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ISOMORPHISMS OF SOME CYCLIC ABELIAN COVERS OF SYMMETRIC DIGRAPHS

By

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Abstract. Let D be a connected symmetric digraph, $\boldsymbol{Z}_p$ a cyclic group of prime order $p\ (>2)$ and Γ a group of automorphisms of D. We enumerate the number of Γ-isomorphism classes of g-cycic $\boldsymbol{Z}_p \times \boldsymbol{Z}_p$-covers of D for any nonunit $g \in \boldsymbol{Z}_p \times \boldsymbol{Z}_p$.

1. Introduction

Graphs and digraphs treated here are finite and simple.

Let D be a symmetric digraph and A a finite group. A function $\alpha : A(D) \rightarrow A$ is called *alternating* if $\alpha(y,x) = \alpha(x,y)^{-1}$ for each $(x,y) \in A(D)$. For $g \in A$, a *g-cyclic A-cover* (or *g-cyclic cover*) $D_g(\alpha)$ of D is the digraph as follows:

$$V(D_g(\alpha)) = V(D) \times A, \text{ and } ((u,h),(v,k)) \in A(D_g(\alpha)) \quad \text{if and only if}$$
$$(u,v) \in A(D) \text{ and } k^{-1}h\alpha(u,v) = g.$$

The *natural projection* $\pi : D_g(\alpha) \rightarrow D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a *cyclic A-cover* of D if D' is a g-cyclic A-cover of D for some $g \in A$. In the case that A is abelian, then $D_g(\alpha)$ is simply called a *cyclic abelian cover.*

Let α and β be two alternating functions from $A(D)$ into A, and let Γ be a subgroup of the automorphism group $\operatorname{Aut} D$ of D, denoted $\Gamma \leq \operatorname{Aut} D$. Let $g,h \in A$. Then two cyclic A-covers $D_g(\alpha)$ and $D_h(\beta)$ are called *Γ-isomorphic*, denoted $D_g(\alpha) \cong_\Gamma D_h(\beta)$, if there exist an isomorphism $\Phi : D_g(\alpha) \rightarrow D_h(\beta)$ and a $\gamma \in \Gamma$ such that $\pi\Phi = \gamma\pi$, i.e., the diagram

$$\begin{tikzcd} D_g(\alpha) \arrow[r, "\Phi"] \arrow[d, "\pi"'] & D_h(\beta) \arrow[d, "\pi"] \\ D \arrow[r, "\gamma"] & D \end{tikzcd}$$

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commutes. Let $I = \{1\}$ be the trivial subgroup of automorphisms.

Cheng and Wells [1] discussed isomorphism classes of cyclic triple covers (1-cyclic $\boldsymbol{Z}_3$-covers) of a complete symmetric digraph. Mizuno and Sato [16] gave a formula for the characteristic polynomial of a cyclic A-cover of a symmetric digraph D, for any finite group A. Mizuno and Sato [15,17] enumerated the number of I-isomorphism classes of g-cyclic $\boldsymbol{Z}_p^n$-covers and g-cyclic $\boldsymbol{Z}_{p^n}$-covers, and Γ-isomorphism classes of g-cyclic $\boldsymbol{Z}_p$-covers of D for any prime $p\,(> 2)$. Furthermore, Mizuno, Lee and Sato [14] gave a formula for the the number of I-isomorphism classes of connected g-cyclic $\boldsymbol{Z}_p^n$-covers and connected g-cyclic $\boldsymbol{Z}_{p^n}$-covers of D for any prime $p\,(> 2)$.

A graph H is called a *covering* of a graph G with projection $\pi : H \to G$ if there is a surjection $\pi : V(H) \to V(G)$ such that $\pi|_{N(v')} : N(v') \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. The projection $\pi : H \to G$ is an *n-fold covering* of G if π is n-to-one. A covering $\pi : H \to G$ is said to be *regular* if there is a subgroup B of the automorphism group $\operatorname{Aut} H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and A a finite group. Let $D(G)$ be the arc set of the symmetric digraph corresponding to G. Then a mapping $\alpha : D(G) \to A$ is called an *ordinary voltage assignment* if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in D(G)$. The (*ordinary*) *derived graph* G^α derived from an ordinary voltage assignment α is defined as follows:

$$V(G^\alpha) = V(G) \times A, \text{ and } ((u,h),(v,k)) \in D(G^\alpha) \quad \text{if and only if}$$

$$(u,v) \in D(G) \text{ and } k = h\alpha(u,v).$$

The graph G^α is called an *A-covering* of G . The A-covering G^α is an $|A|$-fold regular covering of G. Every regular covering of G is an A-covering of G for some group A (see [3]). Furthermore the 1-cyclic A-cover $D_1(\alpha)$ of a symmetric digraph D can be considered as the A-covering $\widetilde{D}^\alpha$ of the underlying graph $\widetilde{D}$ of D.

A general theory of graph coverings is developed in [4]. $\boldsymbol{Z}_2$-coverings (double coverings) of graphs were dealed in [5] and [19]. Hofmeister [6] and, independently, Kwak and Lee [11] enumerated the I-isomorphism classes of n-fold coverings of a graph, for any $n \in \boldsymbol{N}$. Dresbach [2] obtained a formula for the number of strong isomorphism classes of regular coverings of graphs with voltages in finite fields. The I-isomorphism classes of regular coverings of graphs with voltages in finite dimensional vector spaces over finite fields were enumerated by Hofmeister [7]. Hong, Kwak and Lee [9] gave the number of I-isomorphism classes of $\boldsymbol{Z}_n$-coverings, $\boldsymbol{Z}_p \bigoplus \boldsymbol{Z}_p$-coverings and D_n-coverings, n: odd, of graphs, respectively.

In the case of connected coverings, Kwak and Lee [13] enumerated the I-

isomorphism classes of connected n-fold coverings of a graph G. Furthermore, Kwak, Chun and Lee [12] gave some formulas for the number of I-isomorphism classes of connected A-coverings of a graph G when A is a finite abelian group or D_n.

We present the number of Γ-isomorphism classes of g-cycic $\boldsymbol{Z}_p \times \boldsymbol{Z}_p$-covers of connected symmetric digraphs for any element $g \neq 0 \in \boldsymbol{Z}_p \times \boldsymbol{Z}_p$, where 0 is the unit of $\boldsymbol{Z}_p \times \boldsymbol{Z}_p$.

2. Isomorphisms of cyclic $\boldsymbol{Z}_p \times \boldsymbol{Z}_p$-covres

Let D be a connected symmetric digraph and A a finite abelian group. The group Γ of automorphisms of D acts on the set $C(D)$ of alternating functions from $A(D)$ into A as follows:

$$\alpha^{\gamma}(x,y) = \alpha(\gamma(x), \gamma(y)) \text{ for all } (x,y) \in A(D),$$

where $\alpha \in C(D)$ and $\gamma \in \Gamma$.

Let G be the underlying graph of D. The set of ordinary voltage assignments of G with voltages in A is denoted by $C^1(G;A)$. Note that $C(D) = C^1(G;A)$. Furthhremore, let $C^0(G;A)$ be the set of functions from $V(G)$ into A. We consider $C^0(G;A)$ and $C^1(G;A)$ as additive groups. The homomorphism $\delta : C^0(G;A) \to C^1(G;A)$ is defined by $(\delta s)(x,y) = s(x) - s(y)$ for $s \in C^0(G;A)$ and $(x,y) \in A(D)$. The *1-cohomology group* $H^1(G;A)$ with coefficients in A is defined by $H^1(G;A) = C^1(G;A)/\operatorname{Im}\delta$. For each $\alpha \in C^1(G;A)$, let $[\alpha]$ be the element of $H^1(G;A)$ which contains α.

