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A REMARK ON THE SPECTRUM OF MAGNETIC LAPLACIAN ON A GRAPH

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Abstract. For the magnetic Laplacian on an abelian covering graph, we show the Bloch property. In addition, we investigate several spectral properties for the maximal abelian covering graph of some finite graph.

1. Introduction

The concept of a discrete analogue of the magnetic Schrödinger operator was originally introduced for the 2-dimensional lattice $\boldsymbol{Z}^2$ ([6]) as the classical Harper operator which is a discrete model with a uniform magnetic field in $\boldsymbol{R}^2$. After that, many kinds of generalization of Harper operator have been introduced and studied (cf.[13, 17]). We also studied the spectral properties of a kind of discrete magnetic Schrödinger operators ([7, 8]). In this note, as a continuation of our previous works [7, 8], we give some further results on the spectral properties in terms of geometries of a graph.

Let $G=(V(G),A(G))$ be a connected, locally finite graph. Here $A(G)$ is the set of oriented edges of G. For a given 1-form θ, we define a discrete magnetic Schrödinger operators on $\ell^2(V(G))$ as follows:

$$H_{\theta,G}f(x)=\sum_{e\in A_x(G)}p(e)\exp\left(\sqrt{-1}\theta(e)\right)f(t(e))-f(x), \tag{1.1}$$

where p is assumed to be a transition probability that has a reversible measure m and $A_x(G)=\{e\in A(G)|o(e)=x\}$. More precise description of our setting and notation will be given in Appendix.

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Now, we put assumptions for periodicity as follows:

ASSUMPTION 1. (Periodicity for graphs) A graph G has a group Γ of automorphisms which is finitely generated and acts on G freely; $\sigma x \neq x$ and $\sigma e \neq \bar{e}$ for any $x \in V(G)$, $e \in A(G)$ and $\sigma(\neq 1) \in \Gamma$. Moreover, the quotient graph $M = \Gamma \backslash G$ is finite.

ASSUMPTION 2. (Periodicity for operators) The function p, m and θ on $G = (V(G), A(G))$ are invariant under the Γ-action.

Under these assumptions, a graph G can be considered as a normal covering graph of a finite graph M with the covering transformation group Γ, and $H_{\theta,G}$ as the lift of $H_{\theta,M}$ by the natural projection $\pi : G \to M$. We call G an abelian covering graph of M if Γ is abelian. Then, we give the following theorem:

THEOREM A. *Let G be an abelian covering of a finite graph M. Then, the lift $H_{\theta,G}$ has the Bloch property. Namely, the set of ℓ^∞-eigenvalues coincides with the set of ℓ^2-spectrum.*

This is a discrete analogue of a result for Laplacian on a manifold [10]. Also we will give some informations about the bottom of the spectrum in Section 2 and Appendix.

If the transformation group Γ is $H_1(M, \boldsymbol{Z})$, a graph G is called the maximal abelian covering graph, or the homology universal covering graph, of a finite graph M; we denote it by M^{ab}. Maximality of coverings gives us various special properties.

The first property is as follows:

THEOREM B. *Let G be M^{ab} for a finite graph M. Then, for any 1-form $\theta \in C^1(M, \boldsymbol{R})$, the lift $H_{\theta,G}$ of $H_{\theta,M}$ is unitarily equivalent to the Laplacian $\Delta_G = H_{0,G}$.*

We will also remark on other properties coming from the maximality of abelian coverings in Section 3.

The second one is concerned with the following problem which is a discrete analogue of one of the problems raised in [10].

PROBLEM C. *For $\Delta_G = H_{0,G}$, we set $p(e) = (\deg_G o(e))^{-1}$, that is,*

$$\Delta_G f(x) = (\deg_G x)^{-1} \sum_{e \in A_x(G)} f(t(e)) - f(x). \tag{1.2}$$

In general, $Spec(-\Delta_G)$ is contained the closed interval $[0,2]$. Does it always hold that $Spec(-\Delta_G) = [0,2]$ when $G = M^{ab}$ for a finite graph M?

