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# THE "WALK CALCULUS" OF REGULAR LIFTS OF GRAPH AND MAP AUTOMORPHISMS

By

JOZEF ŠIRÁŇ

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**Abstract.** Regular coverings of graphs have been treated in the literature in various ways. In this paper we develop an elementary theory of regular lifts of graphs and automorphisms by means of walks and their voltages, building thus up certain type of "walk calculus" associated with regular coverings.

## 1. Introduction

In the theory of graphs and maps (that is, cellular embeddings of graphs) one often tries to produce objects that are highly symmetric. The most frequently used method for such a purpose seems to be the construction of *regular covering spaces*. Originating in algebraic topology long time ago, the construction was used in the sixties (in disguise, however) in the solution of the famous Heawood map colouring problem. Since then it was combinatorialized in various ways, and we just refer to [4] for a survey. Besides various applications in calculating the genus of a graph (see [4] again), the covering construction was also successfully used in other areas. To list a few, we mention the construction of highly symmetric trivalent graphs [2], regular maps (which, in a sense, are the "most symmetric" maps; see e.g. [3, 8]), regular maps with forbidden external automorphisms [1], and finally the largest known vertex-transitive graphs of diameter 2 and given degree [7].

A very general approach (based on group actions and their morphisms) to coverings of topological spaces with applications to graphs can be found (together with an extensive bibliography) in [6]. For a treatment of graph coverings aiming towards their enumeration we refer to the recent paper [5] and references therein. Nevertheless, the theory associated with regular coverings of graphs can be developed on a purely elementary level, considering walks in voltage graphs. Elements of the "walk calculus" that arises this way have proved useful in all the applications of the covering graph construction mentioned above. The purpose

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of this paper is to further develop the ideas of such a "walk calculus", continuing thereby the research initiated in [3].

The paper is organized as follows. We first review some concepts necessary for our exposition, such as graphs, walks, voltage assignments on graphs, lifts of graphs and regular coverings, lifts of automorphisms, and so on. In Section 3 we revisit a necessary and sufficient condition for a group of graph automorphisms to have a lift and discuss in detail the associated group homomorphisms. Sometimes the lift of a group of automorphisms of a graph turns out to be a split extension of the voltage group by the original group; necessary and sufficient conditions for this to happen are given in Section 4. The final Section 5 is devoted to discussing the limitations to possible extensions of the semidirect product results to lifts of automorphism groups of regular maps. Throughout we develop a number of formulae involving walks and voltages; this is what we call the "walk calculus" associated with regular coverings of graphs.

## 2. Preliminaries

In what follows we give a brief introduction to graphs, voltage assignments, and regular coverings; for many more details the reader is invited to consult the excellent book [4]. All graphs and groups considered here will be finite.

As usual in topological graph theory, with every edge of a graph $\Gamma$ we will associate a pair of oppositely directed edges called *arcs*; if $x$ is an arc with initial vertex $u$ and terminal vertex $v$ then we write $u = i(x)$ and $v = t(x)$. The symbol $x^{-1}$ will denote the reverse arc to $x$, with $i(x^{-1}) = t(x)$ and $t(x^{-1}) = i(x)$. We do not exclude the case when $u = v$, i.e., we allow lopps (as well as parallel edges) in our graphs. A *walk* in $\Gamma$ is a sequence $x_1x_2\ldots x_k$ of arcs of $\Gamma$ such that $t(x_j) = i(x_{j+1})$ for $1 \leq j < k$. The *reverse walk* to $W$ is the walk $W^{-1} = x_k^{-1}\ldots x_2^{-1}x_1^{-1}$. A walk $W = x_1x_2\ldots x_k$ is said to be a $u \to v$ *walk* if $i(x_1) = u$ and $t(x_k) = v$; in addition, the walk will be called *closed* and *$u$-based* if $u = v$. For the sake of convenience we admit at each vertex $u$ also the *trivial* closed walk consisting only of $u$ (and no arcs at all). If $U_1 = y_1y_2\cdots y_m$ and $U_2 = z_1z_2\ldots z_n$ are two walks in $\Gamma$ such that $t(y_m) = i(z_1)$ then their *concatenation* is the walk $U_1U_2 = y_1y_2\cdots y_m z_1z_2\ldots z_n$. In connection with the "walk calculus" we intend to develop later, note that $(U_1U_2)^{-1} = U_2^{-1}U_1^{-1}$. Also, if $A$ is an arbitrary automorphism of $\Gamma$ and if $W = x_1x_2\ldots x_k$ is a walk in $\Gamma$ then the image of $W$ under $A$ is the walk $A(W) = A(x_1)A(x_2)\ldots A(x_k)$. The identities $A(W^{-1}) = (A(W))^{-1}$ and $A(U_1U_2) = A(U_1)A(U_2)$ are obvious. For brevity we sometimes write $AW$ instead of $A(W)$.

