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ON THE FLIP OPERATIONS OF CLIQUE-ACYCLIC ORIENTATIONS OF GRAPHS

By

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Abstract. In this paper, we study the edge flip operations on some classes of clique-acyclic digraphs (that is, digraphs containing no directed triangle), especially we show that if G is a simple undirected graph whose every induced subgraph has a vertex v whose degree $\delta(v) \leqq 7$, any two clique-acyclic orientations π and π' have a sequence of clique-acyclic orientations $\pi = \pi_0, \pi_1, \ldots, \pi_t = \pi'$ such that we obtain π_i by reversing the orientation of one single edge of π_{i-1}, (then we call that π' is attainable from π). The latter bound "$\delta(G) \leqq 7$" is sharp. Actually, if G is a connected 8-regular graph, then there are exactly five examples of G each of which has a clique-acyclic orientation such that, if we flip any edge of it, the resulting new orientation has a directed triangle. Last, we show that, except for the above five examples of G, any two clique-acyclic orientations of a connected graph G whose maximum degree $\Delta(G) \leqq 8$, are attainable from one to another.

1. Introduction

In this paper, a graph is assumed to be finite, connected and free of loops and parallel edges and arcs oriented in both directions.

Let G be an undirected graph. The orientations of G and the resulting directed graphs (digraphs), whose underlying undirected graph is G, will be confused, and with the same abuse of terminology, we will often use the same notation π defined before as an orientation of G to denote also the resulting digraph, without previous notice.

Let π be an orientation of an undirected graph G, and H a subgraph of G. Then $\pi(H)$ denotes the restriction of π on G to the one on H. Particularly, $\pi(G) = \pi$.

An orientation π of G is called *acyclic* if the resulting digraph $\pi(G)$ contains no directed cycle. In the same way, an orientation π of G is called *clique-acyclic*

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if no clique of the resulting digraph $\pi(G)$ has directed cycles. It should be noted that an orientation is clique-acyclic if and only if it contains no directed triangle.

For any orientation π of G and for any edge e of G, we denote by $\mathrm{flip}(\pi, e)$ the orientation obtained from π by reversing the orientation of e, and $\mathrm{Diff}(\pi, \pi')$ denotes the set of all edges whose orientation are different in π and π'.

For a given acyclic orientation π of a graph, an edge is called *flippable* if $\mathrm{flip}(\pi, e)$ is again acyclic. We use the same term for clique-acyclic case as well: for a given clique-acyclic orientation π of a graph, an edge is called *flippable* if $\mathrm{flip}(\pi, e)$ is again clique-acyclic.

For acyclic orientations, the following result is known. For its simple graph-theoretical proof, see [1].

LEMMA 1.1. (Acyclic Orientation Lemma) *Let G be a graph and let π and π' be any two distinct acyclic orientations of G. Then there exists an edge $e \in \mathrm{Diff}(\pi, \pi')$ such that $\mathrm{flip}(\pi, e)$ is again acyclic.*

Let us suppose that two acyclic orientations π and π', of an undirected graph G, have a sequence of acyclic orientations $\pi = \pi_0, \pi_1, \ldots, \pi_t = \pi'$ such that $\pi_i = \mathrm{flip}(\pi_{i-1}, e_i)$ for some edge e_i of G. Then let us call π' *attainable from π by t steps.* Again, we use the same term for clique-acyclic case as well: for two given distinct clique-acyclic orientations π and π' of a graph G, π' is called *attainable from π by t steps*, if there exists a sequence of clique-acyclic orientations $\pi = \pi_0, \pi_1, \ldots, \pi_t = \pi'$ such that $\pi_i = \mathrm{flip}(\pi_{i-1}, e_i)$ for some edge e_i of π_{i-1}.

On this terminology, Lemma 1.1 states that, for any two acyclic orientations π and π' of a graph, π' is attainable from π by $|\mathrm{Diff}(\pi, \pi')|$ steps.

Now, let us consider whether similar statements in Lemma 1.1 hold in the cases of clique-acyclic orientations or not. In these cases, the situation is not so simple. First, there is an easy example in which no edge in $\mathrm{Diff}(\pi, \pi')$ is flippable. See Figure 1.

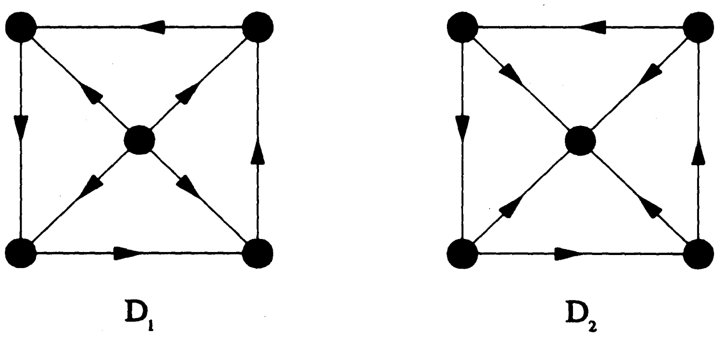


Figure 1. We need $(|\mathrm{Diff}(D_1, D_2)| + 2)$ flips to attain D_2 to D_1.

So the condition that the edges flipped in each step is taken from $\mathrm{Diff}(\pi_{i-1}, \pi_i)$ must be removed. Thus, the next to be considered is whether two clique-acyclic orientations of an undirected graph are attainable from one to another. In other words, the problem is whether the clique-acyclic flip graph of G is always connected or not, where a *clique-acyclic flip graph* of G means an undirected graph whose vertices represent the clique-acyclic orientations of G and two vertices σ and τ are connected by an edge if τ can be obtained from σ by flipping only one edge.

However, the statement also fails in general. For its counterexample, we construct clique-acyclic digraphs in which no edge is flippable in Section 5.

Now arises the following question: *Are there any interesting classes of graphs whose clique-acyclic flip graph is connected?* The main purpose of the present paper is to show that the class of graphs whose every induced subgraph has a vertex v whose degree $\delta(v) \leqq 7$ is such a kind of class.

THEOREM 1. *Let G be a simple undirected graph such that every induced subgraph of G has a vertex v whose degree $\delta(v) \leqq 7$. Then any two clique-acyclic orientations of G are always attainable from each other by at most $2|E(G)|$ steps.*

The number 7 of the above condition in Theorem 1 is sharp. That is, we show the following:

THEOREM 2. *Let G be an 8-regular simple undirected graph. Then, there are exactly five examples of G each of which has a clique-acyclic orientation whose every edge is non-flippable. The vertex-size of these graphs are $15, 18, 18, 20$, and 24.*

From Theorem 1 and 2 together, we can easily deduce that:

THEOREM 3. *Let G be a connected graph whose maximum degree $\Delta(G) \leqq 8$. Then, except for the five examples of G mentioned in Theorem 2, any two clique-acyclic orientations of G are attainable from one to another by at most $2|E(G)|$ steps.*

2. Deletion process

Let $G := (V(G), E(G))$ ($G = (V, E)$, for short) be an undirected graph. Then, we will consider the following two properties of G.