So far, we have not obtained complete answers but some affirmative results.

This note is organized as follows. We will prove Theorem A with reviewing our results [7, 8] in Section 2. The proof of Theorem B and some topics concerning about maximal abelian covering graphs, and some results for Problem C will be shown in Section 3. Our setting and some remarks will be given in Appendix.

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2. The Bloch property

In [7], we define a decomposition D_G of $V(G)$ called *a 1-dim decomposition* and a growth function for D_G called *a boundary area growth*:

DEFINITION 2.1. A decomposition D_G is called a 1-dim decomposition if it satisfies the following:

a) suppose $V(G) = \cup_{k=0}^{\infty} V_k$ where V_k is a non-empty finite set of vertices for every k;

b) V_k's are mutually disjoint;

c) for any $e \in A(G)$, $o(e) \in V_k$ implies $t(e) \in V_{k-1} \cup V_k \cup V_{k+1}$.

For a 1-dim decomposition D_G, put

$$\begin{aligned} \partial B_n &= \{ e \in A(G) \mid o(e) \in V_n, t(e) \in V_{n+1} \}, \\ Area\,(\partial B_n) &= \sum_{e \in \partial B_n} m(o(e))p(e), \end{aligned} \tag{2.1}$$

and define the boundary area growth for D_G by

$$\mu(D_G) = \liminf_{n \to \infty} \frac{1}{n} \log Area\,(\partial B_n). \tag{2.2}$$

In terms of the boundary area growth, we showed a criterion for a weak version of Bloch property.

THEOREM 2.2. ([7]) *Suppose that an infinite graph G has a 1-dim decomposition D_G whose boundary area growth $\mu(D_G) \le 0$. Then, all ℓ^∞-eigenvalues of $-H_{\theta,G}$ are contained in its ℓ^2-spectrum set for any 1-form θ on $A(G)$.*

Now, let G be a graph satisfying Assumption 1 with an *abelian* group Γ and $H_{\theta,G}$ an operator satisfying Assumption 2. Then, we get the operator $H_{\theta,G}$ has the Bloch property (Theorem A in Section 1):

PROPOSITION 2.3. *Let G be an abelian covering of a finite graph M and $H_{\theta,G}$ be the lift of $H_{\theta,M}$ for any 1-form θ on M. Then, the equation $-H_{\theta,G}u = \lambda u$ has a non-trivial bounded solution if and only if λ is the spectrum of $-H_{\theta,G}$.*

Proof. For a fixed vertex $x_0 \in V(G)$, set $V_k = \{x \in V(G) \,|\, dist(x, x_0) = k\}$. Here the distance $dist(x, y)$ is the number of edges in the shortest path joining x and y. This gives the most typical 1-dim decomposition D_G. If a group Γ is abelian, it is of subexponential growth [15] and then $\mu(D_G) \leq 0$ for the above D_G (cf. [7]). Therefore, the "only if "part is a direct consequence of Theorem 2.2.

To show the "if"part, we will equip a representation-theoretic technique employed in [8, 11]. Using it, we imitate the procedure in the continuous case [10]. Let $\widehat{\Gamma}$ be the group of unitary characters of the covering transformation group Γ. For any $\chi \in \widehat{\Gamma}$, we put

$$\ell^2_\chi = \{ f : V(G) \to \boldsymbol{C} \mid f(\sigma x) = \chi(\sigma) f(x) \text{ for any } \sigma \in \Gamma \} \qquad (2.3)$$

with the inner product $\langle f, g \rangle_\chi = \sum_{x \in \mathcal{D}} f(x)\overline{g(x)} m(x)$, where $\mathcal{D}$ is a fundamental set of $V(G)$ for Γ. Since $H_{\theta,G}$ commutes with the Γ-action, we can restrict it to ℓ^2_χ, and denote the restriction operator by $H_{\theta,\chi} = H_{\theta,G}|_{\ell^2_\chi}$. Then, it follows from the theory of direct integral that

$$Spec(-H_{\theta,G}) = \bigcup_{\chi \in \widehat{\Gamma}} Spec(-H_{\theta,\chi}). \qquad (2.4)$$