The automorphism group $\operatorname{Aut} A$ acts on $C^0(G;A)$ and $C^1(G;A)$ as follows:

$$(\sigma s)(x) = \sigma(s(x)) \text{ for } x \in V(D),$$

$$(\sigma\alpha)(x,y) = \sigma(\alpha(x,y)) \text{ for } (x,y) \in A(D),$$

where $s \in C^0(G;A)$, $\alpha \in C^1(G;A)$ and $\sigma \in \operatorname{Aut} A$. A finite group $\mathcal{B}$ is said to have the *isomorphism extension property* (IEP), if every isomorphism between any two isomorphic subgroups $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{B}$ can be extened to an automorphism of $\mathcal{B}$ (see [9]). For example, the cyclic group $\boldsymbol{Z}_n$ for $n \in \boldsymbol{N}$, the dihedral group $\boldsymbol{D}_n$ for odd $n \geq 3$, and the direct sum of m copies of $\boldsymbol{Z}_p$(p: prime) have the IEP.

Mizuno and Sato [17] gave a characterization for two cyclic A-covers of D to be Γ-isomorphic.

THEOREM 1. (17, Corollary 3) *Let D be a connected symmetric digraph, G the underlying graph of D, A a finite abelian group with the IEP, $g \in A$, $\alpha, \beta \in C(D)$ and $\Gamma \leq \operatorname{Aut} D$. Assume that the order of g is odd. Then the following are equivalent:*

1. $D_g(\alpha) \cong_\Gamma D_g(\beta)$.
2. *There exist* $\gamma \in \Gamma$, $\sigma \in \operatorname{Aut} A$ *and* $s \in C^0(G; A)$ *such that*

$$\beta = \sigma\alpha^\gamma + \delta s \text{ and } \sigma(g) = g.$$

Let $\operatorname{Iso}(D, A, g, \Gamma)$ denote the number of Γ-isomorphism classes of g-cyclic A-covers of D. The following result holds.

THEOREM 2. (17, Theorem 3) *Let D be a connected symmetric digraph, A a finite abelian group with the IEP, $g, h \in A$ and $\Gamma \leq \operatorname{Aut} D$. Assume that the orders of g and h are equal and odd, and $\rho(g) = h$ for some $\rho \in \operatorname{Aut} A$. Then*

$$\operatorname{Iso}(D, A, g, \Gamma) = \operatorname{Iso}(D, A, h, \Gamma).$$

Let p be odd prime and $\boldsymbol{Z}_p$ the cyclic group of order p. Then $\boldsymbol{Z}_p^2 = \boldsymbol{Z}_p \times \boldsymbol{Z}_p$ has the IEP. Since $\boldsymbol{Z}_p$ is the 2-dimensional vector space over $\boldsymbol{Z}_p$, the general linear group $GL_2(\boldsymbol{Z}_p)$ is the automorphism group of $\boldsymbol{Z}_p^2$. Furthermore, $GL_2(\boldsymbol{Z}_p)$ acts transively on $\boldsymbol{Z}_p^2 \setminus \{\mathbf{0}\}$. Set $\boldsymbol{e} = \boldsymbol{e}_1 = {}^t(1\ 0) \in \boldsymbol{Z}_p^2$. By Theorem 2, we have $\operatorname{Iso}(D, A, g, \Gamma) = \operatorname{Iso}(D, A, \boldsymbol{e}, \Gamma)$ for any element $g \in \boldsymbol{Z}_p^2 \setminus \{\mathbf{0}\}$. Thus we consider the number of Γ-isomorphism classes of $\boldsymbol{e}$-cyclic $\boldsymbol{Z}_p^2$-covers of D.

Let $\Gamma \leq \operatorname{Aut} D$ and $\Pi = GL_2(\boldsymbol{Z}_p)$. Furthermore, set

$$\Pi_{\boldsymbol{e}} = \{\sigma \in \Pi \mid \sigma(\boldsymbol{e}) = \boldsymbol{e}\}.$$

An action of $\Pi_{\boldsymbol{e}} \times \Gamma$ on $H^1(G; \boldsymbol{Z}_p^2)$ is defined as follows:

$$(\boldsymbol{A}, \gamma)[\alpha] = [\boldsymbol{A}\alpha^\gamma] = \{\boldsymbol{A}\alpha^\gamma + \delta s \mid s \in C^0(G; \boldsymbol{Z}_p^2)\},$$

where $\boldsymbol{A} \in \Pi_{\boldsymbol{e}}$, $\gamma \in \Gamma$ and $\alpha \in C^1(G; \boldsymbol{Z}_p^2)$. By Theorem 1, the number of Γ-isomorphism classes of $\boldsymbol{e}$-cyclic $\boldsymbol{Z}_p^2$-covers of D is equal to that of $\Pi_{\boldsymbol{e}} \times \Gamma$-orbits on $H^1(G; \boldsymbol{Z}_p^2)$.

Let D be a connected symmetric digraph, G the underlying graph of D, $\Gamma \leq \operatorname{Aut} D$, $\gamma \in \Gamma$ and $\lambda \in \boldsymbol{Z}_p^*$. A $\langle\gamma\rangle$-orbit σ of length k on $E(G)$ is called *diagonal* if $\sigma = \langle\gamma\rangle\{x, \gamma^k(x)\}$ for some $x \in V(G)$. The vertex orbit $\langle\gamma\rangle x$ and the arc orbit $\langle\gamma\rangle(x, \gamma^k(x))$ are also called *diagonal*. A diagonal arc orbit of length $2k$ (the corresponding edge orbit of length k and the corresponding vertex orbit of length $2k$) is called *type-1* if $\lambda^k = -1$ (or $m = 2k$), and *type-2* otherwise, where m is the order $\operatorname{ord}(\lambda)$ of λ.

For $\gamma \in \Gamma$, let $G(\gamma)$ be a simple graph whose vertices are the $\langle\gamma\rangle$-orbits on $V(G)$, with two vertices adjacent in $G(\gamma)$ if and only if some two of their representatives are adjacent in G. The kth *p-level* of $G(\gamma)$ is the induced subgraph of $G(\gamma)$ on the vertices ω such that $\theta(|\omega|) = p^k$, where $\theta(i)$ is the largest power of p dividing i. A *p-level component* H of $G(\gamma)$ is a connected component of some p-level of $G(\gamma)$, where H is considered as a subset of $V(G(\gamma))$. A p-level component H is called *minimal* if there exists no vertex σ of H which is adjacent in $G(\gamma)$ to a vertex ω such that $\theta(|\sigma|) > \theta(|\omega|)$ (see [12]).

Let $k \in \boldsymbol{N}$. Then a $\langle\gamma\rangle$-orbit σ on $V(G), E(G)$ or $A(D)$ is called *k-divisible* if $|\sigma| \equiv 0 \pmod{k}$. A vertex orbit σ is called *edge-induced* if there exists a orbit $\langle\gamma\rangle\{x, y\}$ on $E(G)$ with $x, y \in \sigma$. A k-divisible $\langle\gamma\rangle$-orbit σ on $V(G)$ ia called *strongly k-divisible* if σ is edge-induced and satisfies the following condition:

$$\text{If } \Omega = \langle\gamma\rangle(x, y) \text{ is any } \langle\gamma\rangle\text{-orbit on } A(D), \text{ and}$$
$$y = \gamma^j(x),\ x, y \in \sigma, \text{ then } j \equiv 0 \pmod{k}.$$

For $k \geq 1$, let H be a k th p-level component of $G(\gamma)$. Then H is called *p-favorable* if H is minimal and there exists a $\sigma \in H$ which is not strongly p-divisible. Furthermore, H is called *p-defective* if H is minimal and each vertex σ of H is strongly p-divisible.