PROPERTY 1. *Every acyclic orientation of G is attainable from any other clique-acyclic orientation of G by at most $\frac{1}{2}|E(G)|$ steps.*

PROPERTY 2. *Any two clique-acyclic orientations of G are always attainable from one to another by at most $2|E(G)|$ steps.*

LEMMA 2.1. *G satisfies Property 2 if it satisfies Property 1.*

Proof. Let π and θ be any two clique-acyclic orientations of G. From Property 1, there exist two acyclic orientations π' and θ' of G such that π (θ', resp.) is attainable from π_1' (θ, resp.) by at most $\frac{1}{2}|E(G)|$ steps. On the other hand, from Lemma 1.1, we have that any two acyclic orientations of G are attainable from one to another by at most $|E(G)|$ steps. Thus there exists a sequence of clique-acyclic orientations

$$\pi = \pi_0, \pi_1, \dots, \pi_{k-1} = \pi', \pi_k, \dots, \pi_{k+l-1} = \theta', \pi_{k+l}, \dots, \pi_{k+l+m-1} = \theta$$

such that $k, m \leq \frac{1}{2}|E(G)|$, $l \leq |E(G)|$ and $\pi_i = \text{flip}(\pi_{i-1}, e_i)$ for some flippable edge e_i of π_{i-1}. And hence $k + l + m \leq 2|E(G)|$, which is just the statement of Property 2. ■

Thus, from now on, in order to prove Theorem 1, we only have to prove the following:

(1) *Let G be such a simple undirected graph that any induced subgraph of G has a vertex v whose degree $\delta(v) \leqq 7$. Then G satisfies Property 1.*

Let $\pi(G) := (V(\pi(G)), A(\pi(G)))$ ($\pi(G) = (V, A)$, for short) denote a digraph whose underlying undirected graph is G, where $A(\pi(G))$ is the arc-set of $\pi(G)$ corresponding to $E(G)$. For $V' \subseteq V(G)$, $G[V']$($\pi(G)[V']$, resp.) denotes the subgraph of G ($\pi(G)$, resp.) induced by V'.

Let $e \in E$ be an undirected edge of G and $a \in A$ the arc of $\pi(G)$ corresponding to e of G. If u, v are the end-vertices of e, we use the notation $uv(= vu)$ to denote e and $a = \overrightarrow{uv}$ means that u and v are the tail and the head of a, respectively. For a subset T of V, $N_G(T)$ ($N(T)$, for short) denote the set $\{v \in V \mid \exists u \in T, \text{ either } \overrightarrow{uv} \text{ or } \overrightarrow{vu} \in A\}$.

Then, we prove here the following lemma which is the key of the proof of (1).

LEMMA 2.2. *Let $\pi(G) := (V, A)$ be a clique-acyclic digraph, T a subset of V. Suppose that $\pi(G)[T]$ is acyclic and that $\pi(G)$ has no directed path between two vertices in $N(T) \setminus T$ passing through some vertex of T. Then there exists an acyclic digraph to be attainable from $\pi(G)$ if and only if there exists an acyclic digraph to be attainable from $\pi(G)[V \setminus T]$.*

Proof. The "only if"-part is obvious. In order to prove the "if"-part, it is enough to notice that, if we flip any one flippable edge of $\pi(G)[V \setminus T]$ (such a edge is also a flippable edge of $\pi(G)$, because $\pi(G)$ has no directed path between two vertices in $N(T) \setminus T$ passing through some vertex of T.), we have another clique-acyclic orientation of G, say $\pi(G)'$, such that $\pi(G)'[T]$ is also acyclic and that there is no directed path between two vertices in $N(T) \setminus T$ passing through some vertex of T in $\pi(G)'$. ■

3. Fans, wheels and inverters

Let $P := v_0 v_1 \cdots v_{k-1} v_k$ be a chordless undirected path such that $v_i v_{i+1}$ $(0 \leq i \leq k-1)$ are the edges of P and $\forall i, j$ $v_i \neq v_j$. And suppose that a vertex o is adjacent to every vertex of P. Then the resulting graph $F := P * o$ is called an *undirected fan* and o is called the *center* of F. An edge incident with o is called a *rib* of F.

Now, let us put an orientation on the undirected fan $F = P * o$ so that P is directed so that it has a simple directed path $\overrightarrow{P}$ from v_0 to v_k and that o is the tail of every edge incident with o (denote the tail as o_t). The resulting directed graph $\overrightarrow{F_O} := \overrightarrow{P} \overleftarrow{*} o_t$ is called a *directed outer fan*, o_t its *center tail* and an arc incident with the center tail o_t an *outer rib* of $\overrightarrow{F_O}$. Especially, $\overrightarrow{o_t v_0}$ and $\overrightarrow{o_t v_k}$ are called the *first outer rib* and the *last outer rib* of $\overrightarrow{F_O}$, respectively. A *directed inner fan* $\overrightarrow{F_I} := \overrightarrow{P} \overrightarrow{*} o_h$ will be defined in the same way except for o is the head of every arc incident with o (denote it as o_h). Then, o_t is called its *center head*, an edge incident with the center head o_h an *inner rib* of $\overrightarrow{F_I}$. Especially, $\overrightarrow{v_0 o_h}$ and $\overrightarrow{v_k o_h}$ are called the *first inner rib* and the *last inner rib* of $\overrightarrow{F_I}$, respectively. See Figure 2.

If we identify the first rib and the last rib of an undirected fan $F = P * o$, we obtain an *undirected wheel* $W := C * o$, where $C := v_0 v_1 \cdots v_{k-1} v_0$ is the chordless simple undirected cycle corresponding to P of F. Then o is also called the *center* of the wheel and an edge incident with the center is called a *spoke*.

In the same way, we also obtain a *directed outer wheel* $\overrightarrow{W_O} := \overrightarrow{C} \overleftarrow{*} o_t$ by identifying the first outer rib and the last outer rib of a directed outer fan $\overrightarrow{F_O}$. Then the vertex o_t is called the *center tail* of $\overrightarrow{W_O}$, an arc incident with o_t an *outer spoke* of $\overrightarrow{W_O}$. Similarly, we obtain *directed inner wheel* $\overrightarrow{W_I} := \overrightarrow{C} \overrightarrow{*} o_h$ by identifying the first inner rib and the last inner rib of a directed inner fan $\overrightarrow{F_I}$. Then the vertex o_h is called the *center head* of $\overrightarrow{W_I}$, an arc incident with o_h an *inner spoke* of $\overrightarrow{W_I}$. See Figure 3.

Last we define an *inverter*. Let $P := v_0 v_1 \cdots v_{k-1} v_k$ be a chordless undirected path, $C := v_0 v_1 \cdots v_{2l-1} v_0$ a chordless undirected cycle. And let $F = P * o_1$ be an undirected fan, $W = C * o_2$ an undirected wheel. Then a clique-acyclic orientation I_O on F is called an *open inverter* if either it has $\overrightarrow{o v_i}$ and $\overrightarrow{v_{i+1} o}$

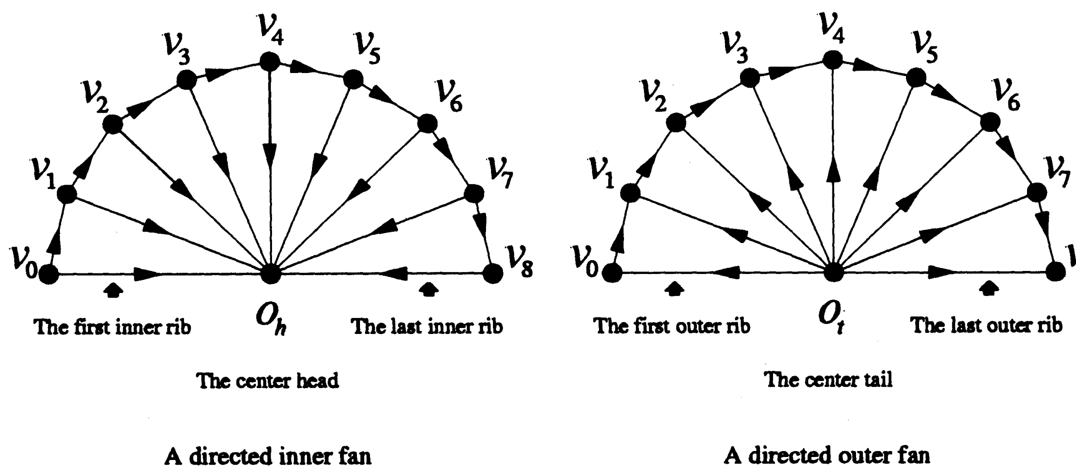


Figure 2. Directed Fans.

for all $i \equiv 0(\text{modulo}\,2)$ or it has $\overrightarrow{v_i o}$ and $\overrightarrow{o v_{i+1}}$ for all $i \equiv 0(\text{modulo } 2)$. In the same way, a clique-acyclic orientation I_C on W is called a *closed inverter* if either it has $\overrightarrow{o v_i}$ and $\overrightarrow{v_{i+1} o}$ for all $i \equiv 0(\text{modulo } 2)$ or it has $\overrightarrow{v_i o}$ and $\overrightarrow{o v_{i+1}}$ all $i \equiv 0(\text{modulo } 2)$. See Figure 4.

