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THE $Y\Delta$-EQUIVALENCE OF KERNELS ON THE TORUS WITHOUT TAKING DUALS

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Abstract. Let G be an embedding on the torus and let ℓ be an essential closed curve on the torus. Let $f_G(\ell)$ be the minimum number of intersections of G and a closed curve ℓ', where ℓ' ranges over all closed curves homotopic to ℓ. An embedding G on the torus is said to be a *kernel* if for any proper minor T of G, $f_G(\ell) > f_T(\ell)$ for some ℓ. In this paper, we show that any two kernels G and G' on the torus with $f_G = f_{G'}$ can be transformed into each other by a sequence of $Y\Delta$-exchanges.

1. Introduction

In this paper, we suppose that a *graph* is always 2-cell embedded in some closed surface F^2 other than the sphere. We denote the vertex set, the edge-set and the face-set of a graph G by $V(G)$, $E(G)$ and $F(G)$, respectively. An *isotopy* over F^2 is a continuous map $\Psi : [0,1] \times F^2 \to F^2$ such that the map $\Psi_t : F^2 \to F^2$, defined by $\Psi_t(x) = \Psi(t,x)$, is a homeomorphism over F^2 for each $t \in [0,1]$ with Ψ_0 the identity map of F^2. In this paper, we regard two graphs G and G' on F^2 as the same one if there is an isotopy Ψ over F^2 such that $\Psi_1(G) = G'$.

Let G and G' be graphs on a closed surface F^2. We say that G' is a *minor* of G if G' can be obtained from G by a sequence of edge-contractions and edge-deletions on the surface. In particular, if $G \neq G'$, then G' is called a *proper* minor of G.

Let D be a *closed curve* on a closed surface F^2, that is, a continuous map $D : S^1 \to F^2$ or its image, where S^1 is the unit circle in the complex plane. We say that D is *trivial* if $D(S^1)$ bounds a 2-cell on F^2, while D is *essential* otherwise. Two closed curves (or loops) D_1 and D_2 on F^2 are said to be *homotopic* if there is a continuous map $\Psi : [0,1] \times S^1 \to F^2$ such that $\Psi_0(x) = D_1(x)$ and $\Psi_1(x) = D_2(x)$ for each $x \in S^1$.

Let G be a graph on a closed surface F^2. The *representativity* of G, denoted

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by $r(G)$, is the minimum number of intersections of G and an essential closed curve on F^2. In particular, G is called *minor-minimal k-representative* or simply *k-minimal* if

(i) $r(G) = k$, and

(ii) for any proper minor G' of G, $r(G') < k$.

Let G be a graph on a closed surface F^2. Suppose that G has a vertex v of degree 3 with neighbors x, y and z. The *$Y\Delta$-exchange* at v is to delete v and add three edges xy, yz and xz, or its inverse operation. See Figure 1.

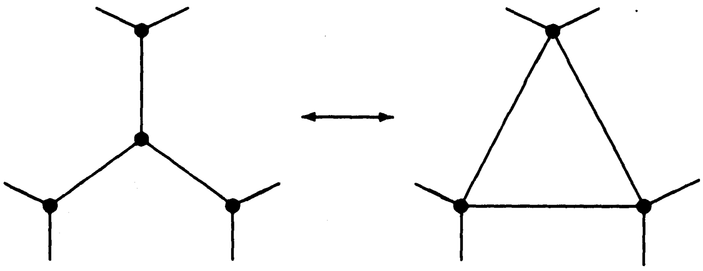

Figure 1. $Y\Delta$-exchange

The k-minimality can be preserved by (i) a $Y\Delta$-exchange and (ii) taking the dual. When G and G' are transformed into each other by a sequence of (i) and (ii), they are said to be *$Y\Delta$-equivalent.*

For the projective plane, Vitray determined the complete list of 2- and 3-minimal graphs, and showed that for $k = 2$ and 3, any two of k-minimal graphs can be transformed into each other by a sequence of only $Y\Delta$-exchanges [8]. Moreover, Randby showed that this fact holds for any natural number k [4].