Let $\lambda \in \boldsymbol{Z}_p^*$ and $\operatorname{ord}(\lambda) = m$. Then, let $G_\lambda(\gamma)$ be the subgraph of $G(\gamma)$ induced by the set of m-divisible $\langle\gamma\rangle$-orbits on $V(G)$. The kth *p-level* and *p-level components* of $G_\lambda(\gamma)$ are defined similarly to the case of $G(\gamma)$. A p-level component H of $G_\lambda(\gamma)$ is called *defective* if each vertex σ of H is strongly m-divisible, not type-1 diagonal, and H is minimal. Note that, if $\sigma = \langle\gamma\rangle x$ is strongly m-divisible, $|\sigma| = t$ and there exists a diagonal $\langle\gamma\rangle$-orbit $\Omega = \langle\gamma\rangle(x, \gamma^{t/2}(x))$ on $A(D)$, then Ω is type-2.

THEOREM 3. *Let D be a connected symmetric digraph, G its underlying graph, p odd prime, $g \in \boldsymbol{Z}_p^2 \setminus \{0\}$, and $\Gamma \leq \operatorname{Aut} G$. For $\gamma \in \Gamma$, let $\epsilon(\gamma)$, $\rho(\gamma)$ and $\epsilon_1(\gamma)$ be the number of $\langle\gamma\rangle$-orbits, diagonal $\langle\gamma\rangle$-orbits and, not diagonal and p-divisible $\langle\gamma\rangle$-orbits on $E(G)$, respectively. Let $\nu(\gamma)$ and $\nu_0(\gamma)$ be the number of $\langle\gamma\rangle$-orbits and not p-divisible $\langle\gamma\rangle$-orbits on $V(G)$, respectively. Moreover, let $c(\gamma)$, $\xi(\gamma)$, $d(\gamma)$ and $d_1(\gamma)$ be the number of p-level components, minimal p-level components, p-defective p-level components and not minimal p-level components with p-divisible orbits in $G(\gamma)$, respectively. For $\gamma \in \Gamma$ and $\lambda \in \boldsymbol{Z}_p^*$, let $\nu_0(\gamma, \lambda)$, $\mu(\gamma, \lambda)$ and $d(\gamma, \lambda)$ be the number of not m-divisible $\langle\gamma\rangle$-orbits on $V(G)$, type-2 diagonal $\langle\gamma\rangle$-orbits on $E(G)$ and defective p-level components in $G_\lambda(\gamma)$, respectively, where $m = \operatorname{ord}(\lambda)$. Furthermore, let $\kappa(\gamma, \lambda)$ be the number of not m-divisible $\langle\gamma\rangle$-orbits on $E(G)$ which are not diagonal. Then the number of Γ-isomorphism classes of g-cyclic $\boldsymbol{Z}_p^2$-covers of D is*

$$\operatorname{Iso}(D, \boldsymbol{Z}_p^2, g, \Gamma) = \frac{1}{p(p-1)|\Gamma|} \sum_{\gamma\in\Gamma} \{p^{2(\epsilon(\gamma)-\nu(\gamma)+\xi(\gamma)-\rho(\gamma))}$$
$$+ (p-1)p^{\epsilon(\gamma)-2\nu(\gamma)+\nu_0(\gamma)-\rho(\gamma)+\epsilon_1(\gamma)+c(\gamma)-d_1(\gamma)+d(\gamma)}$$

$$+p \sum_{\lambda \in \boldsymbol{Z}_p^* \setminus \{1\}} p^{2(\epsilon(\gamma)-\nu(\gamma))+\xi(\gamma)-\rho(\gamma)+\nu_0(\gamma,\lambda)-\kappa(\gamma,\lambda)-\mu(\gamma,\lambda)+d(\gamma,\lambda)}\}.$$

Proof. By the preceding remark and Burnside' Lemma, the number of Γ-isomorphism classes of e-cyclic $\boldsymbol{Z}_p^2$-covers of D is

$$\frac{1}{|\Pi_{\boldsymbol{e}}| \cdot |\Gamma|} \sum_{(\boldsymbol{A},\gamma) \in \Pi_{\boldsymbol{e}} \times \Gamma} \left| H^1(G; \boldsymbol{Z}_p^2)^{(\boldsymbol{A},\gamma)} \right|,$$

where $U^{(\boldsymbol{A},\gamma)}$ is the set consisting of the elements of U fixed by $(\boldsymbol{A}, \gamma)$.

Now, we have

$$\Pi_{\boldsymbol{e}} = \left\{ \begin{bmatrix} 1 & \mu \\ 0 & \lambda \end{bmatrix} \mid \lambda = 1, 2, \cdots, p-1;\ \mu = 0, 1, \cdots, p-1 \right\}.$$

Then there exist $p+1$ conjugacy classes of $\Pi_{\boldsymbol{e}}$:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & p-1 \\ 0 & 1 \end{bmatrix} \right\},$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix}, \cdots, \begin{bmatrix} 1 & p-1 \\ 0 & \lambda \end{bmatrix} \right\} (\lambda = 2, \cdots, p-1).$$

Let $\boldsymbol{A}, \boldsymbol{B} \in \Pi_{\boldsymbol{e}}$ be conjugate. Then there exists an element $\boldsymbol{C} \in \Pi_{\boldsymbol{e}}$ such that $\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^{-1} = \boldsymbol{B}$. Thus $[\alpha] \in H^1(G; \boldsymbol{F}_p^2)^{(\boldsymbol{A},\gamma)}$ if and only if $\boldsymbol{A}\alpha^\gamma = \alpha + \delta s$ for some $s \in C^0(G; \boldsymbol{Z}_p^2)$. But $\boldsymbol{A}\alpha^\gamma = \alpha + \delta s$ if and only if $\boldsymbol{B}(\boldsymbol{C}\alpha)^\gamma = \boldsymbol{C}\alpha + \delta(\boldsymbol{C}s)$, i.e., $[\boldsymbol{C}\alpha] \in H^1(G; \boldsymbol{Z}_p^2)^{(\boldsymbol{B},\gamma)}$. By the fact that a mapping $[\alpha] \longmapsto [\boldsymbol{C}\alpha]$ is bijective, we have

$$\left| H^1(G; \boldsymbol{Z}_p^2)^{(\boldsymbol{A},\gamma)} \right| = \left| H^1(G; \boldsymbol{Z}_p^2)^{(\boldsymbol{B},\gamma)} \right|.$$

Therefore the number of Γ-isomorphism classes of e-cyclic $\boldsymbol{Z}_p^2$-covers of D is

$$\frac{1}{|\Pi_{\boldsymbol{e}}| \cdot |\Gamma|} \sum_{\gamma \in \Gamma} \{ \left| H^1(G; \boldsymbol{Z}_p^2)^{(\boldsymbol{I},\gamma)} \right| + (p-1) \left| H^1(G; \boldsymbol{Z}_p^2)^{(\boldsymbol{A}_1,\gamma)} \right|$$
$$+ p \sum_{\lambda=2}^{p-1} \left| H^1(G; \boldsymbol{Z}_p^2)^{(\boldsymbol{A}_\lambda,\gamma)} \right| \},$$

where

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{A}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \boldsymbol{A}_\lambda = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}.$$

Let $(\boldsymbol{A}, \gamma) \in \Pi_{\boldsymbol{e}} \times \Gamma$.

Case 1: $\boldsymbol{A} = \boldsymbol{I}$.

Then $[\alpha] \in H^1(G;\boldsymbol{Z}_p^2)^{(\boldsymbol{I},\gamma)}$ if and only if $\alpha^\gamma = \boldsymbol{I}\alpha^\gamma = \alpha + \delta s$ for some $s \in C^0(G;\boldsymbol{Z}_p^2)$.