LEMMA 3.1. *Let $G = (V, E)$ be an undirected graph such that $\exists o \in V$, $N(o) = V \setminus \{o\}$. And let π be a clique-acyclic orientation of G such that $\{a \in A(\pi(G)) \mid \exists u \in V(\pi(G)), a = \overrightarrow{uo}\} \neq \emptyset$ ($\{a \in A(\pi(G)) \mid \exists u \in V(\pi(G)), a = \overrightarrow{ou}\} \neq \emptyset$, resp.) and $\pi(G)$ has no directed inner (outer, resp.) wheel whose center head (tail, resp.) is o, as its subgraph. Then, $\{a \in A(\pi(G)) \mid \exists u \in V(\pi(G)), a = \overrightarrow{uo}\}$ ($\{a \in A(\pi(G)) \mid \exists u \in V(\pi(G)), a = \overrightarrow{ou}\}$, resp.) contains at least one flippable arc.*

Proof. Let $A_I(o) := \{a \in A(\pi(G)) \mid \exists u \in V(\pi(G)), a = \overrightarrow{uo}\}$. Then $\forall a \in A_I(o)$, either a is an inner rib of a directed inner fan or a is an inner rib of an inverter or the tail of a is adjacent to only o. Thus $\exists a' \in A_I(o)$, no directed inner fan whose center head is o has a' as its first inner rib, for otherwise, $\pi(G)$ has a directed inner wheel whose center head is o as its subgraph, it is a contradiction. Obviously a' is flippable. The other case can be shown in the same way. ■

A pair of undirected wheels $(F_1 := C_1 * o, F_2 := C_2 * o)$ of an undirected graph G is called an *undirected double-wheel* of G if $(|C_1| - 4)(|C_2| - 4) \geq 0$ and they have no common spoke. If $(F_1 := C_1 * o, F_2 := C_2 * o)$ is an undirected double-wheel, then o is called the *center* of the undirected double-wheel. A pair of directed wheels $(\overrightarrow{F_1} := \overrightarrow{C_1} \overrightarrow{*} o_d, \overrightarrow{F_2} := \overrightarrow{C_2} \overleftarrow{*} o_d)$ of a directed graph $\pi(G)$ is called a *directed double-wheel* of $\pi(G)$ if $\overrightarrow{F_1}$ is a directed inner wheel and $\overrightarrow{F_2}$ is a

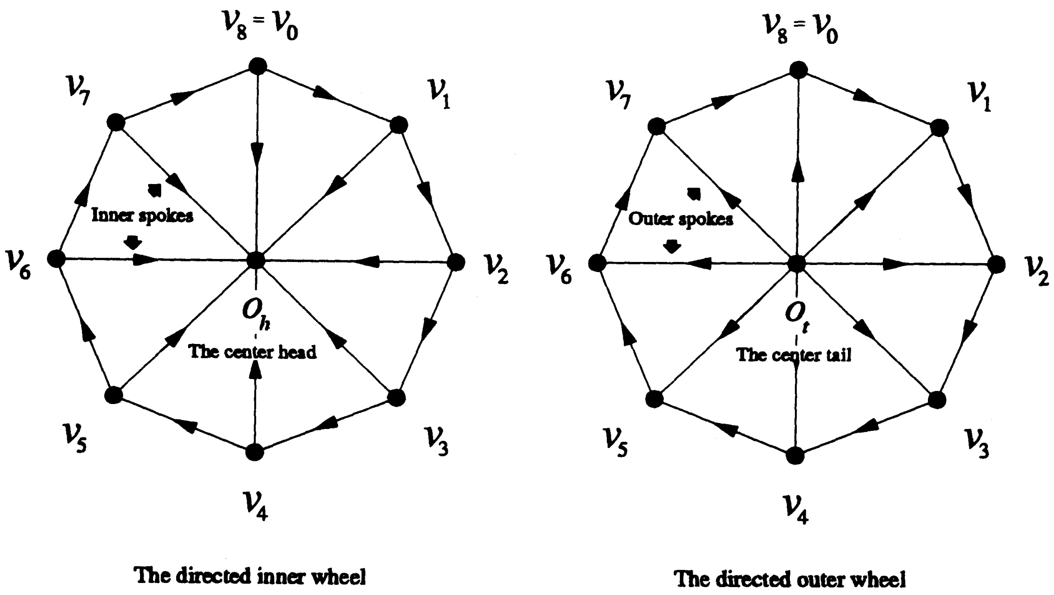


Figure 3. Directed Wheels.

directed outer wheel and $(|\overrightarrow{C_1}|-4)(|\overrightarrow{C_2}|-4) \geq 0$ holds. Clearly, the underlying undirected graph of a directed double-wheel is an undirected double-wheel. If $(\overrightarrow{F_1} := \overrightarrow{C_1} \overrightarrow{*} o_d, \overrightarrow{F_2} := \overrightarrow{C_2} \overleftarrow{*} o_d)$ is a directed double-wheel, then o_d is called the *center* of the directed double-wheel. See Figure 5.

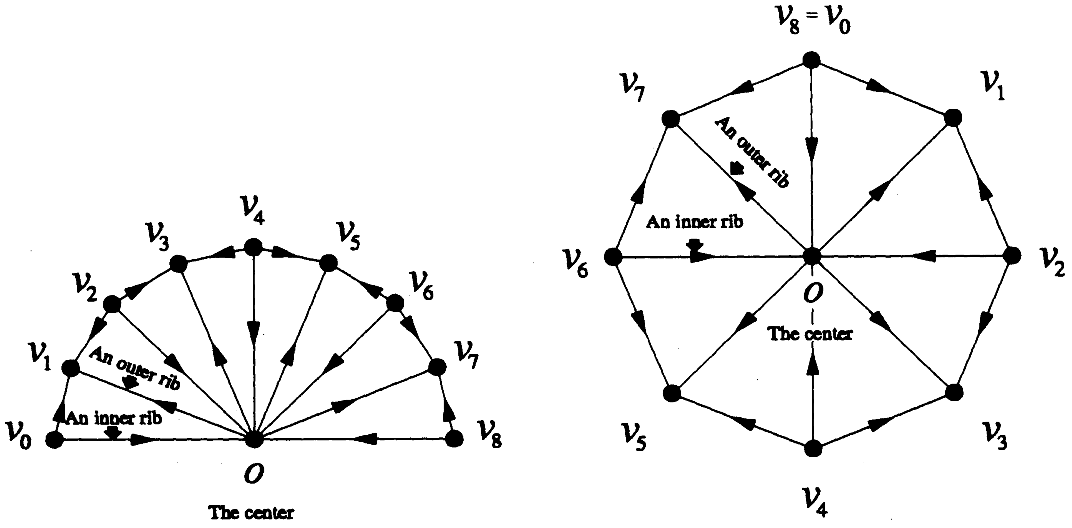
Let G and H be two simple undirected graphs. Then we define the *union* $G \cup H$ of G and H as

$$(V(G \cup H), E(G \cup H)) := (V(G) \cup V(H), E(G) \cup E(H)).$$

Moreover, let $\pi(G)$ and $\pi(H)$ be directed graphs such that, for any two distinct vertices $x, y \in V(\pi(G)) \cap V(\pi(H))$, $\overrightarrow{xy} \in A(\pi(G)[V(\pi(G)) \cap V(\pi(H))])$ only if $\overrightarrow{yx} \notin A(\pi(H)[V(\pi(G)) \cap V(\pi(H))])$. Then we also define the *union* $\pi(G) \cup \pi(H)$ of $\pi(G)$ and $\pi(H)$ as

$$(V(\pi(G) \cup \pi(H)), A(\pi(G) \cup \pi(H))) := (V(\pi(G)) \cup V(\pi(H)), A(\pi(G)) \cup A(\pi(H))).$$

LEMMA 3.2. *Let π be a clique-acyclic orientation of G whose vertex-size is n. And let $\{v_1, v_2, \ldots, v_n\}$ be a sequence of the vertices of G, $G_1 := G$ and $G_{i+1} := G_i - v_i$ $(1 \leq i \leq n-1)$. Now suppose that, for all i $(1 \leq i \leq n-1)$, $\pi(G_i)$ has no directed double-wheel whose center is v_i. Then there exists some acyclic orientation $\theta(G)$ of G such that $\theta(G)$ is attainable from $\pi(G)$ by at most $|E(G)|$ steps.*



An open inverter A closed inverter

Figure 4. Inverters.