Let $\Gamma$ be a connected graph and let $H$ be a group. We say that a mapping $\alpha: D(\Gamma) \to H$ is a *voltage assignment* if $\alpha(x^{-1}) = (\alpha(x))^{-1}$ for each arc

$x \in D(\Gamma)$, that is, if mutually reverse arcs are assigned mutually inverse elements in $H$. We briefly say that $\alpha$ is a voltage assignment *on* $\Gamma$ *in* $H$ and call $H$ the *voltage group*. For any walk $W = x_1x_2\dots x_k$ in $\Gamma$ the *net voltage* of $W$ is the product $\alpha(W) = \alpha(x_1)\alpha(x_2)\dots\alpha(x_k)$; observe that $\alpha(W^{-1}) = (\alpha(W))^{-1}$. The voltage assignment $\alpha$ is said to be *proper* if for some (and hence each) vertex $u$ of $\Gamma$ and for each $h \in H$ there exists a closed $u$-based walk in $\Gamma$ such that $\alpha(W) = h$. It is easy to see that if $\alpha$ is proper then for any pair of vertices $u, v \in V(\Gamma)$ and for any $h \in H$ there exists a $u \to v$ walk $U$ in $\Gamma$ with $\alpha(U) = h$; we will often make use of this fact without an explicit reference.

Let $\Gamma$ be a connected graph endowed with a voltage assignment $\alpha$ in a group $H$. The *lift* $\Gamma^\alpha$ is the graph with vertex set $\{v_h;\ v \in V(G),\ h \in H\}$ and arc set $\{x_h;\ x \in D(\Gamma),\ h \in H\}$, where an arc $x_h$ emanates from the vertex $i(x)_h$ and terminates at the vertex $t(x)_{h\alpha(x)}$. Note that reverse arcs in the lift are given by $(x_h)^{-1} = (x^{-1})_{h\alpha(x)}$. The lift $\Gamma^\alpha$ is a *regular covering graph* of $\Gamma$, and the mapping $\pi:\ D(\Gamma^\alpha) \to D(\Gamma)$ which erases subscripts, that is, $\pi(x_h) = x$ for each $x \in D(\Gamma)$ and each $h \in H$, is a graph homomorphism called the *(regular) covering projection*. It is easy to see that the lift $\Gamma^\alpha$ is a connected graph if and only if the voltage assignment $\alpha$ is proper.

Along with lifting graphs we shall also be interested in lifting graph automorphisms. Let again $\Gamma$ be a connected graph and let $\alpha$ be a proper voltage assignment on $\Gamma$ in a group $H$. Consider an automorphism $A$ of the graph $\Gamma$. We say that $A$ *lifts* to an automorphism $\tilde{A}$ of the lift $\Gamma^\alpha$ if $\pi\tilde{A} = A\pi$ (the composition is to be read from the right to the left). Leaving a more detailed discussion of lifts of automorphisms to the next section, at this point we just note that the identity automorphism $id$ of $\Gamma$ has $|H|$ lifts, namely, the mappings $id_h:\ \Gamma^\alpha \to \Gamma^\alpha$, $h \in H$, where $id_h(x_g) = x_{hg}$. Observe also that this way the voltage group $H$ has a regular action on each *fibre* $\pi^{-1}(x)$, $x \in D(\Gamma)$; hence the adjective "regular" in the term for $\Gamma^\alpha$ and $\pi$.

Continuing in the above exposition, let us now consider a group $G$ of automorphisms of $\Gamma$ and let us assume that each automorphism of $G$ has a lift. Let $G^\alpha$ denote the collection of *all* lifts of all automorphisms of $G$. It is easy to see that $G^\alpha$ again is a group, called the *lift* of $G$ (with respect to the voltage assignment $\alpha$). Also, an easy calculation shows that the lift of the trivial group $\{id\}^\alpha = \{id_h;\ h \in H\} \simeq H$ is a normal subgroup of the lifted group $G^\alpha$, and hence the lift $G^\alpha$ is always isomorphic to some extension of the voltage group $H$. The question of when this extension is a semidirect product will be treated in detail in Section 4.

For notational convenience, in order to distinguish between unit elements of the groups that appear frequently in our exposition we keep using the symbol $id$ for the identity automorphism of $\Gamma$ and 1 for the unit element of the voltage group $H$. Also, we will occasionally omit brackets when no confucion is likely; for example, we can write $\alpha(AW)$ instead of $\alpha(A(W))$, etc.

## 3. $G$-compatible voltage assignments and the associated group homomorphisms

Let $\Gamma$ be a connected graph and let $\alpha : D(\Gamma) \to H$ be a voltage assignment on $\Gamma$ in a group $H$. Let $Aut(\Gamma)$ denote the (full) automorphism group of $\Gamma$ and let $G$ be a subgroup of $Aut(\Gamma)$. We shall say that the voltage assignment $\alpha$ is *$G$-compatible* at a vertex $u$ if for each closed $u$-based walk $W$ in $\Gamma$ and each automorphism $A \in G$,

$$(1) \qquad \alpha(W) = 1 \quad \text{if and only if} \quad \alpha(AW) = 1 \ .$$

We first show that the choice of the vertex $u$ in the above definition is immaterial. Indeed. let $v$ be any other vertex of our connected graph $\Gamma$ and let $U$ be a fixed $u \to v$ walk in $\Gamma$. Now let $W$ be an arbitrary closed $v$-based walk in $\Gamma$. Then, $\alpha(W) = 1$ if and only if $\alpha(UWU^{-1}) = 1$. The walk $W' = UWU^{-1}$ is closed and $u$-based, so by (1) we have $\alpha(W') = 1$ if and only if $\alpha(AW') = 1$ for each $A \in G$. The latter is equivalent to $\alpha((AU)(AW)(AU^{-1})) = 1$, which holds if and only if $\alpha(AW) = 1$. Thus, we have (1) for closed walks based at *any* vertex, as claimed. From now on we will therefore speak about $G$-compatibility without specifying a reference vertex.