Now, let $\alpha = a\boldsymbol{e}_1 + b\boldsymbol{e}_2,\ a,b \in C^1(G;\boldsymbol{Z}_p)$, where $\boldsymbol{e}_1 = {}^t(1\ 0)$ and $\boldsymbol{e}_2 = {}^t(0\ 1)$. Furthermore, let $s = w\boldsymbol{e}_1 + z\boldsymbol{e}_2,\ w,z \in C^0(G;\boldsymbol{Z}_p)$. Then $\alpha^\gamma = \alpha + \delta s$ if and only if $a^\gamma = a + \delta w$ and $b^\gamma = b + \delta z$, i.e., $[a]^\gamma = [a]$ and $[b]^\gamma = [b]$. Note that $[a],[b] \in H^1(G;\boldsymbol{Z}_p)^\gamma$. Since $[a\boldsymbol{e}_1 + b\boldsymbol{e}_2] = [a]\boldsymbol{e}_1 + [b]\boldsymbol{e}_2$, we have

$$\left|H^1(G;\boldsymbol{z}_p^2)^{(\boldsymbol{I},\gamma)}\right| = \left|H^1(G;\boldsymbol{Z}_p)^\gamma\right|^2.$$

By Theorem 5 of [8], it follows that

$$\left|H^1(G;\boldsymbol{Z}_p^2)^{(\boldsymbol{I},\gamma)}\right| = (p^{\epsilon(\gamma)-\nu(\gamma)+\xi(\gamma)-\rho(\gamma)})^2 = p^{2(\epsilon(\gamma)-\nu(\gamma)+\xi(\gamma)-\rho(\gamma))}.$$

Case 2: $\boldsymbol{A} = \boldsymbol{A}_\lambda$.

Then $[\alpha] \in H^1(G;\boldsymbol{Z}_p^2)^{(\boldsymbol{A}_\lambda,\gamma)}$ if and only if $\boldsymbol{A}_\lambda\alpha^\gamma = \alpha + \delta s$ for some $s \in C^0(G;\boldsymbol{Z}_p^2)$. Let $\alpha = a\boldsymbol{e}_1 + b\boldsymbol{e}_2,\ a,b \in C^1(G;\boldsymbol{Z}_p)$ and $s = w\boldsymbol{e}_1 + z\boldsymbol{e}_2,\ w,z \in C^0(G;\boldsymbol{Z}_p)$. Then $\boldsymbol{A}_\lambda\alpha^\gamma = \alpha + \delta s$ if and only if $a^\gamma\boldsymbol{e}_1 + \lambda b^\gamma\boldsymbol{e}_2 = (a+\delta w)\boldsymbol{e}_1 + (b+\delta z)\boldsymbol{e}_2$, i.e., $a^\gamma = a + \delta w$ and $\lambda b^\gamma = b + \delta z$. Thus $(\boldsymbol{A}_\lambda,\gamma)[\alpha] = [\alpha]$ if and only if $[a]^\gamma = [a]$ and $\lambda[b]^\gamma = [b]$. Therefore, we have

$$\left|H^1(G;\boldsymbol{Z}_p^2)^{(\boldsymbol{A}_\lambda,\gamma)}\right| = \left|H^1(G;\boldsymbol{Z}_p)^\gamma\right| \cdot \left|H^1(G;\boldsymbol{Z}_p)^{(\gamma,\lambda)}\right|.$$

By Theorem 5 of [8] and Theorem 3.3 of [18], it follows that

$$\left|H^1(G;\boldsymbol{Z}_p^2)^{(\boldsymbol{A}_\lambda,\gamma)}\right| = p^{2\epsilon(\gamma)-2\nu(\gamma)+\xi(\gamma)-\rho(\gamma)+\nu_0(\gamma,\lambda)-\kappa(\gamma,\lambda)-\mu(\gamma,\lambda)+d(\gamma,\lambda)}.$$

Case 3: $\boldsymbol{A} = \boldsymbol{A}_1$.

Then we have

$$\left|H^1(G;\boldsymbol{Z}_p^2)^{(\boldsymbol{A}_1,\gamma)}\right| = p^{\epsilon(\gamma)-2\nu(\gamma)+\nu_0(\gamma)-\rho(\gamma)+\epsilon_1(\gamma)+c(\gamma)-d_1(\gamma)+d(\gamma)}.$$

The detail is developed in Section 3.

By cases 1,2 and 3, the result follows. ■

COROLLARY 1. (15, Corollary 4.6) *Let D be a connected symmetric digraph and p odd prime. Then the number of I-isomorphism classes of g-cyclic $\boldsymbol{Z}_p^2$-covers of D is*

$$\operatorname{Iso}(D,\boldsymbol{Z}_p^2,g,I) = p^{B(D)} + \frac{p^{B(D)-1}(p^{B(D)}-1)}{p-1}.$$

where $B(D) = \frac{1}{2}|A(D)| - |V(G)| + 1$ is the Betti-number of D.

Proof. Since $I = \{1\}$, we have $\epsilon(1) = \kappa(1,\lambda) = |E(G)|$, $\rho(1) = \epsilon_1(1) = \mu(1,\lambda) = 0$, $\nu(1) = \nu_0(1) = \nu_0(1,\lambda) = |V(G)|$, $c(1) = \xi(1) = 1$ and $d_1(1) = d(1) = d(1,\lambda) = 0$. ■

3. The elements of $H^1(G; \boldsymbol{Z}_p^2)$ fixed by $(\boldsymbol{A}_1, \gamma)$

Let D be a connected symmetric digraph, G its underlying graph and $\Gamma \leq \operatorname{Aut} G$. We present the number of elements on $H^1(G; \boldsymbol{Z}_p^2)$ fixed by $(\boldsymbol{A}_1, \gamma)$ for each $\gamma \in \Gamma$. The argument is an analogue of Hofmeister's method [5].

Let

$$\boldsymbol{A} = \boldsymbol{A}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Note that the order ord$(\boldsymbol{A})$ of $\boldsymbol{A}$ is p and

$$\boldsymbol{A}^j = \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}$$

for any j. Set $C^0 = C^0(G; \boldsymbol{Z}_p^2)$, $C^1 = C^1(G; \boldsymbol{Z}_p^2)$, and $H^1 = H^1(G; \boldsymbol{Z}_p^2)$. We consider the following exact sequence:

$$0 \longrightarrow \operatorname{Ker} \delta \xrightarrow{\delta^0} C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta^1} H^1 \longrightarrow 0,$$

where δ^0 is the canonical monomorphism and δ^1 is the canonical epimorphism. For $\gamma \in \Gamma$, two endomorphisms $\mu_\gamma : C^1 \to C^1$ and $\nu_\gamma : H^1 \to H^1$ are defined as follows: $\mu_\gamma(\alpha) = \boldsymbol{A}\alpha^\gamma - \alpha$ and $\nu_\gamma([\alpha]) = [\boldsymbol{A}\alpha^\gamma - \alpha]$, where $\alpha \in C^1$. Then, note that $\nu_\gamma \delta^1 = \delta \mu_\gamma$ and $\operatorname{Ker} \nu_\gamma = (H^1)^{(\boldsymbol{A}, \gamma)}$.

Now, let $C_\gamma^0 = \delta^{-1}(\operatorname{Im} \mu_\gamma)$ and $C_\gamma^1 = \mu^{-1}(\operatorname{Im} \delta)$.

Let (x, y) be any arc of $A(D)$, $\gamma \in \Gamma$ and $t = |\langle\gamma\rangle(x, y)|$ the length of the arc $\langle\gamma\rangle$-orbit containing (x, y). Furthermore, let $s = {}^t(u, v) = u\boldsymbol{e}_1 + v\boldsymbol{e}_2 \in C^0$, $u, v \in C^0(G; \boldsymbol{Z}_p)$.

LEMMA 1. *Let $\gamma \in \Gamma$, $s \in C^0$. Then $s \in C_\gamma^0$ if and only if*

$$v(x) + v^\gamma(x) + \cdots + v^{\gamma^{t-1}}(x) = v(y) + v^\gamma(y) + \cdots + v^{\gamma^{t-1}}(y) \qquad \cdots (*)_1$$

for each $(x, y) \in A(D)$. Specially, if $p \mid t$, then

$$\sum_{j=0}^{t-1} \boldsymbol{A}^j s^{\gamma^j}(x) = \sum_{j=0}^{t-1} \boldsymbol{A}^j s^{\gamma^j}(y) \qquad \cdots (*)_2.$$

Proof. Set $t = |\langle\gamma\rangle(x, y)|$.