Let $\pi : G \to M$ be the natural projection. For a function f on M, define the lift $\widetilde{f}$ to G by $\widetilde{f} = f \circ \pi$. We can find a unitary map $U : \ell^2(V(M)) \to \ell^2_\chi$ such that $Uf = \widetilde{f} s_0$, where s_0 is a bounded function on $V(G)$. Denote by $\mathcal{L}_{\theta,\chi}$ the operator $U^{-1} H_{\theta,\chi} U : \ell^2(V(M)) \to \ell^2(V(M))$. Then, we have

$$Spec(-H_{\theta,G}) = \bigcup_{\chi \in \widehat{\Gamma}} Spec(-\mathcal{L}_{\theta,\chi}). \qquad (2.5)$$

Detailed explanations should be referred to [8, 11, 16].

If λ is in the spectrum of $-H_{\theta,G}$, then λ is an eigenvalue of $-\mathcal{L}_{\theta,\chi}$ for some $\chi \in \widehat{\Gamma}$. It follows that there exists a non-zero function $f \in \ell^2(V(M))$ such that $-\mathcal{L}_{\theta,\chi} f = \lambda f$. Then $g = Uf = \widetilde{f} s_0 \in \ell^2_\chi$ is a bounded function on G satisfying $-H_{\theta,G}\, g = \lambda g$. This completes the proof. ∎

The next proposition is also a discrete and magnetic analogue of a result in [10].

PROPOSITION 2.4. *Let G be an abelian covering of a finite graph M and $H_{\theta,G}$ the lift of $H_{\theta,M}$. For a sufficiently small $\epsilon > 0$ and for any 1-form θ such that $\|\theta\|_A < \epsilon$, there exists $\lambda(> \lambda_G(\theta))$ such that the interval $[\lambda_G(\theta), \lambda]$ is contained in the continuous spectrum, where $\|\cdot\|_A$ and $\lambda_G(\theta)$ are the same as in Theorem 4.1 in Appendix.*

Proof. For $\chi \in \widehat{\Gamma}$, denote by $\lambda_0(\theta,\chi)$ the smallest eigenvalue of the operator $-\mathcal{L}_{\theta,\chi}$ and put $\lambda' = \max_{\chi\in\widehat{\Gamma}_1} \lambda_0(\theta,\chi)$, where $\widehat{\Gamma}_1$ is the component of $\widehat{\Gamma}$ containing the trivial character $\mathbf{1}$. It follows from Corollary 4.2 in Appendix (cf.[8]) and the analyticity of $\lambda_0(0,\chi)$ ([8, 11]) in χ that, for any 1-form θ with the sufficiently small $\|\theta\|_A$, there exists a neighbourhood $U(\mathbf{1})$ of the trivial character $\mathbf{1}$ such that $\lambda_0(\theta,\chi)$ is analytic in $\chi \in U(\mathbf{1})$, $\lambda_G(\theta) = \inf_{\chi\in U(\mathbf{1})} \lambda_0(\theta,\chi)$ and the Hessian of $\lambda_0(\theta,\chi)$ does not vanish in $U(\mathbf{1})$. Thus we have $\lambda' > \lambda_G(\theta)$ since $\lambda_0(\theta,\chi)$ is not a constant function, and that the interval $[\lambda_G(\theta), \lambda']$ is contained in the spectrum.