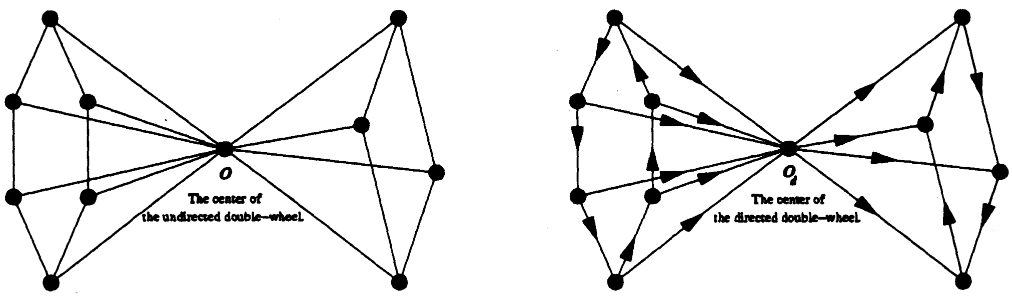


Figure 5. Double wheels.

Proof. For any simple undirected graph H and its orientation ϕ, let $\mathrm{NA}_I(v, \phi(H)) = \{a \in A(\phi(H)) \mid \exists u \in V(\phi(H)), a = \overrightarrow{uv}\}$ and $\mathrm{NA}_O(x, \phi(H)) := \{a \in A(\phi(H)) \mid \exists u \in V(\phi(G)), a = \overrightarrow{vu}\}$ where v is a vertex of a graph H. And let $\mathrm{NA}(v, \phi(H)) := \mathrm{NA}_I(v, \phi(H)) \cup \mathrm{NA}_O(v, \phi(H))$. Then from Lemma 3.1, by flipping either all the arcs in $\mathrm{NA}_I(v_1, \pi(G))$ or all the arcs in $\mathrm{NA}_O(v_1, \pi(G))$, we have another clique-acyclic digraph $\pi_2(G_1)$ which has either no arc whose head is v_1 or has no arc whose tail is v_1. In other words, $\pi_2(G_1)$ is attainable from $\pi(G)$ by at most $\max\{|\mathrm{NA}_I(v_1, G)|, |\mathrm{NA}_O(v_1, G)|\}$ steps. Then from Lemma 2.2, we can only consider $\pi_2(G_2) := \pi_1(G_1) - v_1$ instead of $\pi_2(G_1)$. And hence, in the same way, we can obtain a sequence of clique-acyclic digraphs $\{\pi_1(G_1) := \pi(G_1), \pi_2(G_2), \dots, \pi_n(G_n)\}$, so that, for each i, we make $\pi_{i+1}(G_i)$

attainable from $\pi_i(G_i)$ by flipping at most $\max\{|\mathrm{NA}_I(v_i, G_i)|, |\mathrm{NA}_O(v_i, G_i)|\}$ arcs of $\pi_i(G_i)$ incident with v_i where $\pi_{i+1}(G_i)$ has either no arc whose head is v_i or has no arc whose tail is v_i. (Here, $d_{G_i}(v_i)$ denotes the number of arcs of G_i incident with v_i.) Obviously, $\pi_n(G_n)$ is merely the vertex v_n itself. Then,

$$\theta(G) := \bigcup_{i=1}^{n} \mathrm{NA}(v_i, \pi_{i+1}(G_i))$$

is an acyclic orientation of G. And since

$$\sum_{i=1}^{n} \{\max\{|\mathrm{NA}_I(v_i, G_i)|, |\mathrm{NA}_O(v_i, G_i)|\}\} \leqq |E(G)|,$$

the above orientation of G is attainable from $\pi(G)$ by at most $|E(G)|$ steps. ■

Here we note that the following immediate consequences of the above lemma:

LEMMA 3.3. *Let G be an undirected graph whose no clique-acyclic orientation has a directed double-wheel. Then any two clique-acyclic orientations of G are always attainable from one to another by at most $3|E(G)|$ steps.*

Proof. Since G has no clique-acyclic orientation which has a directed double-wheel, it is easily deduced from Lemma 3.2 that G satisfies Property 1. And hence, from Lemma 2.1, we have that any two clique-acyclic orientations of G are always attainable from one to another by at most $3|E(G)|$ steps. ■

THEOREM 4. *Let G be an undirected graph which has no undirected double-wheel. Then any two clique-acyclic orientations of G are always attainable from one to another by at most $3|E(G)|$ steps.*

4. Proof of Theorem 1

In this section, we suppose that G be a simple undirected graph such that every induced subgraph of G has a vertex v whose degree $\delta(v) \leqq 7$.

Proof of Theorem 1. Let π be an arbitrary clique-acyclic orientation of a graph G whose every induced subgraph has a vertex v whose degree $\delta(v) \leqq 7$. Then, we have a numbering $\{v_1, v_2, \ldots, v_n\}$ of the vertices of G and the corresponding sequence $\{G_1 := G, G_2 := G_1 - v_1, \ldots, G_n := G_{n-1} - v_{n-1} = v_n\}$ of subgraphs of G so that $\delta_{G_i}(v_i) \leqq 7$ $(i = 1, 2, \ldots, n)$, where $\delta_{G_i}(v_i)$ denotes the degree of v_i in G_i.

On the other hand, since a cycle of every directed wheel of $\pi(G)$ could not be a directed triangle and hence it has at least 4 vertices. It means that:

- *for every subset S of $V(G)$, every vertex v in S for which $\delta_{G[S]}(v_i) \leqq 7$ holds cannot be a center of any directed double-wheel of $\pi(G[S])$.*

Thus, we can apply Lemma 3.2 to the sequence $\{G_1, G_2, \ldots, G_n\}$. Moreover,

- *at each step of (i, v_i, G_i), by flipping at most $\lfloor \frac{d_{G_i}(v_i)}{2} \rfloor$ arcs of $\pi(G_i)$ incident with v_i, we have another clique-acyclic digraph $\pi_{i+1}(G_i)$ such that either v_i sees no vertex of $\pi_{i+1}(G_i)$ or no vertex of $\pi_{i+1}(G_i)$ sees v_i:*

In order to prove the above statement, first we will note that, for all i, G_i has at most one directed wheel whose center is v_i. If G_i has no such directed wheel, then the statement is obvious. If such wheel exists, it uses at least 4 edges of the (at most 7) edges adjacent to v_i as its spokes, and hence, the statement also holds.

And since

$$\sum_{i=1}^{n} \frac{d_{G_i}(v_i)}{2} = \frac{1}{2}|E(G)|,$$

the above orientation of G is attainable from $\pi(G)$ by at most $\frac{1}{2}|E(G)|$ steps. ∎

5. The Case of $\Delta(G) = 8$

Let G be a simple undirected graph, π be a clique-acyclic orientation, and v be an arbitrary vertex of G. Then let us use the notation $N(v)$ to denote the *neighborhood* of v in $\pi(G)$, that is the vertex-set defined by $\{u \in V(\pi(G)) \mid$ either $\overrightarrow{uv}$ or $\overrightarrow{vu} \in A(\pi(G))\}$.

From now on, we will show the proof of Theorem 2.