Suppose that $s \in C_\gamma^0$. Then there exists $\alpha \in C^1$ such that $\boldsymbol{A}\alpha^\gamma - \alpha = \delta s$. Thus

$$\boldsymbol{A}^i \alpha^{\gamma^i} - \alpha = \delta(\boldsymbol{A}^{i-1}\alpha^{\gamma^{i-1}} + \cdots + \boldsymbol{A}\alpha^\gamma + s), \ i \geq 1.$$

Let $(x,y) \in A(D)$. Then we have

$$\sum_{j=0}^{t-1} \boldsymbol{A}^j s^{\gamma^j}(x) - \sum_{j=0}^{t-1} \boldsymbol{A}^j s^{\gamma^j}(y) = \boldsymbol{A}^t \alpha^{\gamma^t}(x,y) - \alpha(x,y) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \alpha(x,y).$$

For the $(2,1)$-array of the above equation, we have

$$v(x) + v^{\gamma}(x) + \cdots + v^{\gamma^{t-1}}(x) - \{v(y) + v^{\gamma}(y) + \cdots + v^{\gamma^{t-1}}(y)\} = 0.$$

Specially, if $p \mid t$, then we have

$$\sum_{j=0}^{t-1} \boldsymbol{A}^j s^{\gamma^j}(x) - \sum_{j=0}^{t-1} \boldsymbol{A}^j s^{\gamma^j}(y) = 0.$$

Conversely, assume that $s = {}^t(u,v)$ satisfies $(*)_1$ and $(*)_2$ for each $(x,y) \in A(D)$. Let Ω be any $\langle\gamma\rangle$-orbit on $A(D)$, $|\Omega| = t$ and $(x,y) \in \Omega$. If Ω is not diagonal, then let $\alpha(x,y) = 0 \cdot e_1 + b(x,y)e_2$, $b \in C^1(G;\boldsymbol{Z}_p)$ be defined as follows:

$$\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ b(x,y) \end{bmatrix} = \sum_{j=0}^{t-1} \boldsymbol{A}^j s^{\gamma^j}(x) - \sum_{j=0}^{t-1} \boldsymbol{A}^j s^{\gamma^j}(y).$$

If $p \mid t$, then we may set $b(x,y) = 0$. Furthermore, if Ω is diagonal, then let

$$\alpha(x,y) = -(\boldsymbol{A}^l + \boldsymbol{I})^{-1} \sum_{j=0}^{l-1} (\boldsymbol{A}^j s^{\gamma^j}(x) - \boldsymbol{A}^j s^{\gamma^j}(y)),$$

where $l = \frac{t}{2}$.

Now, let

$$\boldsymbol{A}^i \alpha^{\gamma^i}(x,y) = \alpha(x,y) + \sum_{j=0}^{i-1} (\boldsymbol{A}^j s^{\gamma^j}(x) - \boldsymbol{A}^j s^{\gamma^j}(y)), \ i \geq 1 \qquad \cdots (1).$$

If Ω is not diagonal and $t \not\equiv 0 \ (mod\ p)$, then we have

$$\begin{aligned} &\boldsymbol{A}^{pt} \alpha^{\gamma^{pt}}(x,y) \\ &= \alpha(x,y) + \sum_{j=0}^{pt-1} (\boldsymbol{A}^j s^{\gamma^j}(x) - \boldsymbol{A}^j s^{\gamma^j}(y)) \\ &= \alpha(x,y) + (\boldsymbol{I} + \boldsymbol{A}^t + \cdots + \boldsymbol{A}^{(p-1)t}) \sum_{j=0}^{t-1} (\boldsymbol{A}^j s^{\gamma^j}(x) - \boldsymbol{A}^j s^{\gamma^j}(y)) = \alpha(x,y). \end{aligned}$$

Furthermore, let Ω is not diagonal and $p \mid t$. Then we have

$$\boldsymbol{A}^t \alpha^{\gamma^t}(x,y) = \alpha(x,y) + \sum_{j=0}^{t-1} (\boldsymbol{A}^j s^{\gamma^j}(x) - \boldsymbol{A}^j s^{\gamma^j}(y)) = \alpha(x,y).$$

Next, let Ω be diagonal. Then we have

$$\begin{aligned}\sum_{j=0}^{t-1}(\boldsymbol{A}^j s^{\gamma^j}(x) - \boldsymbol{A}^j s^{\gamma^j}(y)) &= (\boldsymbol{I} - \boldsymbol{A}^l)\sum_{j=0}^{l-1}(\boldsymbol{A}^j s^{\gamma^j}(x) - \boldsymbol{A}^j s^{\gamma^j}(y)) \\ &= -(\boldsymbol{I} - \boldsymbol{A}^l)(\boldsymbol{I} + \boldsymbol{A}^l)\alpha(x, y) \\ &= \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\alpha(x, y),\end{aligned}$$

where $t = 2l$. If $p \mid t$, then $\boldsymbol{A}^t \alpha^{\gamma^t}(x, y) = \alpha(x, y)$ similarly to the case that Ω is not diagonal. Otherwise,

$$\begin{aligned}&\boldsymbol{A}^{2pl}\alpha^{\gamma^{2pl}}(x, y) \\ &= \alpha(x, y) + (\boldsymbol{I} - \boldsymbol{A}^l + \boldsymbol{A}^{2l} + \cdots - \boldsymbol{A}^{2(p-1)l+l})\sum_{j=0}^{l-1}(\boldsymbol{A}^j s^{\gamma^j}(x) - \boldsymbol{A}^j s^{\gamma^j}(y)) \\ &= \alpha(x, y).\end{aligned}$$

Therefore it follows that (1) is well-defined.

By (1), we have

$$\boldsymbol{A}^r \alpha^{\gamma^{r+i}}(x, y) = \alpha^{\gamma^i}(x, y) + \sum_{j=0}^{r-1}(\boldsymbol{A}^j s^{\gamma^j}(\gamma^i(x)) - \boldsymbol{A}^j s^{\gamma^j}(\gamma^i(y))), \ r, i \geq 1.$$

If Ω is not diagonal, then we define $\alpha(v, u) = -\alpha(u, v), (u, v) \in \Omega$.

In the case that Ω is diagonal, we have

$$\begin{aligned}\alpha(y, x) &= \alpha^{\gamma^l}(x, y) \\ &= \boldsymbol{A}^{-l}(\alpha(x, y) + \sum_{j=0}^{l-1}\delta \boldsymbol{A}^j s^{\gamma^j}(x, y)) \\ &= \boldsymbol{A}^{-l}\alpha(x, y) - \boldsymbol{A}^{-l}(\boldsymbol{A}^l + \boldsymbol{I})\alpha(x, y) \\ &= -\alpha(x, y).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\boldsymbol{A}^{i+l}\alpha^{\gamma^i}(y, x) &= \boldsymbol{A}^l\{\boldsymbol{A}^i \alpha^{\gamma^{l+i}}(x, y)\} \\ &= \boldsymbol{A}^l\{\alpha^{\gamma^l}(x, y) + \sum_{j=0}^{i-1}(\boldsymbol{A}^j s^{\gamma^j}(\gamma^l(x)) - \boldsymbol{A}^j s^{\gamma^j}(\gamma^l(y)))\} \\ &= -\boldsymbol{A}^l\{\alpha(x, y) + \sum_{j=0}^{i-1}(\boldsymbol{A}^j s^{\gamma^j}(x) - \boldsymbol{A}^j s^{\gamma^j}(y))\} \\ &= -\boldsymbol{A}^{l+i}\alpha^{\gamma^i}(x, y),\end{aligned}$$

i.e.,

$$\alpha^{\gamma^i}(y,x) = -\alpha^{\gamma^i}(x,y), i \geq 1.$$

Therefore we obtain an $\alpha \in C^1$ such that $\boldsymbol{A}\alpha^\gamma - \alpha = \delta s$, i.e., $s \in C^0_\gamma$. ■

LEMMA 2. *For* $\gamma \in \Gamma$,

$$|C^0_\gamma| = p^{2n-2\nu(\gamma)+\nu_0(\gamma)+c(\gamma)-d_1(\gamma)+d(\gamma)}, \ n = |V(G)|.$$

Proof. We count the number of $s \in C^0_\gamma$ which satisfy both $(*)_1$ and $(*)_2$ for each $(x,y) \in A(D)$.