If λ is an eigenvalue of $-H_{\theta,G}$, it follows from the analyticity of $-\mathcal{L}_{\theta,\chi}$ in χ that there exists a component A of $\widehat{\Gamma}$ such that λ is an eigenvalue of $-\mathcal{L}_{\theta,\chi}$ for any $\chi \in A$. Assume that there exists a sequence $\{\lambda_n\}$ of eigenvalues of $-H_{\theta,G}$ accumulating to $\lambda_G(\theta)$. So, we can choose a subsequence $\{\lambda_{n_i}\}$ and a character χ such that λ_{n_i} are eigenvalues of $-\mathcal{L}_{\theta,\chi}$. This contradicts the finiteness of the number of eigenvalues of $-\mathcal{L}_{\theta,\chi}$. ∎

3. Maximal abelian covering

In this section, let us put $\Gamma = H_1(M, \mathbf{Z})$ under Assumption 1 and 2 in Section 1. Before proving Theorem B in Section 1, we give some necessary definitions in our discussions as follows: A path $p = (e_1 e_2 \cdots e_n)$ of length n is a sequence of oriented edges with $t(e_i) = o(e_{i+1})$ for $i = 1, \ldots, n-1$. For a path p, we define

$$\int_p \theta = \sum_{i=1}^{n} \theta(e_i). \tag{3.1}$$

Moreover, for a closed path c, which is a path such that $t(e_n) = o(e_1)$, we call $\int_c \theta$ the magnetic flux of θ through c.

THEOREM 3.1. *Let G be the maximal abelian covering of a finite graph M. Then, for any 1-form $\theta \in C^1(M, \mathbf{R})$, the lift $H_{\theta,G}$ of $H_{\theta,M}$ is unitarily equivalent to the Laplacian $\Delta_G = H_{0,G}$. Moreover, for any 1-form $\theta \in C^1(M, \mathbf{R})$, the spectrum set $Spec(-H_{\theta,M})$ is contained in $Spec(-\Delta_G)$.*

Proof. The second part is obvious from the first part and Theorem A since the lift of an eigenfunction of $H_{\theta,M}$ is bounded. So we shall show only the first part of this theorem.

Let $\theta \in C^1(M, \boldsymbol{R})$ and $\widetilde{\theta}$ its lift to $G = M^{ab}$ by the covering map $\pi : G \to M$. For any closed path c in G, put c_0 the image of c by π. Regarding c_0 as a 1-cycle, we have c_0 is null-homologous in M since the lift of c_0 is closed in G and the transformation group is $H_1(M, \boldsymbol{Z})$. Then we obtain

$$\int_c \widetilde{\theta} = \int_{c_0} \theta = 0, \tag{3.2}$$

for any closed path c in G. By Proposition 3.2, this implies that $H_{\theta,G}$ is unitarily equivalent to the Laplacian $\Delta_G = H_{0,G}$ on G. ■

PROPOSITION 3.2. (cf.[7]) *Let $\theta_1, \theta_2 \in C^1(G, \boldsymbol{R})$ be 1-forms on G. If the magnetic flux of θ_1 equals to the one of θ_2 in modulo 2π for every closed path of G, then $Spec(-H_{\theta_1,G}) = Spec(-H_{\theta_2,G})$.*

The proof of Theorem 3.1 is also applicable to the following two propositions.

PROPOSITION 3.3. *Let G be the maximal abelian covering of a finite graph M. Then, G is bipartite, that is, G has no closed path of odd length.*

Proof. We choose a 1-form $\theta \in C^1(M, \boldsymbol{R})$ such that $|\theta(e)| = \alpha \in \boldsymbol{R}\backslash\{0\}$ and $\theta(e) = -\theta(\overline{e})$ for every $e \in A(M)$. Assume that G is not bipartite, then G has a closed path c of odd length. For the lift $\widetilde{\theta}$ of θ to G, it is easy to see that $\int_c \widetilde{\theta} \neq 0$ from the odd parity of the length c. This contradicts the equality (3.2) since G is the maximal abelian covering of M. ■

Remark. M. Kotani and T. Sunada proved this proposition independently [12].