How about k-minimal graphs on the torus or other closed surfaces? The torus however admits 4-regular quadrangulations, to which no $Y\Delta$-exchange can be applied. (A *quadrangulation* of a closed surface F^2 is a graph on F^2 whose faces are all quadrilateral.) For example, consider the 4-regular quadrangulation on the torus isomorphic to the Cartesian product $C_k \times C_k$, where C_k is a cycle of length k. Clearly, the quadrangulation $C_k \times C_k$ is k-minimal, but this is not $Y\Delta$-equivalent to other k-minimal graphs. In case of $k = 2$, the author verified that any two k-minimal graphs on the torus, not 4-regular quadrangulations, are transformed into each other by a sequence of $Y\Delta$-exchanges.

Shrijver gave a nice characterization of two graphs on the same non-spherical orientable closed surface being $Y\Delta$-equivalent, as follows.

For a graph G on a closed surface F^2 and an essential closed curve ℓ on F^2, let $f_G(\ell)$ be the minimum number of intersections of G and a closed curves ℓ', where ℓ' ranges over all closed curves homotopic to ℓ. We call the function f_G

the *width* of G. This is a generalization of the representativity. We say that G is a *kernel* if for any proper minor G' of G, $f_G(\ell) > f_{G'}(\ell)$ for some closed curve ℓ. The minimum value of $f_G(l)$ coincides with the representativity of G. Clearly, if G is k-minimal, then G is a kernel.

Clearly, if two graphs G and G' are $Y\Delta$-equivalent, then $f_G = f_{G'}$. Moreover, if G is a kernel, then G' is also a kernel in this case.

THEOREM 1. (A. Schrijver [6]) *Two kernels on the same non-spherical orientable closed surface are $Y\Delta$-equivalent if and only if they have the same width.*

He also determined the number of $Y\Delta$-equivalence classes of k-minimal graphs on the torus for every natural number k [5]. Theorem 1 seems to hold for non-orientable closed surface.

As mentioned before, for any fixed k, any two k-minimal graphs on the projective plane can be transformed into each other only by $Y\Delta$-exchanges (i.e., $Y\Delta$-equivalent without taking duals). The author [2, 3] verified that any two 2-minimal graphs on the torus or the Klein bottle with the same width are $Y\Delta$-equivalent without taking duals, related to the irreducible bipartite quadrangulations. Moreover, Hirachi [1] checked that any two 3-minimal graphs on the torus with the same width can be transformed into each other by a sequence of only $Y\Delta$ exchanges.

In this paper, we would like to discuss whether or not taking duals is actually needed to transform two kernels on the torus with the same width, and show the following theorem.

THEOREM 2. *Any two kernels on the torus with the same width can be transformed into each other by a sequence of only $Y\Delta$-exchanges.*

Our theorem describes only toroidal case, but we conjecture the following.

CONJECTURE 3. *Any two kernels on the same non-spherical orientable closed surface with the same width can be transformed into each other by a sequence of only $Y\Delta$-exchanges.*

2. Preliminaries

For a graph G on a closed surface, the *medial graph* $M(G)$ of G is constructed, as follows.

(i) Add a vertex v_e to the middle point of each edge e of G,

(ii) for any two edges e and e' which are consecutive with respect to the rotation of some vertex of G, add an edge $v_e v_{e'}$, and

(iii) delete all the original vertices of G.

It is obvious that $M(G)$ is 4-regular, and it has a 2-face-coloring with black and white so that each black region contains a vertex of G.

Let G^* denote the dual of G. Note that $M(G) = M(G^*)$ and their 2-face-colorings are *opposite*, that is, each black (or white) region of $M(G)$ is a white (or black) region of $M(G^*)$.