Let $(x,y) \in A(D)$, $\Omega = \langle\gamma\rangle(x,y)$ and $|\Omega| = t$.

Case 1: x, y are in the same $\langle\gamma\rangle$-orbit σ on $V(G)$.

Then Ω is not diagonal and $|\sigma| = t$. Let $y = \gamma^j(x)$ $(1 \leq j < t)$. By Lemma 1, we have

$$v(x) + v^\gamma(x) + \cdots + v^{\gamma^{t-1}}(x) = v(y) + v^\gamma(y) + \cdots + v^{\gamma^{t-1}}(y).$$

This is an identical equation.

Case 1.1: σ is p-divisible.

Then $\boldsymbol{A}^t = \boldsymbol{I}$. By Lemma 1, we have

$$s(x) + \boldsymbol{A}s^\gamma(x) + \cdots + \boldsymbol{A}^{t-1}s^{\gamma^{t-1}}(x) = s(y) + \boldsymbol{A}s^\gamma(y) + \cdots + \boldsymbol{A}^{t-1}s^{\gamma^{t-1}}(y),$$

i.e.,

$$(\boldsymbol{A}^{t-j} - \boldsymbol{I})(s(x) + \boldsymbol{A}s^\gamma(x) + \cdots + \boldsymbol{A}^{t-1}s^{\gamma^{t-1}}(x)) = 0.$$

$$\begin{bmatrix} 0 & -j \\ 0 & 0 \end{bmatrix}(s(x) + \boldsymbol{A}s^\gamma(x) + \cdots + \boldsymbol{A}^{t-1}s^{\gamma^{t-1}}(x)) = \boldsymbol{0}.$$

If σ is strongly p-divisible or $p \mid j$, then there are p^{2t} possible choices for the $s(w)$ with $w \in \sigma$. If σ is not strongly p-divisible, then we have

$$-j(v(x) + v^\gamma(x) + \cdots + v^{\gamma^{t-1}}(x)) = 0.$$

Since $u(w)$ is any, there are p^{2t-1} possible choices for the $s(w)$ with $w \in \sigma$.

Case 1.2: σ is not p-divisible.

Since $u(w)$ and $v(w)$ are any, there p^{2t} possible choices for the $s(w)$ with $w \in \sigma$.

Case 2: x and y are in different vetex $\langle\gamma\rangle$-orbits σ_1, σ_2 of length t_1, t_2, respectively.

Then t is the least common multiple $[t_1, t_2]$ of t_1 and t_2. Let $t_i = p^{a_i}q_i$, $(p, q_i) = 1$ $(i = 1,2)$, and $a = max\{a_1, a_2\}$. Then $t = p^a[q_1, q_2]$. Let $t'_1 = [q_1, q_2]/q_i (i = 1,2)$. By Lemma 1, we have

$$p^{a-a_1}t'_1(v(x) + v^\gamma(x) + \cdots + v^{\gamma^{t_1-1}}(x)) = p^{a-a_2}t'_2(v(y) + v^\gamma(y) + \cdots + v^{\gamma^{t_2-1}}(y)).$$

If $p \mid t$, then we have

$$(\boldsymbol{I}+\boldsymbol{A}^{t_1}+\cdots+\boldsymbol{A}^{t_1(p^{p-a_1}t_1'-1)})(s(x)+\boldsymbol{A}s^{\gamma}(x)+\cdots+\boldsymbol{A}^{t_1-1}s^{\gamma^{t_1-1}}(x))$$
$$=(\boldsymbol{I}+\boldsymbol{A}^{t_2}+\cdots+\boldsymbol{A}^{t_2(p^{a-a_2}t_2'-1)})(s(y)+\boldsymbol{A}s^{\gamma}(y)+\cdots+\boldsymbol{A}^{t_2-1}s^{\gamma^{t_2-1}}(y)).$$

Case 2.1: σ_1 is p-divisible and σ_2 is not p-divisible.

Since $\boldsymbol{A}^{t_1}=\boldsymbol{I}$ and $\boldsymbol{A}^{t_2}\neq\boldsymbol{I}$, we have

$$\boldsymbol{I}+\boldsymbol{A}^{t_2}+\cdots+\boldsymbol{A}^{t_2(p^{a-a_2}t_2'-1)}=\boldsymbol{0}.$$

Thus

$$p^{a-a_1}t_1'(s(x)+\boldsymbol{A}s^{\gamma}(x)+\cdots+\boldsymbol{A}^{t_1-1}s^{\gamma^{t_1-1}}(x))=\boldsymbol{0}.$$

Since $a=a_1, p^{a-a_1}t_1'=t_1'\neq 0$, and so

$$s(x)+\boldsymbol{A}s^{\gamma}(x)+\cdots+\boldsymbol{A}^{t_1-1}s^{\gamma^{t_1-1}}(x)=\boldsymbol{0}.$$

Case 2.2: Both σ_1 and σ_2 are p-divisible.

Then $\boldsymbol{A}^{t_1}=\boldsymbol{A}^{t_2}=\boldsymbol{I}$. If $a_1=a_2$, then we have

$$t_1'(s(x)+\boldsymbol{A}s^{\gamma}(x)+\cdots+\boldsymbol{A}^{t_1-1}s^{\gamma^{t_1-1}}(x))=t_2'(s(y)+\boldsymbol{A}s^{\gamma}(y)+\cdots+\boldsymbol{A}^{t_2-1}s^{\gamma^{t_2-1}}(y)).$$

If $a_1>a_2$, then we have

$$s(x)+\boldsymbol{A}s^{\gamma}(x)+\cdots+\boldsymbol{A}^{t_1-1}s^{\gamma^{t_1-1}}(x)=\boldsymbol{0}.$$

Case 2.3: Both σ_1 and σ_2 are not p-divisible.

Since $t\not\equiv 0 \pmod p$, we have

$$t_1'(v(x)+v^{\gamma}(x)+\cdots+v^{\gamma^{t_1-1}}(x))=t_2'(v(y)+v^{\gamma}(y)+\cdots+v^{\gamma^{t_2-1}}(y)).$$

Let H be a 0 th p-level component of $G(\gamma)$. Then some vertex σ of H admits $p^{2|\sigma|}$ choices according to Cases 1.2 and 2.1, while any other vertex ω of H admits $p^{2|\omega|-1}$ choices by Case 2.3.

Let H be a k th p-level component of $G(\gamma)$ for $k\geq 1$. If H is not minimal, then any vertex σ of H admits $p^{2|\sigma|-2}$ choices for the $s(w)$ with $w\in\sigma$ by cases 1.1, 2.1 and 2.2. If H is p-favorable, then some vertex σ of H admits $p^{2|\sigma|-1}$ choices according to Case 1.1, while any other vertex ω of H admits $p^{2|\omega|-2}$ choices by Case 2.2. If H is p-defective, then some vertex σ of H admits $p^{2|\sigma|}$ choices according to Case 1.1, while any other vertex ω of H admits $p^{2|\omega|-2}$ choices by Case 2.2.