The importance of the concept of $G$-compatibility is the following theorem stated in [3] for maps (with a proof that readily extends to graphs); see also [6] for a list of various predecessors.

**THEOREM 1.** *Let $\Gamma$ be a connected graph and let $\alpha$ be a voltage assignment on $\Gamma$ in a group $H$. Let $G$ be a subgroup of $Aut(\Gamma)$. Then each automorphism $A \in G$ lifts to an automorphism of $G^\alpha$ if and only if the voltage assignment $\alpha$ is $G$-compatible. If, in addition, the assignment $\alpha$ is proper then, fixing a vertex $u$, all lifts of an automorphism $A \in G$ have the form $A_{h;u}$, $h \in H$ where*

$$(2) \qquad A_{h;u}(x_{\alpha(U)}) = (Ax)_{h\alpha(AU)} \quad \textit{for any} \quad u \to i(x) \ \textit{walk}\ U \ .$$

We emphasize that the explicit form of lifts of automorphisms given in (2) depends on the choice of the fixed vertex $u$ (therefore, to apply it correctly we always have to make sure that the corresponding walk $U$ in (2) indeed emanates from $u$ and terminates at $i(x)$). To see this, let $A$ be an automorphism from $G$, let $v$ be another vertex of $\Gamma$ and let $V$ be a $u \to v$ walk in $\Gamma$ such that $\alpha(V) = 1$ (the existence of $V$ follows from the fact that $\alpha$ is proper). It is an easy consequence of $G$-compatibility of $\alpha$ that the value of $\alpha(AV)$ does not depend on a particular choice of the $u \to v$ walk $V$ with $\alpha(V) = 1$. Now, for any arc $x$ of $\Gamma$ and for any $u \to i(x)$ walk $U$ in $\Gamma$ the walk $V^{-1}U$ is a $v \to i(x)$ walk with $\alpha(V^{-1}U) = \alpha(U)$, and according to (2) we have

$$\begin{aligned} A_{h;v}(x_{\alpha(V^{-1}U)}) &= (Ax)_{h\alpha(A(V^{-1}U))} \\ &= (Ax)_{h\alpha(AV^{-1})\alpha(AU)} = A_{h\alpha(AV^{-1});u}(x_{\alpha(U)}) \, . \end{aligned}$$

It follows that

(3) $\quad A_{h;v} = A_{h\alpha(AV^{-1});u}$ for any $u \to v$ walk $V$ such that $\alpha(V) = 1$ .

The mapping $\alpha(U) \mapsto \alpha(AU)$ that appears in the explicit formula (2) for a lifted automorphism is worth further study. We begin with the following observation.

**PROPOSITION 1.** *Let $\Gamma$ be a connected graph, let $G < Aut(\Gamma)$, and let $\alpha$ be a proper voltage assignment on $\Gamma$ in a group $H$. Then, the following two statements are equivalent:*

1. *The voltage assignment $\alpha$ is $G$-compatible.*
2. *For each vertex $u$, the formula*

   (4) $\quad \Phi_{A;u}(\alpha(W)) = \alpha(AW)$ *for any closed $u$-based walk in* $\Gamma$

   *well-defines a bijection $H \to H$ such that $\Phi_{A;u}(1) = 1$.*

*Moreover, if either of the above conditions is satisfied then the mapping $\Phi_{A;u}$ is automatically an automorphism of the voltage group $H$.*

*Proof.* First, assume that $\alpha$ is a $G$-compatible voltage assignment. If $W$ and $W'$ are two closed $u$-based walks of $\Gamma$ such that $\alpha(W) = \alpha(W')$, then for the walk $W'W^{-1}$ (which, again, is closed and $u$-based) we have $\alpha(W'W^{-1}) = 1$. By (1) we have $\alpha(A(W'W^{-1})) = 1$, which quickly implies that $\alpha(AW) = \alpha(AW')$, that is, the value of $\Phi_{A;u}(\alpha(W))$ does not depend on the choice of $W$. A similar but reverse argument shows that $\Phi_{A;u}$ is one-to-one. From the fact that $\alpha$ is a *proper* voltage assignment we now see that $\Phi_{A;u}$ is a *bijection* $H \to H$; that $\Phi_{A;u}(1) = 1$ is a consequence of $G$-compatibility. Conversely, if (4) well-defines a bijection of $H$ such that $\Phi_{A;u}(1) = 1$ then we obviously have $G$-compatibility of $\alpha$. Finally, if $\Phi_{A;u}$ is a well-defined bijection of $H$ that fixes 1 then it is automatically an automorphism of $H$, as for any two closed $u$-based walks $W$, $W'$ we have $\Phi_{A;u}(\alpha(W)\alpha(W')) = \Phi_{A;u}(\alpha(WW')) = \alpha(A(WW')) = \alpha(AW)\alpha(AW') = \Phi_{A;u}(\alpha(W))\Phi_{A;u}(\alpha(W'))$. ∎