PROPOSITION 3.4. *Let G be the maximal abelian covering of a finite graph M. Then, for any transition probability $p : A(M) \to \boldsymbol{R}^+$, its lift $\widetilde{p}$ on G is reversible even if p is not reversible on M.*

Proof. We set the 1-form ω by

$$\omega(e) = \log p(e) - \log p(\overline{e}) \tag{3.3}$$

for any $e \in A(M)$ and $\widetilde{\omega} = \omega \circ \pi$. It is sufficient to show that

$$\int_c \widetilde{\omega} = 0 \tag{3.4}$$

for any closed path $c = (e_1 e_2 \cdots e_n)$ in $G = M^{ab}$. See Proposition 4.3 in Appendix.

Now it follows from (3.2) that

$$\int_c \tilde{\omega} = \int_{c_0=\pi(c)} \omega = 0, \tag{3.5}$$

since c_0 is null-homologous in M. Thus we get the conclusion. ■

Now, we give another kind of application of Theorem 3.1.

From now on, we put

$$p(e) = (\deg_G o(e))^{-1} \tag{3.6}$$

for any $e \in A(G)$, which gives the transition probability of a simple random walk on G. It is obvious that the function m, $m(x) = \deg_G x = \#A_x(G)$ for any $x \in V(G)$, is a reversible measure for the above p. Then, for a function f on $V(G)$,

$$H_{\theta,G} f(x) = (\deg_G x)^{-1} \sum_{e \in A_x(G)} \exp\left(\sqrt{-1}\theta(e)\right) f(t(e)) - f(x). \tag{3.7}$$

Also, the Laplacian Δ_G on G is as follows:

$$\Delta_G f(x) = (\deg_G x)^{-1} \sum_{e \in A_x(G)} f(t(e)) - f(x), \tag{3.8}$$

which is one of the standard Laplacians on graphs (cf.[3, 4, 5]). Remark that, under Assumption 1, this Laplacian Δ_G automatically satisfies Assumption 2.

We have a conjecture (Problem C in Section 1) which is a discrete analogue of one of the problems in [10]:

CONJECTURE 3.5. *Assume that $G = M^{ab}$ for a finite graph M. Then, $Spec(-\Delta_G) = [0, 2]$.*

This is still open up to now, but we shall give some partial solutions using Theorem 3.1.

PROPOSITION 3.6. *Under the assumption of Conjecture 3.5, let $\deg_G x \in 2\mathbf{N}$ for any vertex $x \in V(G)$. Then, $Spec(-\Delta_G) = [0, 2]$.*

Proof. Let us first recall a notion in graph theory. A closed path $c = (e_1 e_2 \ldots e_n)$ in a finite graph M is called an *Euler circuit* if $n = \#A(M)/2$ and, for each $e \in A(M)$, there exists $i\,(1 \le i \le n)$ such that e_i or $\overline{e_i}$ equals to e. It is known as a classical fact in graph theory that a finite graph M has an Euler circuit if and only if the degree of each vertex is even (cf.[1, 2]).

Now, since the quotient graph M also satisfies $\deg_M x \in 2\boldsymbol{N}$ for any vertex $x \in V(M)$, so M has an Euler circuit $c = (e_1 e_2 \cdots e_n)$. Then, we put a 1-form $\theta \in C^1(M, \boldsymbol{R})$ as $\theta(e_i) = \alpha \in \boldsymbol{R}$ and $\theta(\overline{e_i}) = -\alpha$ for every i. It is easy to see that $\{e_i, \overline{e_i}\}_{i=1}^n = A(M)$ and

$$\#\{\,\theta(e) = \alpha \mid e \in A_x(M)\,\} = \#\{\,\theta(e) = -\alpha \mid e \in A_x(M)\,\} = (\deg_M x)/2 \tag{3.9}$$

for every vertex $x \in V(M)$. Denote by $\mathbf{1}$ a constant function on $V(M)$ such that $\mathbf{1}(x) = 1$. For a 1-form θ set in the above, by (3.9),

$$\begin{aligned} -H_{\theta,M}\mathbf{1}(x) &= -(\deg_M x)^{-1} \sum_{e \in A_x(M)} \exp\left(\sqrt{-1}\theta(e)\right) \mathbf{1}(t(e)) + \mathbf{1}(x) \\ &= 1 - (\deg_M x)^{-1} \cdot (\deg_M x)/2 \cdot \left(\exp\left(\sqrt{-1}\alpha\right) + \exp\left(-\sqrt{-1}\alpha\right)\right) \\ &= (1 - \cos\alpha)\mathbf{1}(x). \end{aligned} \tag{3.10}$$