Schrijver gave a nice characterization of kernels on orientable closed surfaces, using the medial graphs, as follows. The union of closed curves $D_0, \ldots, D_{k-1}$ is called a *minimally crossing* system if for any i and j, D_i and D_j have a minimal number of intersections among all closed curves D_i' and D_j' homotopic to D_i and D_j, respectively. (Thus, each intersection of two closed curves must be a "crossing" and not a "touching".)

Let G be a graph on a closed surface F^2 such that each vertex of G has even degree. The *straight decomposition* of G is the decomposition of G into several closed walks $C_0, \ldots, C_{n-1}$ such that

1. $E(G) = E(C_0) \cup \cdots \cup E(C_{n-1})$,
2. for any i and j with $i \neq j$, $E(C_i) \cap E(C_j) = \emptyset$, and
3. for each vertex v of G with neighbors $v_0, \ldots, v_{2k-1}$ in this cyclic order, vv_i and vv_{i+k} belong to the same closed walk for each i, where the subscripts are taken modulo $2k$.

Note that this decomposition is unique.

THEOREM 4. (A. Schrijver [7]) *A graph G on a non-spherical orientable closed surface S is a kernel if and only if the straight decomposition of the medial graph $M(G)$ of G is a minimally crossing system of essential closed curves. Moreover, if S is the torus, each closed curve can be taken to be simple (i.e., without self-intersections).*

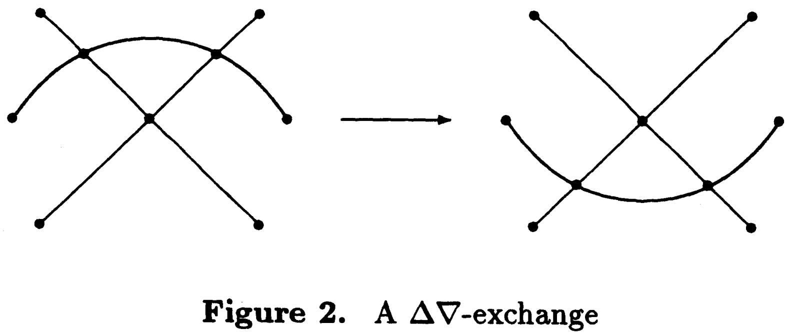

Figure 2. A $\Delta\nabla$-exchange

Let G be a 4-regular graph on a closed surface F^2. The operation shown in Figure 2 is called a $\Delta\nabla$-*exchange*. Observe that if $D_0 \cup \cdots \cup D_{k-1}$ is a minimally crossing system of closed curves, then the resulting system by a $\Delta\nabla$-exchange is also a minimally crossing system.

It is obvious that a $Y\Delta$-exchange in a graph G corresponds to a $\Delta\nabla$-exchange in its medial graph. Thus, in order to show Theorem 1, Schrijver showed that the medial graphs of two kernels on the same orientable closed surface with the same width can be transformed into each other by a sequence of $\Delta\nabla$-exchanges. However, since $M(G) = M(G^*)$ for any graph G, his argument cannot deny the necessity of taking duals.

3. Proof of theorems

This section is devoted to proving our main theorem. For our purpose, it suffices to prove the following lemma.

LEMMA 5. *If G is a kernel on the torus, then G can be transformed into the dual G^* of G by a sequence of $Y\Delta$-exchanges.*

Proof. To prove the lemma, we use the medial graphs $M(G)$ and $M(G^*)$ of G and G^*, respectively. Since $M(G) = M(G^*)$, we distinguish $M(G)$ and $M(G^*)$ with their 2-face-colorings. We shall show that the 2-face-coloring of $M(G)$ can be transformed into its opposite 2-face-coloring by a sequence of $\Delta\nabla$-exchanges.