In order to determine the way the automorphism $\Phi_{A;u}$ of Proposition 1 depends on the fixed vertex $u$, let $v$ be another vertex of the (connected) graph $\Gamma$. As before, we let $\alpha$ be a proper voltage assignment on $\Gamma$ in a group $H$. Let $h$ be an element of $H$ and let $U$ be a $u \to v$ walk in $\Gamma$ such that $\alpha(U)$ belongs to $C_H(h)$, the centralizer of $h$ in $H$. Let $W$ be an arbitrary closed $v$-based walk in $\Gamma$ such that $\alpha(W) = h$. Then the walk $W' = UWU^{-1}$ is closed, $u$-based, and since $\alpha(U) \in C_H(h)$, for the net voltage of $W'$ it holds that $\alpha(W') = \alpha(W) = h$. Now, assuming that $\alpha$ is a $G$-compatible voltage assignment for some $G < Aut(\Gamma)$, for any $A \in G$ we have $\Phi_{A;u}(h) = \Phi_{A;u}(\alpha(W')) = \alpha(AW') = \alpha(AU)\alpha(AW)\alpha(AU^{-1}) = \alpha(AU)\Phi_{A;v}(\alpha(W))\alpha(AU^{-1}) = \alpha(AU)\Phi_{A;v}(h)\alpha(AU^{-1})$. It follows that

$$\Phi_{A;v}(h) = \alpha(AU^{-1})\Phi_{A;u}(h)\alpha(AU) \tag{5}$$

for any $u \to v$ walk $U$ such that $\alpha(U) \in C_H(h)$.

We see that in general for distinct vertices $u, v$ the automorphisms $\Phi_{A;u}$ and $\Phi_{A;v}$ may be different. Therefore one has to be careful when evaluating $\Phi_{A;v}$ at a particular $h \in H$ – the element $h$ must be represented as the voltage of a closed walk based at *the* vertex $v$ that appears in the subscript of $\Phi_{A;v}$. Of course, it is of interest to look for conditions which would guarantee independence of $\Phi_{A;v}$ on a particular vertex $v$ – at least within a subset of vertices which is, in some sense, naturally related to our problem. This is the subject of our next result; to state it we introduce the concept of a *G-orbit* of a vertex $u$ of $\Gamma$, which is the set $\{A(u);\ A \in G\}$.

**PROPOSITION 2.** *Let $\Gamma$ be a connected graph, let $G < Aut(\Gamma)$ and let $u$ be a vertex of $\Gamma$. Let $\alpha$ be a proper $G$-compatible voltage assignment on $\Gamma$ in a group $H$. Then the following statements are equivalent:*

1. *For any vertex $v$ in the $G$-orbit of $u$, the mapping $\Phi_{A;v}$ defined by (4) does not depend on $v$; that is, $\Phi_{A;v}(h) = \Phi_{A;u}(h)$ for each $h \in H$.*
2. *The function $\Phi_u : G \to Aut(H)$ such that, for each $A \in G$, $\Phi_u(A) = \Phi_{A;u}$ where $\Phi_{A;u}$ is given by (4), is a group homomorphism $G \to Aut(H)$.*
3. *For any vertex $v$ in the $G$-orbit of $u$ and for any $u \to v$ walk $U$ in $\Gamma$, for any closed $v$-based walk $W$ and any $A \in G$ we have*

$$\alpha(U) \in C_H(\alpha(W)) \ \text{ if and only if } \ \alpha(AU) \in C_H(\alpha(AW)) . \tag{6}$$

*Proof.* $1 \Rightarrow 2$. Let $A, B$ be two automorphisms from $G$ and let $v = B(u)$. Let $W$ be a closed $u$-based walk in $\Gamma$; then $B(W)$ is a closed walk that is $v$-based.

Now, assuming the statement 1 with $h = \alpha(BW)$ and using the definition of $\Phi_{A;u}$ given in (4) we obtain

$$\Phi_{A;u}\Phi_{B;u}(\alpha W) = \Phi_{A;u}(\alpha(BW)) = \Phi_{A;v}(\alpha(BW)) = \alpha(ABW) = \Phi_{AB;u}(\alpha W) ,$$

which proves the statement 2.

$2 \Rightarrow 3$. Let $v$ be an arbitrary vertex of the $G$-orbit of $u$, that is, $v = B(u)$ for some $B \in G$. Let $W$ be a closed $v$-based walk in $\Gamma$ and let $U$ be a $u \to v$ walk in $\Gamma$ such that $\alpha(U) \in C_H(\alpha(W))$. Note that then for the closed $u$-based walk $UWU^{-1}$ we have $\alpha(UWU^{-1}) = \alpha(W)$. For any $A \in G$ we have, according to (4) and the statement 2:

$$\begin{aligned}\alpha(AW) &= \Phi_{AB;u}(\alpha(B^{-1}W)) = \Phi_{A;u}\Phi_{B;u}(\alpha(B^{-1}W)) \\ &= \Phi_{A;u}(\alpha W) = \Phi_{A;u}(\alpha(UWU^{-1})) = \alpha(A(U)A(W)A(U^{-1})) .\end{aligned}$$

This shows that $\alpha(U) \in C_H(\alpha(W))$ implies that $\alpha(AU) \in C_H(\alpha(AW))$. Conversely, assume that $\alpha(AU) \in C_H(\alpha(AW))$. Then, setting $h = \alpha(W)$ and $h' = \alpha(UWU^{-1})$ we successively obtain (using the statement 2 in the middle of our computation and making sure that the evaluation is carried out at appropriate closed walks):

$$\begin{aligned}\Phi_{A;u}(h') &= \Phi_{A;u}(\alpha(UWU^{-1})) = \alpha(A(U)A(W)A(U^{-1})) = \alpha(AW) \\ &= \Phi_{AB;u}(\alpha(B^{-1}W)) = \Phi_{A;u}\Phi_{B;u}(\alpha(B^{-1}W)) = \Phi_{A;u}(\alpha(W)) \\ &= \Phi_{A;u}(h) .\end{aligned}$$

Since, by Proposition 1, $\Phi_{A;u}$ is an automorphism of $H$, we have $h = h'$, that is, $\alpha(U) \in C_H(\alpha(W))$, which completes the proof of the statement 3.