This implies that the quantity $1 - \cos\alpha$ is an eigenvalue of $-H_{\theta,M}$. Thus, by Theorem 3.1, $1 - \cos\alpha \in Spec(-\Delta_G)$ for any $\alpha \in \boldsymbol{R}$, and we get the conclusion. ■

By an argument similar to that in the above, we can get some results. Before stating them, we recall the notion of k-factor.

DEFINITION 3.7. (k-factor cf.[1, 2]) Let $G = (V(G), E(G))$ be a finite graph. Here $E(G)$ is the set of unoriented edges. A graph $H = (V(H), E(H))$ is called a spanning subgraph or a *factor* of G if $V(H) = V(G)$ and $E(H) \subset E(G)$. In addition, a factor H is called a *k-factor* if $\deg_H x = k$ for every vertex $x \in V(H)$.

Of course, a k-factor H may not be connected even if G is connected.

PROPOSITION 3.8. *Under the assumption of Conjecture 3.5, let M be a $(2d+1)$-regular graph having a 1-factor $(d \geq 1)$. Then, $Spec(-\Delta_G) = [0, 2]$.*

Proof. Let H be a 1-factor of M. Then, $M' = (V(M), E(M) \backslash E(H))$, which may consist of some connected components, is a $2d$-regular subgraph of M. So each component M_j of M' has an Euler circuit $c_j = (e_{j1} e_{j2} \cdots e_{jn_j})$ in M_j, and we can set a 1-form $\theta \in C^1(M, \boldsymbol{R})$ as $\theta(e_{ji}) = -\theta(\overline{e_{ji}}) = \alpha$ for every i, j, and $\theta(\cdot) = 0$ for other edges. It is easy to see that

$$\#\{\,\theta(e) = \alpha \mid e \in A_x(M)\,\} = \#\{\,\theta(e) = -\alpha \mid e \in A_x(M)\,\} = d \tag{3.11}$$

for every vertex $x \in V(M)$. As is in the proof of Proposition 3.6,

$$-H_{\theta,M}\mathbf{1}(x) = -(2d+1)^{-1} \sum_{e\in A_x(M)} \exp\left(\sqrt{-1}\theta(e)\right)\mathbf{1}(t(e)) + \mathbf{1}(x)$$
$$= \frac{2d}{2d+1}(1-\cos\alpha)\mathbf{1}(x). \tag{3.12}$$

Thus we have $[0, 4d/(2d+1)] \in Spec(-\Delta_G)$, where $4d/(2d+1) > 1$. The conclusion follows from the bipartiteness of M^{ab} (Proposition 3.3) and the following fact. ■

PROPOSITION 3.9. (cf.[7]) *If a graph G is bipartite, then $Spec(-H_{\theta,G})$ is symmetric with respect to 1.*

Remark. For example, if M satisfies one of the following, the conclusion of Conjecture 3.5 is true.

(1) M is a $2d$-regular graph. (It just follows from Proposition 3.6.)

(2) M is a bipartite $(2d+1)$-regular graph. (Such a graph has a 1-factor by Hall's theorem (cf.[1, 2]).)

(3) M is a non-bipartite $2d$-edge-connected $(2d+1)$-regular graph. (It is known that such a graph has a 1-factor.)

(4) M is a non-bipartite $(2d+1)$-regular graph satisfying that 1) M has a $(2k+1)$-factor such that $2k+1 \leq d$, or, 2) M has a $2k$-factor such that $2k \geq d+1$, where $k \geq 1$. (We can obtain this by imitating the procedure in the proof of Proposition 3.8. It is known that M does not always have a 1-factor even if M has a k-factor $(k \geq 2)$. There exist many criteria for the existence of a k-factor (cf.[1, 2, 9]).)