Therefore it follows that

$$\begin{aligned}|C_{\gamma}^{0}|&=\prod_{H_1}\Big(\prod_{\sigma_1\in H_1}p^{2|\sigma_1|-1}\Big)p\cdot\prod_{H_2}\Big(\prod_{\sigma_2\in H_2}p^{2|\sigma_2|-2}\Big)\\&\quad\times\prod_{H_3}\Big(\prod_{\sigma_3\in H_3}p^{2|\sigma_3|-2}\Big)p\cdot\prod_{H_4}\Big(\prod_{\sigma_4\in H_4}p^{2|\sigma_4|-2}\Big)p^2\\&=p^{2n-2\nu(\gamma)+\nu_0(\gamma)+c(\gamma)-d_1(\gamma)+d(\gamma)},\end{aligned}$$

where H_1, H_2, H_3 and H_4 runs over all 0th p-level components, nonzero th not minimal p-level components, nonzero th p-favorable p-level components and nonzero th p-defective p-level components of $G(\gamma)$, respectively. ■

Each $\langle\gamma\rangle$-orbit Ω' on $E(G)$ corresponds to two $\langle\gamma\rangle$-orbits on $A(D)$ if Ω' is not diagonal, and one $\langle\gamma\rangle$-orbit on $A(D)$ otherwise.

Lemma 3. *For $\gamma \in \Gamma$,*

$$\left|\operatorname{Ker}\mu_\gamma\right| = p^{\epsilon(\gamma)-\rho(\gamma)+\epsilon_1(\gamma)}.$$

Proof. Let $\alpha \in \operatorname{Ker}\mu_\gamma$. Then we have $\alpha = \boldsymbol{A}\alpha^\gamma = \boldsymbol{A}^2\alpha^{\gamma^2} = \cdots$.

Let $\Omega = \langle\gamma\rangle(x,y)$ be any $\langle\gamma\rangle$-orbit on $A(D)$ and $|\Omega| = t$.

Case 1: x and y are in the same $\langle\gamma\rangle$-orbit σ on $V(G)$, and Ω is diagonal.

Let $t = 2k$. Then we have $\alpha^{\gamma^i}(x,y) = \boldsymbol{A}^{-i}\alpha(x,y)$ $(i \geq 1)$, $\boldsymbol{A}^k\alpha(x,y) = -\alpha(x,y)$ and $\boldsymbol{A}^{2k}\alpha(x,y) = \alpha(x,y)$. If Ω is p-divisible, then, since $\boldsymbol{A}^k = \boldsymbol{I}$, $\alpha(x,y) = 0$, i.e., $\alpha(u,v) = 0$ for each $(u,v) \in \Omega$. Otherwise we have

$$\boldsymbol{A}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix},$$

and so $\alpha(u,v) = 0$ for each $(u,v) \in \Omega$.

Case 2: x and y are not in the same $\langle\gamma\rangle$-orbit σ on $V(G)$, or Ω is not diagonal.

Then we have $\alpha^{\gamma^i}(x,y) = \boldsymbol{A}^{-i}\alpha(x,y)(i \geq 1)$ and $\boldsymbol{A}^t\alpha(x,y) = \alpha(x,y)$. If $p \mid t$, then there are p^2 possible choices for $\alpha(x,y)$. Otherwise there are p possible choices for $\alpha(x,y)$.

From the note preceding the lemma, it follows that

$$\left|\operatorname{Ker}\mu_\gamma\right| = p^{\epsilon(\gamma)-\rho(\gamma)+\epsilon_1(\gamma)}.$$

■

Theorem 4. *For $\gamma \in \Gamma$ and odd prime p,*

$$\left|H^1(G; \boldsymbol{Z}_p^2)^{(\boldsymbol{A}_1,\gamma)}\right| = p^{\epsilon(\gamma)-2\nu(\gamma)+\nu_0(\gamma)-\rho(\gamma)+\epsilon_1(\gamma)+c(\gamma)-d_1(\gamma)+d(\gamma)}.$$

Proof. Let $\gamma \in \Gamma$. Set $\epsilon = \epsilon(\gamma)$, $\nu = \nu(\gamma)$, $\nu_0 = \nu_0(\gamma), \cdots$.

Let $C_\gamma = \{(s,\alpha) \mid \delta s = \mu_\gamma(\alpha) = \boldsymbol{A}\alpha^\gamma - \alpha\}$, and consider the two epimorphisms $\gamma^0 : C_\gamma \to C_\gamma^0$ and $\gamma^1 : C_\gamma \to C_\gamma^1$. By Lemmas 2 and 3 and the fact that $\operatorname{Ker}\gamma^0 \cong \operatorname{Ker}\mu_\gamma$, we have

$$\left|C_\gamma\right| = \left|C_\gamma^0\right| \cdot \left|\operatorname{Ker}\gamma^0\right| = p^{2n-2\nu+\nu_0+c-d_1+d+\epsilon-\rho+\epsilon_1}.$$

Since $\operatorname{Ker}\gamma^1 \cong \operatorname{Ker}\delta$ and $|\operatorname{Ker}\delta| = p^2$, it follows that

$$\left|C_\gamma^1\right| = \left|C_\gamma\right| / \left|\operatorname{Ker}\gamma^1\right| = p^{2n-2\nu+\nu_0+c-d_1+d+\epsilon-\rho+\epsilon_1-2}.$$

Set $\tilde{\delta}^1 = \delta^1 \mid C_\gamma^1$. Since $\operatorname{Im}\delta \subset C_\gamma^1$, we have $\operatorname{Ker}\tilde{\delta}^1 = \operatorname{Ker}\delta^1 = \operatorname{Im}\delta$. Thus $|\operatorname{Ker}\tilde{\delta}^1| = p^{2n-2}$. Furthermore, since $\operatorname{Im}\tilde{\delta}^1 = \operatorname{Ker}\nu_\gamma$, it follows that

$$\left|\operatorname{Ker}\nu_\gamma\right| = \left|C_\gamma^1\right| / \left|\operatorname{Ker}\tilde{\delta}^1\right| = p^{\epsilon-2\nu+\nu_0-\rho+\epsilon_1+c-d_1+d}.$$

■

4. Cyclic $Z_p \times Z_p$-covers of special symmetric digraphs

At first, we consider cyclic Z_p^2-covers of a symmetric dipath and a symmetric dicycle. Let PD_n and CD_n be the symmetric dipath and the symmetric dicycle with n vertices, respectively. We enumerate the number of the isomorphism classes of g-cyclic Z_p^2-covers of PD_n and CD_n with respect to its full automorphism group, respectively.

THEOREM 5. *For $n \geq 3$ and odd prime p,*

$$\operatorname{Iso}(CD_n, Z_p^2, g, \operatorname{Aut} CD_n) = \frac{p+3}{2}$$

and

$$\operatorname{Iso}(PD_n, Z_p^2, g, \operatorname{Aut} PD_n) = 1.$$

Proof. Let $V(CD_n) = \{1, 2, \cdots, n\}$. Then the n-cycle is the underline graph of CD_n. Let $\Gamma = \operatorname{Aut} CD_n$. Then we have $\Gamma = \langle \alpha, \beta \rangle$, where $\alpha = (12 \cdots n)$ and

$$\beta = \begin{cases} (1\ n)(2\ n-1)\cdots(\frac{n-1}{2}\ \frac{n+3}{2})(\frac{n+1}{2}) & \text{if } n \text{ is odd ,} \\ (1\ n)(2\ n-1)\cdots(\frac{n}{2}\ \frac{n}{2}+1) & \text{otherwise .} \end{cases}$$

For each $\gamma \in \Gamma$ and $\lambda \in Z_p^*$, all parameters $\epsilon(\gamma), \cdots, \kappa(\gamma, \lambda)$ are constant in each conjugacy class of Γ. The conjugacy classes of Γ are given as follows:

$$\{\alpha^i\}\ (1 \leq i \leq n),\ \{\beta\alpha^i \mid 1 \leq i \leq n\} \quad \text{if } n \text{ is odd,}$$

$$\{\alpha^i\}\ (1 \leq i \leq n),\ \{\beta\alpha^{2j} \mid j = 1, 2, \cdots, \frac{n}{2}\},\ \{\beta\alpha^{2j+1} \mid j = 1, 2, \cdots, \frac{n}{2} - 1\}$$
$$\text{if } n \text{ is even.}$$