The implication $3 \Rightarrow 1$ follows immediately from the analysis leading to the formula (5). ■

For Abelian voltage groups we have the following obvious corollary of the above results.

**COROLLARY 2.** *Let $G$ be a group of automorphisms of a connected graph $\Gamma$ and let $\alpha$ be a proper $G$-compatible voltage assignment on $\Gamma$ in an Abelian group $H$. Then the function $\Phi_u : G \to Aut(H)$ such that, for each $A \in G$, $\Phi_u(A) = \Phi_{A;u}$ where $\Phi_{A;u}$ is given by (4), is a group homomorphism $G \to Aut(H)$.*

## 4. Groups that lift to semidirect products

We continue considering a connected graph $\Gamma$ and a subgroup $G < Aut(\Gamma)$ together with a proper $G$-compatible voltage assignment $\alpha$ on $\Gamma$ in a group $H$.

By Theorem 1, the lifts $A_{g;u}$ of any particular automorphism $A \in G$ are given by the formula (2). Let us now look at a composition of two such lifts, say, $A_{g;u}$ and $B_{h;u}$. In order to evaluate $A_{g;u}B_{h;u}$ for any arc $x_{\alpha(V)}$ where $V$ is a $u \to i(x)$ walk in $\Gamma$, we choose an arbitrary $u \to B(u)$ walk $U$ such that $\alpha(U) = h$. Then, $UB(V)$ is a $u \to i(B(x))$ walk which may be used for computing the image of the arc $(Bx)_{\alpha(UB(V))}$ under $A_{g;u}$ according to (2):

$$\begin{aligned} A_{g;u}B_{h;u}(x_{\alpha(V)}) &= A_{g;u}((Bx)_{h\alpha(B(V))}) = A_{g;u}((Bx)_{\alpha(UB(V))}) \\ &= (ABx)_{g\alpha(A(U)AB(V))} = (AB)_{g\alpha(A(U));u}(x_{\alpha(V)}) \end{aligned}$$

which implies the following composition law in the lifted group $G^\alpha$ with $u$ as a reference vertex:

(7) $$A_{g;u}B_{\alpha(U);u} = (AB)_{g\alpha(A(U));u} \quad \text{for any } u \to B(u) \text{ walk } U\ .$$

As regards inverses in the lifted group, let $A_{g;u} \in G^\alpha$ and let $U$ be an arbitrary $u \to A(u)$ walk in $\Gamma$ such that $\alpha(U) = g$. Then,

(8) $$(A_{g;u})^{-1} = (A_{\alpha(U);u})^{-1} = (A^{-1})_{\alpha(A^{-1}U^{-1});u}$$

which can be verified by a direct computation using (7).

Under the above assumptions, representing the lifted group $G^\alpha$ in the form $G^\alpha = \{A_{h;u};\ A \in G,\ h \in H\}$ we see from (7) that there is an obvious homomorphism $G^\alpha \to G$ which sends $A_{h;u}$ to $A$; its kernel $H' = \{id_{h;u};\ h \in H\}$ is a normal subgroup of $G^\alpha$ isomorphic to $H$. This raises the natural question of when the lifted group $G^\alpha$ is isomorphic to a split extension of the voltage group $H$ by the group of automorphisms $G$ (or, equivalently, a semidirect product of $H$ and $G$). By definition (see e.g. [9]), a group $L$ is a split extension of a normal subgroup $K \lhd L$ by another subgroup $Q < L$ if $K \cap Q = 1_L$ and $K.Q = L$; in our case we have $L = G^\alpha$ and require that $K \simeq H$ and $Q \simeq G$. Now, in general the subgroups (isomorphic to) $H$ and $G$ may appear in $G^\alpha$ in ways which are not naturally connected with the covering construction. Thus, it is unrealistic to expect a reasonable answer to our question without specifying *which copies* of $H$ and $G$ we are interested in. We therefore stipulate that $K = H'$ and ask under what conditions the lifted group $G^\alpha$ is a split extension of $H'$ by a subgroup isomorphic to $G$. Clearly, a necessary condition for this is that $G^\alpha$ *at all* contains a subgroup $G' \simeq G$. In what follows we therefore focus on subgroups $G' \simeq G$ for which $H' \cap G' = 1_{G^\alpha}$ and $H'G' = G^\alpha$. In order to simplify the notation, we will denote the elements of the lifted group $G^\alpha$ by $A_h$ rather than $A_{h;u}$ (bearing in mind the fixed vertex $u$ at all times).