4. Appendix

Let $G = (V(G), E(G))$ be a connected, locally finite graph, where $V(G)$ is the set of its vertices and $E(G)$ the set of its unoriented edges. A graph G may have self-loops and multiple edges. We say G is infinite (finite) if $\#V(G)$ is countably infinite (finite). Considering each edge in $E(G)$ to have two orientations, we introduce the set of all oriented edges; we denote it by $A(G)$. For an edge $e \in A(G)$, the origin vertex and the terminus one of e are denoted by $o(e)$ and $t(e)$, respectively. The inverse edge of e is denoted by $\overline{e}$. Let $p : A(G) \to \boldsymbol{R}^+$ be a transition probability such that, for every vertex x,

$$\sum_{e\in A_x(G)} p(e) = 1, \tag{4.1}$$

where $A_x(G) = \{e \in A(G) | o(e) = x\}$. We assume that p is reversible, that is, there exists a positive valued function $m : V(G) \to \boldsymbol{R}^+$ such that

$$m(o(e))p(e) = m(t(e))p(\bar{e}) \tag{4.2}$$

for every oriented edge $e \in A(G)$. The function m is called a reversible measure for p and it is unique, if exists, up to a multiple constant (cf. Proposition 4.3).

Set the Hilbert space

$$\ell^2(V(G)) = \{f : V(G) \to \boldsymbol{C} \mid \sum_{x \in V(G)} |f(x)|^2 m(x) < \infty\} \tag{4.3}$$

with the inner product $\langle f, g\rangle_V = \sum_{x \in V(G)} f(x)\overline{g(x)}m(x)$. Put

$$C^1(G, \boldsymbol{R}) = \{\theta : A(G) \to \boldsymbol{R} \mid \theta(\bar{e}) = -\theta(e)\} \tag{4.4}$$

and call an element of $C^1(G, \boldsymbol{R})$ a 1-form on G. For a fixed 1-form θ, we define the self-adjoint operator $H_{\theta,G} : \ell^2(V(G)) \to \ell^2(V(G))$ by

$$H_{\theta,G} f(x) = \sum_{e \in A_x(G)} p(e) \exp\left(\sqrt{-1}\theta(e)\right) f(t(e)) - f(x). \tag{4.5}$$

We call it the discrete magnetic Schrödinger operator, or often simply, magnetic Laplacian. We can easily see this operator $H_{\theta,G}$ is a bounded self-adjoint operator on $\ell^2(V(G))$ and the spectrum of $-H_{\theta,G}$, $Spec(-H_{\theta,G})$, is a closed subset in $[0, 2]$.

Under Assumption 1 and 2 in Section 1, we may regard a graph G as a normal covering graph of a finite graph M with the covering transformation group Γ, and $H_{\theta,G}$ as the lift of $H_{\theta,M}$ by the natural projection $\pi : G \to M$.

We call G an *abelian* covering graph of M if Γ is an abelian group. In particular, if Γ is $H_1(M, \boldsymbol{Z})$, the covering graph of M is said to be the *maximal abelian* covering graph of M and denoted by M^{ab}. For any abelian covering graph G of M, there exists a covering map $M^{ab} \to G$ which factorizes the covering map $M^{ab} \to M$. The transformation group Γ_1 of the covering map $M^{ab} \to G$ is a subgroup of $H_1(M, \boldsymbol{Z})$ and Γ is isomorphic to $H_1(M, \boldsymbol{Z})/\Gamma_1$.