Let $1 \le i \le n$ and $d = (n,i)$ the greatest common divisor of n and i. Then we have $\operatorname{ord}(\alpha^i) = \frac{n}{d}$. The cardinality of each $\langle\alpha^i\rangle$-orbit on $V(C_n)$ or $E(C_n)$ is $\frac{n}{d}$. For each $\gamma \in \Gamma$, $\nu(\gamma), \rho(\gamma), \nu_0(\gamma)$ and $\epsilon(\gamma)$ are given as follows:

$$\epsilon(\gamma) = \begin{cases} d & \text{if } \gamma = \alpha^i \ , \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } \gamma = \beta\alpha^i, \\ \frac{n}{2}+1 & \text{if } n \text{ is even and } \gamma = \beta\alpha^{2j}, \\ \frac{n}{2} & \text{if } n \text{ is even and } \gamma = \beta\alpha^{2j+1}, \end{cases}$$

$$\nu(\gamma) = \begin{cases} d & \text{if } \gamma = \alpha^i, \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } \gamma = \beta\alpha^i, \\ \frac{n}{2} & \text{if } n \text{ is even and } \gamma = \beta\alpha^{2j}, \\ \frac{n}{2}+1 & \text{if } n \text{ is even and } \gamma = \beta\alpha^{2j+1}, \end{cases}$$

and

$$\rho(\gamma) = \begin{cases} 1 & \text{if } n \text{ is odd and } \gamma = \beta\alpha^i, \\ 2 & \text{if } n \text{ is even and } \gamma = \beta\alpha^{2j}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $p \ge 3$ and the cardinality of each $\langle\beta\alpha^i\rangle$-orbit on $V(C_n)$ or $E(C_n)$ is at most two,

$$\epsilon_1(\gamma) = \begin{cases} d & \text{if } \gamma = \alpha^i, p \mid n \text{ and } \theta(n) > \theta(i), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nu_0(\gamma) = \begin{cases} 0 & \text{if } \gamma = \alpha^i, p \mid n \text{ and } \theta(n) > \theta(i), \\ \nu(\gamma) & \text{otherwise.} \end{cases}$$

There exists only one p-level component in $C_n(\gamma)$, and so $c(\gamma) = \xi(\gamma) = 1$. Each $\langle\gamma\rangle$-orbit on $V(C_n)$ which is not diagonal is not strongly p-divisible. Thus we have $d(\gamma) = d_1(\gamma) = 0$.

Let $\lambda \in \boldsymbol{Z}_p^*$ and $m = ord(\lambda)$. Then $m = 2$ if $\lambda = -1$, and $m > 2$ otherwise. Thus we have

$$\nu_0(\gamma,\lambda) = \begin{cases} \nu(\gamma) & \text{if } \gamma = \alpha^i, m \mid \frac{n}{d} \text{ or } n \text{ is odd, } \gamma = \beta\alpha^i, \lambda \neq -1, \\ 1 & \text{if } n \text{ is odd and } \gamma = \beta\alpha^i, \lambda = -1, \\ 2 & \text{if } n \text{ is even and } \gamma = \beta\alpha^{2j+1}, \lambda = -1, \\ 0 & \text{otherwise.} \end{cases}$$

In the case of $\gamma = \beta\alpha^i$, each diagonal $\langle\gamma\rangle$-orbit on $A(CD_n)$ is type-1 if $\lambda = -1$, and type-2 otherwise. Thus

$$\mu(\gamma,\lambda) = \begin{cases} 1 & \text{if } n \text{ is odd and } \gamma = \beta\alpha^i, \lambda \neq -1, \\ 2 & \text{if } n \text{ is even and } \gamma = \beta\alpha^{2j}, \lambda \neq -1, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore,

$$\kappa(\gamma,\lambda)=\begin{cases}0 & \text{if } \gamma=\alpha^i, \frac{n}{d}\not\equiv 0 \pmod m \text{ or } n \text{ is odd, } \gamma=\beta\alpha^i, \lambda=-1,\\ \epsilon(\gamma)-1 & \text{if } n \text{ is odd and } \gamma=\beta\alpha^i, \lambda\neq -1,\\ \epsilon(\gamma)-2 & \text{if } n \text{ is even and } \gamma=\beta\alpha^{2j}, \lambda\neq -1,\\ \epsilon(\gamma) & \text{otherwise.}\end{cases}$$

Therefore the result follows.

Similarly to the above, the second formula is obtained. ■

Finally, we shall give an example.

Let KD_n be the complete symmetric digraph of order n, and Γ a subgroup of the symmetric group S_n on n elements. Set $V(KD_n)=\{1,2,\ldots,n\}$. For $\gamma\in\Gamma$, let $(t_1,t_2,\ldots,t_n)$ be the cycle type of γ. Then $\nu(\gamma), \rho(\gamma), \nu_0(\gamma)$ and $\epsilon(\gamma)$ are given as follows:

$$\nu(\gamma)=\sum_{k=1}^{n} t_k,\ \rho(\gamma)=\sum_{k:\ \text{even}} t_k,\ \nu_0(\gamma)=\sum_{k\not\equiv 0 \pmod p} t_k$$

and

$$\epsilon(\gamma)=\sum_{k=1}^{n}\{t_k\lfloor\frac{k}{2}\rfloor+k\tbinom{t_k}{2}\}+\sum_{k=2}^{n}\sum_{l=1}^{k-1}t_kt_l(k,l),$$

where (k,l) is the greatest common divisor of k and l.

Since the graph $K_n(\gamma)$ is complete, we have $\xi(\gamma)=1$. Moreover, since any m-divisible vertex orbit is not strongly m-divisible, we have

$$d(\gamma)=0 \text{ and } c(\gamma)-d_1(\gamma)=1.$$

Let $\lambda\in \boldsymbol{Z}_p^*$ and $m=\mathrm{ord}(\lambda)$. Then $\nu_0(\gamma,\lambda)$ and $\epsilon_1(\gamma)$ are given as follows:

$$\nu_0(\gamma,\lambda)=\sum_{k\not\equiv 0 \pmod m} t_k$$

and

$$\epsilon_1(\gamma)=\sum_{k:\ \text{odd}} t_k\lfloor\frac{k}{2}\rfloor+\sum_{k:\ \text{even}} t_k(\lfloor\frac{k}{2}\rfloor-1)+\sum_{k} k\tbinom{t_k}{2}++\sum_{k=2}^{n}\sum_{l=1}^{k-1}t_kt_l(k,l),$$

where k or l in five $\sum$ runs over multiples of p. Moreover, we have

$$d(\gamma,\lambda)=0,1.$$

Specially, $d(\gamma,\lambda)=1$ if and only if $\lambda=1$.

Let $\lambda = \zeta^i (1 \leq i \leq p-1)$, where ζ is a generator of $\boldsymbol{Z}_p^*$. Set $d = (i, p-1)$. Then we have $m = (p-1)/d$. Furthermore, we have

$$d(\gamma, \lambda) = \begin{cases} \rho(\gamma) - \sum_{p|k;\ k/m:\ \text{odd}} t_k & \text{if } i/d \text{ is odd and } m \text{ is even,} \\ \rho(\gamma) & \text{otherwise.} \end{cases}$$

It is clear that $\mathrm{Iso}\,(KD_1, \boldsymbol{Z}_p^2, g, \Gamma_1) = \mathrm{Iso}\,(KD_2, \boldsymbol{Z}_p^2, g, \Gamma_2) = 1$, where $\Gamma_i \leq S_i (i = 1, 2)$. In Table 1, we give some values of the number $\mathrm{Iso}\,(KD_n, \boldsymbol{Z}_p^2, g, S_n)$ of S_n-isomorphism classes of g-cyclic $\boldsymbol{Z}_p^2$-covers of $KD_n (n \geq 3)$.

$n \backslash p$	3	5	7	11	13	17
3	3	4	5	7	8	10
4	15	57	164	787	1477	4042
5	847	102785	2751778	237812932	1244683409	17850468927
6	812463	6623264490				
7	6811354482					

Table 1.

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