First, observe that if $A_h$ and $A_g$ are elements of a subgroup $G' < G^\alpha$ isomorphic to $G$, then $h = g$. Indeed, let $U$ and $V$ be two $u \to A(u)$ walks in $\Gamma$

such that $\alpha(U) = g$ and $\alpha(V) = h$. Then, by the inversion formula (8) we have $(A_g)^{-1} = (A^{-1})_{\alpha(A^{-1}U^{-1})}$. Using now the composition law (7) we obtain:

$$A_h(A_g)^{-1} = A_{\alpha(V)}(A^{-1})_{\alpha(A^{-1}U^{-1})} = (AA^{-1})_{\alpha(V)\alpha(A(A^{-1}U^{-1}))} = id_{\alpha(VU^{-1})} .$$

However, from the condition $H' \cap G' = 1_{G^\alpha}$ it follows that $\alpha(VU^{-1}) = 1$, that is, $\alpha(U) = \alpha(V)$, and hence $g = h$, as claimed. It is now an easy consequence of the condition $H'G' = G^\alpha$ that $G'$ necessarily has the form

$$G' = \{A_{l(A)};\ A \in G\} \text{ for some function } l:\ G \to H . \tag{9}$$

Our next aim is to investigate the function $l$ that appears in (9). As $G'$ is assumed to be a *subgroup* of $G^\alpha$, by (9) and by the composition law (7) for any $A, B \in G$ we must have

$$A_{l(A)}B_{l(B)} = (AB)_{l(AB)} . \tag{10}$$

If $U$ is a $u \to B(u)$ walk in $\Gamma$ with $\alpha(U) = l(B)$, using (7) we can evaluate the left hand side of (10), which yields $A_{l(A)}B_{l(B)} = (AB)_{l(A)\alpha(AU)}$. This immediately implies that (10) is equivalent with the condition

$$\alpha(U) = l(B) \quad \text{if and only if} \quad \alpha(AU) = (l(A))^{-1}l(AB) \tag{11}$$

for any $A, B \in G$ and any $u \to B(u)$ walk $U$ in the graph $\Gamma$. Obviously, $l(id) = 1$. As regards the values of $l$ for inverses of automorphisms, from the formula (8) we quickly see that

$$l(A^{-1}) = \alpha(A^{-1}U^{-1}) \tag{12}$$

for each $A \in G$ and any $u \to A(u)$ walk $U$ such that $\alpha(U) = l(A)$.

So far we have seen that if the lifted group $G^\alpha$ is a split extension of its subgroup $H' \simeq H$ by a subgroup $G' \simeq G$ then (assuming that $G$ is finite) there exists a mapping $l:\ G \to H$ which satisfies (11); the arguments are clearly reversible. Before stating the result formally (and thus rectifying Theorem 5 of [3]) let us complete the investigation by representing $G^\alpha$ in form of a semidirect product $H' \times_\Psi G'$ for a suitable group homomorphism $\Psi:\ G' \to Aut(H)$ provided that the condition (11) is satisfied. It follows from a general theory of split extensions [9] that the associated group homomorphism $\Psi$ can be recovered by considering conjugation in $G^\alpha$. In order to do so let $A$ be an arbitrary automorphism in $G$ and let $h \in H$. Further, in the graph $\Gamma$ let $U$ be a $u \to A(u)$ walk such that $\alpha(U) = l(A)$ and let $W$ be a closed $u$-based walk with $\alpha(W) = h$. Then, according to the composition law (7) we have

$$\begin{aligned} A_{l(A)}id_h(A^{-1})_{l(A^{-1})} &= A_{l(A)}id_h(A^{-1})_{\alpha(A^{-1}U^{-1})} = A_{l(A)}(idA^{-1})_{h\alpha(idA^{-1}U^{-1})} \\ &= A_{l(A)}(A^{-1})_{\alpha(WA^{-1}(U^{-1}))} = (AA^{-1})_{l(A)\alpha((AW)U^{-1})} = id_{l(A)\alpha(AW)(l(A))^{-1}} . \end{aligned}$$

It follows that the associated group homomorphism $\Psi : G' \to Aut(H)$ is given by the formula

$$A \to \Psi_A \quad \text{where} \quad \Psi_A(\alpha(W)) = l(A)\alpha(AW)(l(A))^{-1} \tag{13}$$

for any automorphism $A \in G$ and any closed $u$-based walk $W$ in $\Gamma$. (The fact that $\Psi$ is well defined is an easy consequence of $G$-compatibility at the vertex $u$.) Using the above computation in conjunction with (13), the fact that $G^\alpha = H'G'$ and the relation (10), we can now express the multiplication in $G^\alpha$ in the form

$$id_g A_{l(A)} . id_h B_{l(B)} = id_g (A_{l(A)} id_h (A^{-1})_{l(A^{-1})}) A_{l(A)} B_{l(B)} = id_{g\Psi_A(h)}(AB)_{l(AB)} \ .$$

Thus, the split extension $G^\alpha$ of $H'$ by $G'$ can now be identified with the semidirect product $H \times_\Psi G'$ whose elements are pairs $(g, A_{l(A)})$ where $g \in H$ and $A \in G$, with the multiplication given by

$$(g, A_{l(A)})(h, B_{l(B)}) = (g\Psi_A(h), (AB)_{l(AB)}) \ . \tag{14}$$

We now summarize all of the preceding discussion as follows.