Proof of Theorem 2. Let G be a simple undirected 8-regular graph. And suppose that there is a clique-acyclic orientation π of G which cannot be attainable from any acyclic orientation of G. Then every vertex o of $\pi(G)$ must be a center of a directed double-wheel whose vertex-set is $\{o\} \cup N(\{o\})$, which is isomorphic to the digraph described in Figure 6. Because otherwise, either o cannot be a center of any directed inner-wheel or o cannot be a center of any directed outer-wheel, and hence, from Lemma 3.1, we have that $\pi(G)$ is attainable from some acyclic orientation of G if and only if $\pi(G - o)$ is. Moreover $G - o$ is a graph such that every induced subgraph of it has a vertex v whose degree $\delta(v) \leqq 7$. Then, from Theorem 1, we have that any two clique-acyclic orientations of G are always attainable from one to another by at most $2|E(G)|$ steps. It is a contradiction.

Now, let us fix an arbitrary vertex o of $\pi(G)$, and let us use the notation $\{a, b, c, d, e, f, g, h\}$, which is described in Figure 6, to denote its neighborhood.

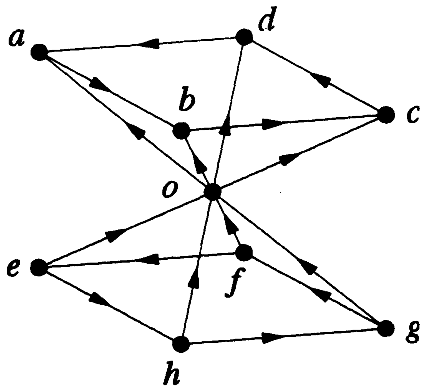


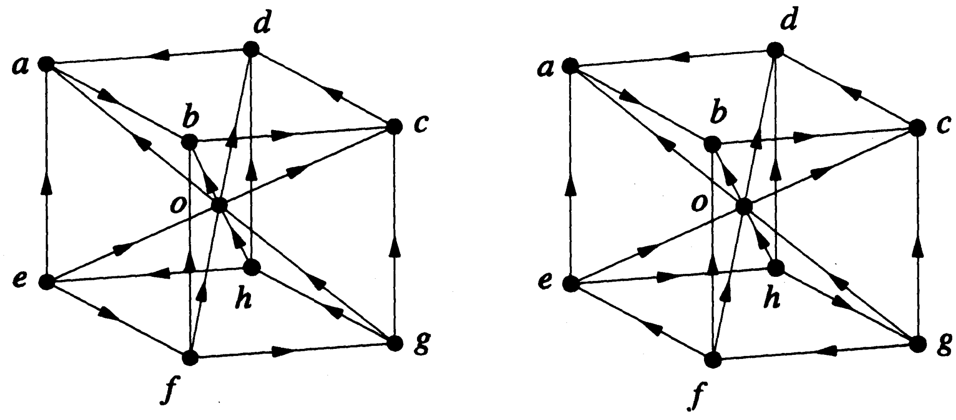
Figure 6. Double Wheel

Here we note that obviously the two 4-cycles $abcda, ehgfe$ must be chord-less cycles.

Then, clearly, each vertex of $\{a,b,c,d\}$ ($\{e,f,g,h\}$, respectively,) must be incident with an directed edge of $\pi(G)$ whose another end-vertex belongs to $\{e,f,g,h\}$ ($\{a,b,c,d\}$, respectively). For otherwise, there exists a vertex in $N(o)$ which cannot be a center of any directed double-wheel in $\pi(G)$, which contradicts our assumption. Moreover, such a directed edge must have a direction from the end-vertex in $\{e,f,g,h\}$ to the end-vertex in $\{a,b,c,d\}$, because otherwise $\pi(G)$ has a directed triangle one of whose vertices is o, which contradicts the clique-acyclicity of π. Furthermore, each two of such edges cannot have common end-vertex. For otherwise, there is a vertex in $N(o)$ which cannot be a center of any directed double-wheel, which again contradicts our assumption. From all of these observations, we have that there are exactly 4 (disjoint) directed edges whose heads are in $\{e,f,g,h\}$ and whose tails are in $\{a,b,c,d\}$. In other words, the subgraph of $\pi(G)$ induced by $\{o\} \cup N(o)$ must be one of the three digraphs described in Figure 7 (Case 1-a, Case 1-b) and Figure 8 (Case 2). Here let us call the above three digraphs *spindles*. And if a spindle has vertex x as its center, let us call it a *spindle with the center* x, or x*'s spindle*.

In this terminology, each vertex of our counterexamples must be the center of its own spindle which is one of the three kinds of spindles, namely Case 1-a spindle, Case 1-b spindle, and Case 2 spindle. In these three spindles, the Case 1 spindles (the Case 1-a and Case 1-b spindles) are more symmetrical than the other, and every Case 1 spindle's center is linked to a directed graph whose underlying undirected graph is a cube-graph.

And here we note that *each counterexample of ours cannot have both a Case 1-a spindle* X *and a Case 1-b spindle* Y *together as its subgraphs so that the center of* X *is adjacent to the center of* Y. Actually, it is easy to check that the above



Case 1–a. Case 1–b.

Figure 7. Case 1-a and 1-b.

restriction causes a directed triangle soon. And hence, *if a counterexample of ours does not have a Case 2 spindle as its subgraph, then either it has only Case 1-a spindles or it has only Case 1-b spindles.*

D. Buset[1] proved that *if an undirected graph G is* locally-cube (*that is,* for each vertex v of G, the subgraph induced by the neighbors of v is isomorphic to a cube-graph) *then G is isomorphic either to the* 1*-skeleton of a* 4*-dimensional regular polytope such that it has* 24 *facets each of which is an octahedron, or to the complement of the* (3×5)*-grid.* (Here, $(p \times q)$-grid denotes *the graph whose vertices are the pq ordered pairs (i,j) with $i = 1, \cdots, p$ and $j = 1, \cdots, q$, two vertices being adjacent if and only if they have one coordinate in common.*)

By using the above fact, we can easily determine all of the counterexamples in the following two subcases.

Case 5.1. *Every vertex of $\pi(G)$ and its neighborhood together induce the subgraph isomorphic to the Case 1-a spindle.*

Case 5.2. *Every vertex of $\pi(G)$ and its neighborhood together induce the subgraph isomorphic to the Case 1-b spindle.*

Claim 5.1. *In Case 5.1, $\pi(G)$ is isomorphic to a unique digraph whose underlying undirected graph is isomorphic to the* 1*-skeleton of a* 4*-dimensional regular polytope such that it has* 24 *facets each of which is an octahedron* (1*-skeleton of the* 24*-cell, for short*).

Claim 5.2. *In Case 5.2, $\pi(G)$ is isomorphic to a unique digraph whose underlying undirected graph is the complement of the* (3×5)*-grid.*

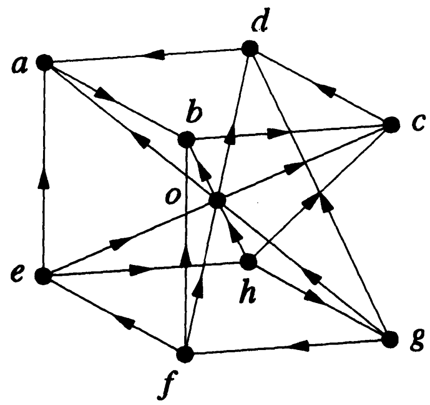


Figure 8. Case 2.

The proof of the above two claims is easily obtained as follows:

Proof of Claim 5.1 and Claim 5.2. First we note that, if $\pi(G)$ is a counterexample of Case 5.1 or Case 5.2, then G turns out to be isomorphic to either the 1-skeleton of the 24-cell or the complement of the (3×5)-grid. And it is easy to check that there is no orientation on the 1-skeleton of the 24-cell (the (3×5)-grid, resp.) whose every spindle is a Case 1-b (Case 1-a, resp.) spindle.