The following is one of the results in a previous paper [8]:

THEOREM 4.1. ([8]) *Let $\lambda_G(\theta) = \inf Spec(-H_{\theta,G})$. If Γ is abelian, then $\lambda_G(\cdot)$ is real analytic in a neighbourhood of $0 \in C^1(M, \boldsymbol{R})$ and $\nabla\lambda_G(0) = 0$. In particular, if θ is a harmonic 1-form, that is, $\sum_{e \in A_x(M)} p(e)\theta(e) = 0$ for any $x \in V(M)$, we have the following expression*

$$\mathrm{Hess}\lambda_G(0)(\theta, \theta) = \frac{2}{Vol(M)} d(\theta, T_1\widehat{\Gamma})^2, \tag{4.6}$$

where $Vol(M) = \sum_{x \in V(M)} m(x)$, $\widehat{\Gamma}$ *is the group of unitary characters of* Γ, $T_{\mathbf{1}}\widehat{\Gamma}$ *is the tangent space at the trivial character* $\mathbf{1}$, *and* $d(\theta, T_{\mathbf{1}}\widehat{\Gamma}) = \inf\{||\theta - \eta||_A \mid \eta \in \widehat{\Gamma}\}$. *The norm* $||\cdot||_A$ *on* $C^1(M, \boldsymbol{R})$ *is defined by*

$$||\theta||_A = \frac{1}{2} \sum_{e \in A(M)} |\theta(e)|^2 m(o(e)) p(e). \tag{4.7}$$

Here $\widehat{\Gamma}$ is identified with a subset of the Jacobian torus $J(M) = H^1(M, \boldsymbol{R})/H^1(M, \boldsymbol{Z})$ by the canonical injection and $d(\theta, \widehat{\Gamma})$ is the natural distance on $J(M)$.

COROLLARY 4.2. *Suppose that* G *is finite and* Γ *is trivial. (In this case,* G *coincides with* M*). If* θ *is a harmonic* 1*-form, we have*

$$\mathrm{Hess}\lambda_M(0)(\theta, \theta) = \frac{2}{Vol(M)} ||\theta||_A^2. \tag{4.8}$$

Remark that by the discrete version of Kodaira-Hodge theorem and Proposition 3.2, we can choose a harmonic 1-form θ' for any θ on M such that $H_{\theta,M}$ and $H_{\theta',M}$ are unitarily equivalent. A representation-theoretic technique was employed to prove this theorem. Details can be seen in [8].

Finally we give a criterion for the existence of a reversible measure for p, which is used in the proof of Proposition 3.4.

PROPOSITION 4.3. *Let* p *be a transition probability on* G *and set the 1-form* ω *by*

$$\omega(e) = \log p(e) - \log p(\overline{e}) \tag{4.9}$$

for any $e \in A(G)$*. Then, there exists a reversible measure* m *on* $V(G)$ *if and only if*

$$\int_c \omega = 0 \tag{4.10}$$

for any closed path $c = (e_1 e_2 \dots e_n)$ *in* G*. If exists,* m *is unique up to a multiple constant.*

Proof. To show the "if"part, we fix an arbitrary vertex x_0 and put

$$m(x) = \exp\left(\int_{p(x_0,x)} \omega\right), \tag{4.11}$$

which is independent of the choice of a path $p(x_0, x)$ joining from x_0 to x by (4.10). It is easy to check that the function m satisfies (4.2). If both m_1 and m_2

satisfy (4.2), then it holds that $m_1(t(e))/m_2(t(e)) = m_1(o(e))/m_2(o(e))$ for any $e \in A(G)$. Since G is connected, m_1/m_2 is a constant function.

Conversely, if m satisfies (4.2), then the ω defined in (4.9) satisfies

$$\omega(e) = \log m(t(e)) - \log m(o(e)), \tag{4.12}$$

for any $e \in A(G)$. Hence we obtain $\int_c \omega = 0$ for any closed path c. ∎

Remark. The equality

$$\prod_{i=1}^{n} p(e_i) = \prod_{i=1}^{n} p(\overline{e_i}) \tag{4.13}$$

is obviously equivalent to (4.10). The expression (4.13) instead of (4.10) is used in the same statement as the above, for example, in [14].

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