**THEOREM 3.** *Let $G$ be a group of automorphisms of a connected graph $\Gamma$ with a fixed vertex $u$. Let $\alpha$ be a proper $G$-compatible voltage assignment on $\Gamma$ in a group $H$. Then the following statements are equivalent:*

1. *The lifted group $G_u^\alpha$ is a split extension of the subgroup $H' = \{id_h;\ h \in H\}$ by some subgroup $G' \simeq G$.*
2. *There exists a mapping $l : G \to H$ with the property that $\alpha(U) = l(B)$ if and only if $\alpha(AU) = (l(A))^{-1} l(AB)$ for any $A, B \in G$ and any $u \to B(u)$ walk $U$ in the graph $\Gamma$.*

*Moreover, if one of the above statements holds true then $G^\alpha$ is isomorphic to the semidirect product $H \times_\Psi G'$ where multiplication is given by* (14) *and the associated group homomorphism $\Psi : G' \to Aut(H)$ is given by* (13).

The calculations preceding Theorem 3 have all been carried out for a fixed vertex $u$. However, the choice of the fixed vertex plays just a role of a "coordinate system". For example, the way the function $l$ associated with the group $G'$ depends on the fixed vertex $u$ can easily be extracted from the formula (3) of Section 3. To see this, let $l = l_u : G \to H$ be the mapping as in the statement 2 of Theorem 3. Let $v$ be another vertex of $\Gamma$ and let $l_v$ be the corresponding function which refers to $v$ as the fixed vertex. That is, returning to the extended notation $A_{h;u}$ for lifts of automorphisms, we now want that $A_{l_u(A);u} = A_{l_v(A);v}$

for each $A \in G$. Let $V$ be an arbitrary $u \to v$ walk in $\Gamma$ with $\alpha(V) = 1$. Then, setting $h = l_v(A)$ in (3) we obtain

$$A_{l_v(A);v} = A_{l_v(A)\alpha(AV^{-1});u} = A_{l_u(A),u}$$

which implies that for the corresponding function $l = l_v$ at the fixed vertex $v$ we have

(15) $\quad l_v(A) = l_u(A)\alpha(AV) \quad$ for any $\quad u \to v$ walk with $\alpha(V) = 1$ .

The reader may have realized some similarity between the group homomorphism $\Phi = \Phi_u : G \to Aut(H)$ introduced in Proposition 2 of Section 3 (see also (4)) and the homomorphism $\Psi : G' \to Aut(H)$ as defined by (13). Indeed, if the mapping $l : G \to H$ referred to in Theorem 3 satisfies $l(A) = 1$ for each $A \in G$, then (identifying $G'$ with $G$) the two homomorphisms $\Phi$ and $\Psi$ are *identical.* We state the consequences of this observation in form of a separate result (which the reader is also invited to compare with statement 2 of Proposition 1). For the ease of the formulation we will still introduce one more concept. Given a connected graph $\Gamma$ and a group $G < Aut(\Gamma)$, a voltage assignment $\alpha$ on the graph $\Gamma$ in a group $H$ will be called *G-supercompatible at u* if for each $A \in G$ and for each walk $U$ in $\Gamma$ with both ends in the $G$-orbit of $u$ we have $\alpha(U) = 1$ if and only if $\alpha(AU) = 1$.

**THEOREM 4.** *Let $\Gamma$ be a connected graph with a fixed vertex $u$ and let $G$ be a group of automorphisms of $\Gamma$. Let $\alpha$ be a proper $G$-compatible voltage assignment on $\Gamma$ in a group $H$. Then the following statements are equivalent:*

1. *The collection $G'_1 = \{A_{1;u};\ A \in G\}$ forms a subgroup of the lifted group $G^\alpha_u$, and, consequently, $G^\alpha_u$ is a split extension of the subgroup $H' = \{id_h;\ h \in H\}$ by $G'_1$.*
2. *The voltage assignment $\alpha$ is $G$-supercompatible at $u$.*
3. *The formula $\Phi_A(\alpha(U)) = \alpha(AU)$ where $U$ is an arbitrary walk in $\Gamma$ with both ends in the $G$-orbit of $u$, well-defines an automorphism of the voltage group $H$ for any $A \in G$.*

*Proof.* $1 \Rightarrow 2$. If $G'_1$ is a subgroup of $G^\alpha$ then for the function $l_u : G \to H$ we have $l_u(A) = 1$ for each $A \in G$. The statement 2 of Theorem 3 implies that for each walk $U$ of $\Gamma$ emanating from the vertex $u$ and terminating at some vertex in the $G$-orbit of $u$ we have $\alpha(U) = 1$ if and only if $\alpha(AU) = 1$. Using the relation (15) we obtain that $l_v(A) = 1$ for each vertex $v$ in the $G$-orbit of $u$. Invoking the statement 2 of Theorem 3 again, this time for the fixed vertex $v$, we obtain the $G$-supercompatibility of $\alpha$.

$2 \Rightarrow 3$. Let $U$ and $V$ be arbitrary two walks in $\Gamma$ with both ends in the $G$-orbit of $u$, such that $\alpha(U) = \alpha(V)$. Assume that $U$ emanates from $u$ and terminates at $u'$ whereas $V$ emanates from $v$ and terminates at $v'$. Let $W$, $W'$ be walks in $\Gamma$ emanating from $u$, $v$ and terminating at $u'$, $v'$, respectively, such that $\alpha(W) = \alpha(W') = 1$. Clearly $\alpha(W^{-1}UW'V^{-1}) = 1$, and by $G$-supercompatibility for each $A \in G$ we have $1 = \alpha(A(W^{-1}UW'V^{-1})) = \alpha(AU)\alpha(AV^{-1})$. Consequently, $\alpha(AU) = \alpha(AV)$ for any two walks $U$, $V$ in $\Gamma$ with both ends in the $G$-orbit of $u$ and such that $\alpha(U) = \alpha(V)$, which proves the statement 3 (the homomorphism part follows from the proof of Lemma 1).