On the other hand, in order to obtain an orientation mentioned in Case 5.1 (Case 5.2, resp.) on a locally-cube graph H, we only fix the directions of edges of a spindle in H as a Case 1-a (Case 1-b, resp.) spindle. Then the directions of other edges must be determined automatically, if they exist. Actually we can show a digraph satisfied the statement of Claim 5.1 and a graph satisfied the statement of Claim 5.2, which are the graph of Figure 9 and the graph of Figure 10, respectively. Thus the proof ended. ■

CASE 5.3. *Some vertex of $\pi(G)$ and its neighborhood together induce the subgraph isomorphic to the Case 2 spindle.*

Here let us consider other counterexamples that have Case 2 spindles as subdigraphs. As a matter of fact, there exist exactly 3 such counterexamples.

CLAIM 5.3. *In Case 5.3, $\pi(G)$ is isomorphic to one of the three digraphs whose vertex-sizes equal to* 18, 18 *and* 20, *respectively.*

In order to enumerate the all counterexamples in Case 5.3, first we start with a Case 2 spindle, then next we will add corresponding vertices and edges to make recursively a new spindle which is one of the type Case 1-a, Case 1-b and Case 2

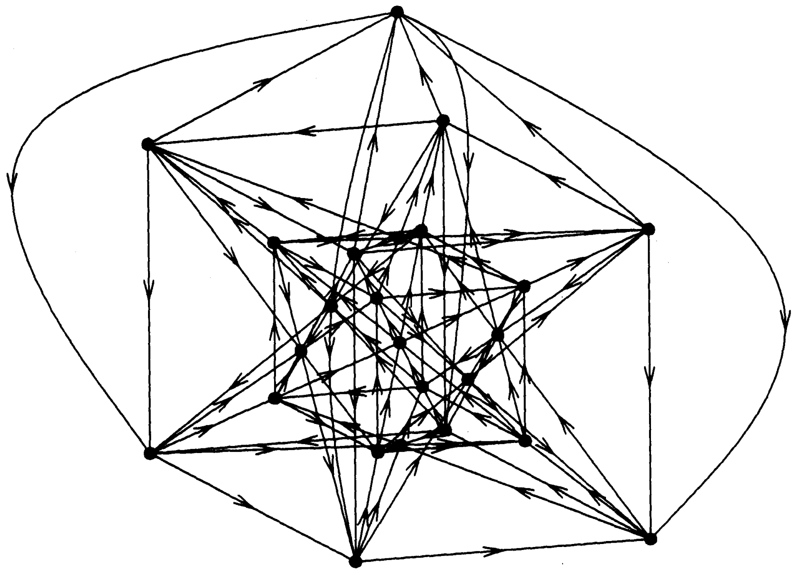


Figure 9. The counterexample which includes only Case 1-a spindles.

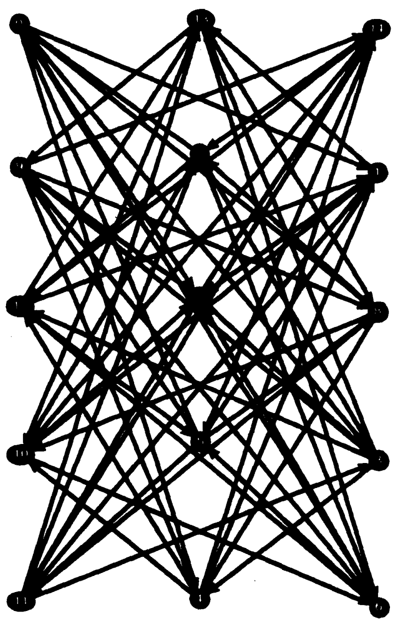


Figure 10. The counterexample which includes only Case 1-b spindles.

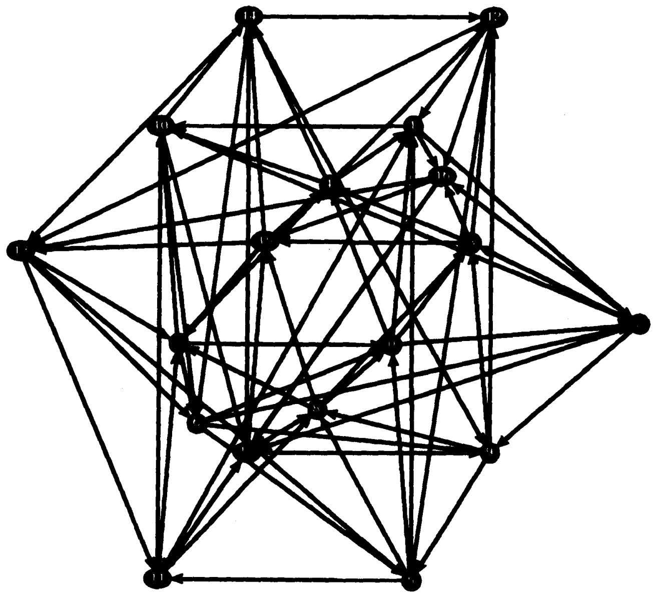


Figure 11. The digraph which has 18 vertices.

whose center is on each of the corners of the existing spindle, with checking the possibility of the identifications of vertices at each step. However, the above process would be quite complicated. Therefore we adopt here a computational proof[1]. The adjacent matrices of those counterexamples are described in the Appendix.

Then, we completed the proof of Theorem 2. ■

Now, combining Theorem 1 and 2 together, we can easily deduce Theorem 3, as follows:

Proof of Theorem 3. If $\pi(G)$ has a vertex v which is incident with a flippable edge, then v cannot be a center of any directed double-wheel of $\pi(G)$. Thus, from Lemma 3.2 we can treat with only the digraph $\pi(G-v)$ instead of $\pi(G)$ itself. And obviously, $G-v$ is a graph satisfying the assumption of Theorem 1. Thus every acyclic orientation of G is attainable from $\pi(G)$ by at most $\frac{1}{2}|E(G)|$ steps. If $\pi(G)$ has no flippable edge, then from Theorem 2, it must be isomorphic to one of the 5 counterexamples mentioned in Theorem 2. ■

[1]The source program of this computational enumeration algorithm has been written by one of the authors, Yoshimasa Hiraishi, with Mathematica. If you need the source program, please contact the authors.

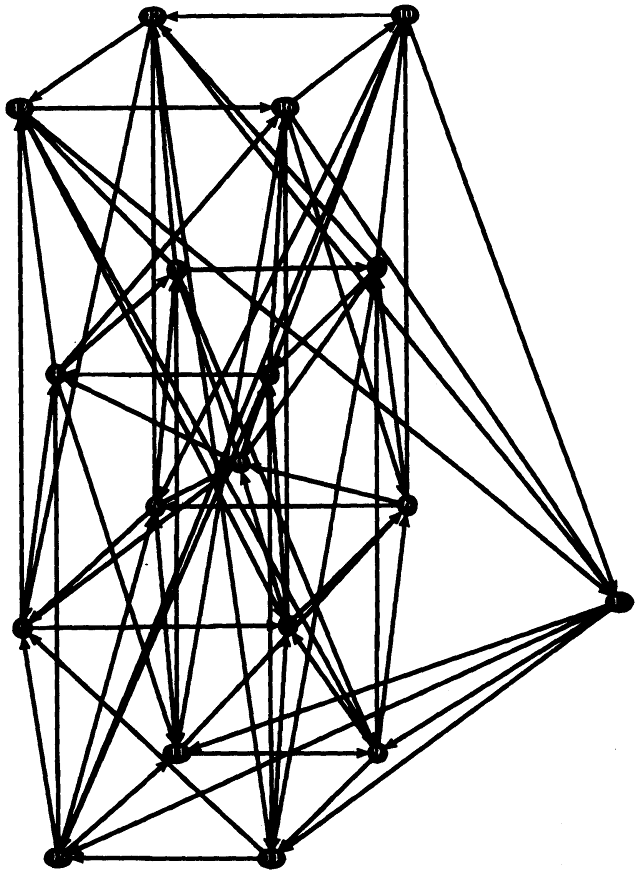


Figure 12. The another digraph mentioned which also has 18 vertices.