By Theorem 4, $M(G)$ can be regarded as a minimally crossing system of simple closed curves, denoted by $\mathcal{L}$. Since two distinct homotopic simple closed curves in a minimally crossing system must be disjoint on the torus, we can put $\mathcal{L} = L_0 \cup \cdots \cup L_{k-1}$, where each L_i is a maximal set of pairwise disjoint, homotopic, essential simple closed curves. We focus on L_0 and let $L_0 = \{l_0, \dots, l_{m-1}\}$, where $m \geq 1$ and the subscript are taken modulo m. Since each l_i is an essential simple closed curve, cutting the torus along $l_0, \dots, l_{m-1}$ yields m annuli. We suppose that $l_0, \dots, l_{m-1}$ appear on the torus to be parallel in this cyclic order. For $i = 0, \dots, m-1$, we denote by A_i the annulus bounded by l_i and l_{i+1}, and let $\operatorname{Int} A_i = A_i - \{l_i, l_{i+1}\}$.

Consider an isotopy Ψ over the torus such that $\Psi_1(l_i) = l_{i+1}$ for $i = 0, \dots, m-1$. We slide only $l_0, \dots, l_{m-1}$ by this isotopy Ψ, fixing $L_1 \cup \cdots \cup L_{k-1}$, so that the resulting 4-regular graph is isomorphic to $M(G)$. We assume that each point on the torus has the same color, black or white, as the region of $M(G)$ which contains it The color of a point are switched when one of $l_0, \dots, l_{m-1}$ jumps it during sliding them by Ψ, and switching its color happens only once. Thus, we obtain the opposite 2-face-coloring of $M(G)$, that is, the 2-face-coloring of $M(G^*)$.

Now we show that the deformation of L_0 by the isotopy Ψ can be obtained by a sequence of $\Delta\nabla$-exchanges. First, we move l_0 toward l_1 so that $\operatorname{Int} A_0$ incudes no vertices afterward. Next, we move l_1 toward l_2. Repeating them, we finally move l_{m-1} to the position where l_0 was.

If $\mathcal{L} = L_0 \cup L_1$, that is, $\mathcal{L}$ consists of only two homotopy classes of closed curves, then each A_i has no vertices inside, and hence the deformation of $\mathcal{L}$ can be realized as an isotopy over the torus which moves the whole of $M(G)$ to itself. In this case, G is a 4-regular quadrangulation of the torus and is self-dual. That is, G and G^* are the same one, up to isotopy.

Suppose that there are n vertices in $\operatorname{Int} A_0$. We use induction on n. If $n = 0$, then we have nothing to do. Suppose that $n > 0$ and there is a vertex v, which is shared by two closed curves x and y. Note that $x, y \notin L_0$ and that each closed curve in $L_1 \cup \cdots \cup L_{k-1}$ appears in A_0 as a union of pairwise parallel and disjoint simple arcs each of whose ends are in l_0 and l_1 respectively. Let p and q be the intersections of x and l_0, and y and l_0, respectively. Then, there is a triangular region Δ bounded by l_0, x and y.

We may assume that Δ is the innermost one among those triangular regions, after reselecting v if we need. Under this assumption, there is no vertex, except v, p and q, inside Δ and also on the boundary of Δ. Thus, we can apply a $\Delta\nabla$-exchange around v and can make v be outside A_0. Now there are $n-1$ vertices inside A_0. By induction hypothesis, we can transform l_0 into a simple closed curve running near along l_1, up to isotopy, by a sequence of $\Delta\nabla$-exchanges.

Similarly, we can generate a sequence of $\Delta\nabla$-exchanges which carries l_i to where l_{i+1} was for all i. This corresponds to a sequence of $Y\Delta$-exchanges which transforms G into G^*. ■

Now we prove our main theorem.

Proof of Theorem 2. Let G and G' be two kernels on the torus with $f_G = f_{G'}$. By Theorem 1, there exists a sequence $G = G_1, \ldots, G_t = G'$ such that for each i, G_i is a kernel, and for $i = 1, \ldots, t-1$, G_i can be transformed into G_{i+1} by either a $Y\Delta$-exchange or taking a dual. When $G_i^* = G_{i+1}$, they can be transformed into each other by a sequence of $Y\Delta$-exchanges, by Lemma 5. Thus, G and G' can be connected only by $Y\Delta$-exchanges. ■

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