformed into one another by diagonal flips and hence $V_{\min}(F^2) < N(F^2)$. So we would like to know when K_n triangulates a closed surface in two or more ways.

Construction of complete triangulations has been studied well, related to Map Color Theorem, and can be found in Ringel's book [30]. However, attempting to construct inequivalent complete triangulations has just begun very recently.

For example, Archa, Bracho and Neumann-Lara [2] have constructed pairs of inequivalent triangulations with the same complete graphs on a series of non-orientable closed surfaces under some conditions. The smallest example is K_{16}, which triangulates the non-orientble closed surface of genus 26 in three inequivalent ways. To recognize their inequivalence, they have developed a theory of tight and untight triangulations. A triangulation G on a closed surface is said to be *tight* if there is a face with three distinct colors at its corners for any surjective color assignment $f : G \to \{1,2,3\}$, and is *untight* otherwise. (It is easy to see that any tight triangulation is complete. Negami and Midorikawa [27] have defined and discussed the tightness adapted for general triangulations, called the *looseness*.)

Lowrencenko, Negami and White [19] also have found three inequivalent complete triangulations with K_{19} on the orientable closed surface of genus 20, but they are all tight. Arocha, Bracho and Neumann-Lara's construction in [2] yields only complete triangulations on non-orientable closed surfaces and cannot be applied to those on orientable closed surfaces. There might be no untight compelete triangulation on any orientable closed surface. Bracho and Strausz [8] have already determined the smallest n for which K_n admits two or more inequivalent triangulations on the same closed surface; $n = 12$ for the orientable surfaces while $n = 9$ for the non-orientable ones.

More successful results can be found in [11] and [12]. Grannell, Griggs and Širáň have discussed Steiner triple systems derived from complete triangulations which have bipartite duals, in these papers. Furthermore, Bonnington, Grannell, Griggs and Širáň [4] have shown that K_n has exponentially many inequivalent triangular embeddings for $n \equiv 7$ or 9 mod 36 and for $n \equiv 19$ or 55 mod 108. The following facts are consequences of their results in [11]:

THEOREM 5. (Grannell, Griggs and Širáň [11])

(i) *If $n \equiv 7$ mod 12 and $n \neq 7$, then K_n triangulates an orientable closed surface in at least two inequivalent ways.*

(ii) *If $n \equiv 1$ mod 6 and $n \neq 7$, then K_n triangulates a non-orientable closed surface in at least two inequivalent ways.*

The next two theorems follow immediately from the above theorem with the well-known formulas of the genus and the non-orientable genus of K_n:

THEOREM 6. *Let F^2 be the orientable closed surface of genus g. If $g = (3k+1)(4k+1)$ for a positive integer $k \geq 1$, then we have $V_{\min}(F^2) < N(F^2)$.* ■

THEOREM 7. *Let F^2 be the non-orientable closed surface of genus q. If $q = (2k-1)(3k-1)$ for a positive integer $k \geq 2$, then we have $V_{\min}(F^2) < N(F^2)$.* ■

Here are the genera of closed surfaces F^2, up to 1000, such that $V_{\min}(F^2) < N(F^2)$, recognized by the theorems in this section.

Orientable	6, 16, 20, 36, 63, 64, 100, 130, 144, 196, 221, 256, 324, 336, 400, 475, 484, 576, 638, 676, 784, 825, 900
Nonorientable	5, 13, 15, 40, 77, 97, 126, 157, 187, 205, 260, 289, 345, 442, 541, 551, 672, 673, 769, 805, 925, 950

Table 1. Genera with non-trivial lower bounds for $N(F^2)$

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