6. Concluding remarks

Last, we will state several remarks and conjectures related to our results.

Now let $\mathfrak{A}$ denote the class of graphs for each of which the minimum degree of its every induced subgraph is $\leqq 7$. And let $\mathfrak{Gen}(n)$ denote the class of graphs each of whose genus is $\leqq n$. Moreover, let $\mathfrak{Cros}(m)$ denote the class of graphs each of whose cross-cap number is $\leqq m$.

As an easy consequence of the following well-known lemma, we have that $\mathfrak{A}$ *contains* $\mathfrak{Gen}(2) \cup \mathfrak{Cros}(4)$.

Lemma 6.1. *Let G be a simple graph embedded in a closed surface F^2 whose Euler characteristic is $\chi(F^2)$. Then the average degree $\bar{d}(G)$ of G is either strictly less than 6 or*

$$\bar{d}(G) \leq \frac{5 + \sqrt{49 - 24\chi(F^2)}}{2}.$$

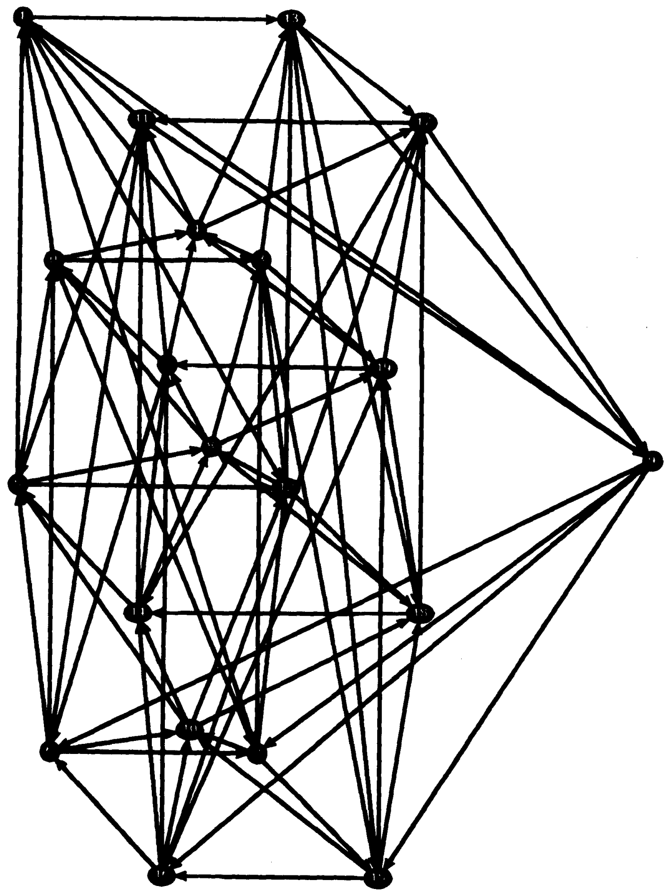


Figure 13. The digraph which has 20 vertices.

Furthermore,

- *For $\forall n \in \mathbb{N}$, there are only finite examples of G such that G's genus (cross-cap number, resp.) is n and that G has a subgraph whose minimum degree is $\geqq 8$.*

And hence, any two clique-acyclic orientations of G whose genus (cross-cap number) is n are attainable from one to another by at most $2|E(G)|$ steps if G has no such finite examples in $\mathfrak{Gen}(n)$ ($\mathfrak{Cros}(n)$, resp.) as its subgraphs. Then, we will propose that:

CONJECTURE 6.1. *Let G be a graph in $\mathfrak{Gen}(3) \cup \mathfrak{Cros}(5)$. Then any two clique-acyclic orientations of G are attainable from each other by at most $2|E(G)|$ steps if G contains none of the five graphs mentioned in Theorem 2 as its subgraph.*

Here we note that $\mathfrak{Gen}(6)$ contains our first example mentioned in Theorem 2 – the graph which has 15 vertices – and hence 5 is one of the non-trivial upper bounds of genus of a graph whose any two clique-acyclic orientations are attainable from each other.

The converse statement of Lemma 3.2 in Section 3 is not true in general. That is; *There is a clique-acyclic orientation π of a graph H whose every vertex is a center of some directed double-wheel and from which some acyclic orientation of H can be attainable by at most $|E(H)|$ steps.*

Thus, it is quite natural and worth while doing to find good characterization of the whole of the connected graphs whose any two clique-acyclic orientations are attainable from one to another. Especially, the settlement of the following conjecture may tell us one of the most important features of these graphs:

CONJECTURE 6.2. *Let G be a graph which has a pair of clique-acyclic orientations (π, θ) such that θ cannot be attainable from π. Then at least one of $\pi(G)$ or $\theta(G)$ has a directed subgraph whose every arc is non-flippable.*

Moreover, we guess that the following stronger statement also be affirmative.

CONJECTURE 6.3. *Let G be a graph which has a pair of clique-acyclic orientations (π, θ) such that θ cannot be attainable from π. Then at least one of $\pi(G)$ or $\theta(G)$ has a directed subgraph $\overrightarrow{D}$ whose every arc is a spoke of a directed double-wheel in $\overrightarrow{D}$.*

References

[1] D. Buset, Graphs which are locally a cube, *Discrete Math.*, **46** (1983), 221–226.

[2] K. Fukuda, A. Prodon and T. Sakuma, Note on acyclic orientations and the shelling lemma, *Theoretical Computer Science*, (to appear).

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Appendix

As we have mentioned in the text, there exist in all 5 counterexamples. We will show the adjacent matrices of these counterexamples. Figures 9 to 13 present their pictures.

The adjacent matrix of a digraph which has only the Case 1-a spindles. See also Figure 9.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0	0	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	-1	0	1	0	-1	-1	0	0	0	1	0	0	-1	1	1	0	0	0	0	0	0	0	0	0
2	-1	-1	0	1	0	0	-1	0	0	-1	1	0	0	1	0	0	0	1	0	0	0	0	0	0
3	-1	0	-1	0	1	0	0	-1	0	0	-1	1	0	1	0	0	0	0	0	0	0	1	0	0
4	-1	1	0	-1	0	0	0	0	-1	0	0	-1	1	1	0	0	0	0	0	1	0	0	0	0
5	1	1	0	0	0	0	1	0	-1	1	0	0	-1	0	0	-1	-1	0	0	0	0	0	0	0
6	1	0	1	0	0	-1	0	1	0	-1	1	0	0	0	0	0	-1	0	-1	0	0	0	0	0
7	1	0	0	1	0	0	-1	0	1	0	-1	1	0	0	0	0	-1	0	0	0	0	0	-1	0
8	1	0	0	0	1	1	0	-1	0	0	0	-1	1	0	0	0	-1	0	0	0	-1	0	0	0
9	0	-1	1	0	0	-1	1	0	0	0	0	0	0	0	-1	-1	0	1	1	0	0	0	0	0
10	0	0	-1	1	0	0	-1	1	0	0	0	0	0	0	0	0	0	-1	-1	0	0	1	1	0
11	0	0	0	-1	1	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1	0
12	0	1	0	0	-1	1	0	0	-1	0	0	0	0	0	1	1	0	0	0	-1	-1	0	0	0
13	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	0	1	0	0
14	0	-1	0	0	0	0	0	0	0	1	0	0	-1	-1	0	1	0	1	0	-1	0	0	0	1
15	0	0	0	0	0	1	0	0	0	1	0	0	-1	0	-1	0	1	0	1	0	-1	0	0	-1
16	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	-1	0	0	-1	0	-1	0	-1	0
17	0	0	-1	0	0	0	0	0	0	-1	1	0	0	-1	-1	0	0	0	1	0	0	1	0	1
18	0	0	0	0	0	0	1	0	0	-1	1	0	0	0	0	-1	1	-1	0	0	0	0	1	-1
19	0	0	0	0	-1	0	0	0	0	0	0	-1	1	-1	1	0	0	0	0	0	1	-1	0	1
20	0	0	0	0	0	0	0	0	1	0	0	-1	1	0	0	1	1	0	0	-1	0	0	-1	-1
21	0	0	0	-1	0	0	0	0	0	0	-1	1	0	-1	0	0	0	-1	0	1	0	0	1	1
22	0	0	0	0	0	0	0	1	0	0	-1	1	0	0	0	0	1	0	-1	0	1	-1	0	-1
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	-1	1	-1	1	-1	1	0

Table 1. The adjacent matrix of the digraph (24 vertices).

The adjacent matrix of a digraph which has only Case 1-b spindles. See also Figure 10.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0
1	-1	0	1	0	-1	-1	0	0	0	0	1	1	-1	1	0
2	-1	-1	0	1	0	0	-1	0	0	1	-1	1	0	0	1
3	-1	0	-1	0	1	0	0	-1	0	1	0	-1	1	1	0
4	-1	1	0	-1	0	0	0	0	-1	-1	1	0	1	0	1
5	1	1	0	0	0	0	-1	0	1	-1	0	1	-1	0	-1
6	1	0	1	0	0	1	0	-1	0	1	-1	0	-1	-1	0
7	1	0	0	1	0	0	1	0	-1	0	-1	-1	1	0	-1
8	1	0	0	0	1	-1	0	1	0	-1	1	-1	0	-1	0
9	0	0	-1	-1	1	1	-1	0	1	0	0	0	0	-1	1
10	0	-1	1	0	-1	0	1	1	-1	0	0	0	0	-1	1
11	0	-1	-1	1	0	-1	0	1	1	0	0	0	0	1	-1
12	0	1	0	-1	-1	1	1	-1	0	0	0	0	0	1	-1
13	0	-1	0	-1	0	0	1	0	1	1	1	-1	-1	0	0
14	0	0	-1	0	-1	1	0	1	0	-1	-1	1	1	0	0

Table 2. The adjacent matrix of a digraph (15 vertices) .

The adjacent matrix of a digraph which has 18 vertices. See also Figure 11.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0
1	-1	0	1	0	-1	-1	0	0	0	0	1	1	-1	0	0	0	1	0
2	-1	-1	0	1	0	0	-1	0	0	1	-1	0	0	0	0	0	1	1
3	-1	0	-1	0	1	0	0	0	-1	-1	0	0	1	0	0	1	1	0
4	-1	1	0	-1	0	0	0	-1	0	0	1	0	1	0	1	-1	0	0
5	1	1	0	0	0	0	-1	0	1	-1	0	1	-1	-1	0	0	0	0
6	1	0	1	0	0	1	0	-1	0	1	-1	0	0	-1	-1	0	0	0
7	1	0	0	0	1	0	1	0	-1	0	1	-1	0	-1	0	-1	0	0
8	1	0	0	1	0	-1	0	1	0	-1	0	-1	0	0	0	1	0	-1
9	0	0	-1	1	0	1	-1	0	1	0	0	0	1	0	-1	0	0	-1
10	0	-1	1	0	-1	0	1	-1	0	0	0	-1	0	0	1	0	0	1
11	0	-1	0	0	0	-1	0	1	1	0	1	0	0	-1	0	0	-1	1
12	0	1	0	-1	-1	1	0	0	0	-1	0	0	0	1	-1	0	1	0
13	0	0	0	0	0	1	1	1	0	0	0	1	-1	0	-1	-1	-1	0
14	0	0	0	0	-1	0	1	0	0	1	-1	0	1	1	0	-1	0	-1
15	0	0	0	-1	1	0	0	1	-1	0	0	0	0	1	1	0	-1	-1
16	0	-1	-1	-1	0	0	0	0	0	0	0	1	-1	1	0	1	0	1
17	0	0	-1	0	0	0	0	0	1	1	-1	-1	0	0	1	1	-1	0

Table 3. The adjacent matrix of the digraph (18 vertices) .

The adjacent matrix of another digraph has also 18 vertices. See also Figure 12.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0
1	-1	0	1	0	-1	-1	0	0	0	1	0	-1	1	1	0	0	0	0
2	-1	-1	0	1	0	0	-1	0	0	-1	1	0	1	0	1	0	0	0
3	-1	0	-1	0	1	0	0	-1	0	0	1	0	0	0	-1	1	1	0
4	-1	1	0	-1	0	0	0	0	-1	0	0	1	0	1	0	-1	1	0
5	1	1	0	0	0	0	-1	0	1	0	-1	-1	1	0	0	-1	0	0
6	1	0	1	0	0	1	0	-1	0	-1	1	-1	0	0	0	0	-1	0
7	1	0	0	1	0	0	1	0	-1	-1	0	0	0	-1	-1	0	1	0
8	1	0	0	0	1	-1	0	1	0	0	0	0	-1	1	-1	-1	0	0
9	0	-1	1	0	0	0	1	1	0	0	0	-1	0	-1	1	0	0	-1
10	0	0	-1	-1	0	1	-1	0	0	0	0	0	1	0	0	1	-1	1
11	0	1	0	0	-1	1	1	0	0	1	0	0	0	0	0	-1	-1	-1
12	0	-1	-1	0	0	-1	0	0	1	0	-1	0	0	1	1	0	0	1
13	0	-1	0	0	-1	0	0	1	-1	1	0	0	-1	0	0	0	1	1
14	0	0	-1	1	0	0	0	1	1	-1	0	0	-1	0	0	1	0	-1
15	0	0	0	-1	1	1	0	0	1	0	-1	1	0	0	-1	0	0	-1
16	0	0	0	-1	-1	0	1	-1	0	0	1	1	0	-1	0	0	0	1
17	0	0	0	0	0	0	0	0	0	1	-1	1	-1	-1	1	1	-1	0

Table 4. The adjacent matrix of the another digraph (18 vertices).

The adjacent matrix of a digraph which has 20 vertices. See also Figure 13.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	0	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0
1	-1	0	1	0	-1	-1	0	0	0	1	0	-1	1	1	0	0	0	0	0	0
2	-1	-1	0	1	0	0	-1	0	0	-1	1	0	1	0	0	1	0	0	0	0
3	-1	0	-1	0	1	0	0	0	-1	0	0	0	-1	1	0	1	0	0	0	1
4	-1	1	0	-1	0	0	0	-1	0	0	0	1	0	1	0	0	0	1	0	-1
5	1	1	0	0	0	0	-1	0	1	0	-1	-1	1	0	-1	0	0	0	0	0
6	1	0	1	0	0	1	0	-1	0	-1	1	-1	0	0	0	0	-1	0	0	0
7	1	0	0	0	1	0	1	0	-1	0	0	1	0	0	-1	0	-1	0	0	-1
8	1	0	0	1	0	-1	0	1	0	0	0	0	-1	0	-1	0	0	0	-1	1
9	0	-1	1	0	0	0	1	0	0	0	0	-1	0	-1	0	1	1	-1	0	0
10	0	0	-1	0	0	1	-1	0	0	0	0	0	1	0	1	-1	-1	0	1	0
11	0	1	0	0	-1	1	1	-1	0	1	0	0	0	0	-1	0	0	-1	0	0
12	0	-1	-1	1	0	-1	0	0	1	0	-1	0	0	1	0	0	0	0	1	0
13	0	-1	0	-1	-1	0	0	0	0	1	0	0	-1	0	0	1	0	1	1	0
14	0	0	0	0	0	1	0	1	1	0	-1	1	0	0	0	0	-1	-1	-1	0
15	0	0	-1	-1	0	0	0	0	0	-1	1	0	0	-1	0	0	1	0	1	1
16	0	0	0	0	0	0	1	1	0	-1	1	0	0	0	1	-1	0	-1	0	-1
17	0	0	0	0	-1	0	0	0	0	1	0	1	0	-1	1	0	1	0	-1	-1
18	0	0	0	0	0	0	0	0	1	0	-1	0	-1	-1	1	-1	0	1	0	1
19	0	0	0	-1	1	0	0	1	-1	0	0	0	0	0	0	-1	1	1	-1	0

Table 5. The adjacent matrix of the digraph (20 vertices).