$3 \Rightarrow 1$. Assuming the existence of $\Phi(A)$ for each $A \in G$ as given in the statement 3 we may rewrite the composition law (7) of Section 4 for the special case $g = 1$ in the form

$$A_{1;u}B_{\alpha(V);u} = (AB)_{1\Phi_A(\alpha(V));u} \quad \text{for any } u \to B(u) \text{ walk } V .$$

Now, if $\alpha(V) = 1$ then, as $\Phi_A$ is a homomorphism, we have $\Phi_A(\alpha(V)) = 1$. This implies that $G'_1$ is a subgroup of $G^\alpha$, which was to be shown. (Note that the mapping $A \mapsto \Phi_A$ is now a homomorphism $G \to Aut(H)$ which, as expected, can be identified with the homomorphism $\Psi$ defined in (13).) ■

The following is an obvious but important corollary of Theorems 3 and 4; roghly speaking it states that lifts of vertex stabilizers are always semidirect products. See also [6] for a different proof.

**COROLLARY 5.** *Let $\Gamma$ be a connected graph endowed with a proper voltage assignment $\alpha$ in a group $H$. Let $u$ be a vertex of $\Gamma$ and let $G$ be a subgroup of $Aut(\Gamma)$ that stabilizes $u$. Then the lifted group $G^\alpha_u$ is a split extension of the group $H' = \{id_h;\ h \in H\}$ by the subgroup $G' = \{A_{1;u};\ A \in G\}$.*

*Proof.* In this case for the mapping $l : G \to H$ we may set $l(A) = 1$ for each $A \in G$, since (by the stabilizer assumption) the condition (11) coincides with $G$-(super)compatibility at $u$. ■

## 5. Remarks and examples

Let us start with recalling the concept of equivalence of voltage assignments (cf. [4]). We say that two voltage assignments $\alpha$ and $\alpha'$ on a graph $\Gamma$ in the same voltage group $H$ are *locally equivalent* if there exists a vertex $v$ of $\Gamma$ and an element $h \in H$ such that $\alpha'(x) = h\alpha(x)$ for each arc $x$ emanating from $v$. Now, two voltage assignments $\alpha, \beta : D(\Gamma) \to H$ are *equivalent* if there is a sequence of voltage assignments $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_k = \beta$ such that $\alpha_j$ and $\alpha_{j+1}$ are locally equivalent for $1 \le j < k$.

Equivalence of voltage assignments is important for various reasons (see e.g. [4, 5]). However, looking at our results concerning the semidirect product we see that neither the $G$-supercompatibility condition of Theorem 4 nor the condition 2 of Theorem 3 are invariant under voltage assignment equivalence. It is therefore tempting to expect here a result that would in one way or another refer to voltage assignment equivalence. To be more concrete, one could expect that (perhaps) for a voltage assignment $\alpha$ satisfying the statement 2 of Theorem 3 there could exist an equivalent $G$-supercompatible assignment. Unfortnately, this is not the case even when restricted to automorphisms of *regular maps* which we now introduce.

A *map* $M$ is a (connected) 2-cell embedded finite graph $\Gamma$ in some closed surface; we confine ourselves here to *orientable* surfaces. A *map automorphism* of $M$ is an automorphism of the *underlying graph* $\Gamma$ which, in addition, preserves the faces of the map and the orientation of the surface. It is well known that the group $Aut(M)$ of all map automorphisms of $M$ always acts freely on the set of arcs of the underlying graph $\Gamma$. We say that a map is *regular* if its map automorphism group acts *regularly* (that is, transitively and freely) on the set of arcs. Thus, informally, regular maps are the "most symmetric" maps with respect to orientation-preserving map automorphisms.

If a proper voltage assignment $\alpha : D(\Gamma) \to H$ on an underlying graph $\Gamma$ of a map $M$ has been specified, then the lifted graph $\Gamma^\alpha$ naturally gives rise to a *lifted map* $M^\alpha$ such that the regular covering $\pi : \Gamma^\alpha \to \Gamma$ extends to a (possibly branched) covering $M^\alpha \to M$; we refer to [4] for (many) details. The important fact to be mentioned is that the necessary and sufficient condition for map automorphisms in a group $G < Aut(M)$ to lift onto map automorphisms of $M^\alpha$ turns out to be the $G$-compatibility of $\alpha$, exactly as in Theorem 1 (see e.g. [3] for a short proof).

Returning back to our project, in what follows we give an example of a regular map $M$ with underlying graph $\Gamma$ endowed with a proper $Aut(M)$-compatible voltage assignment $\alpha$ in a group $H$, such that the lifted map automorphism group $(Aut(M))^\alpha$ of the (regular) lifted map $M^\alpha$ is a split extension of the voltage group $H$ by $Aut(M)$, and yet $\alpha$ is *not* equivalent to any $Aut(M)$-supercompatible voltage assignment in $H$.

**EXAMPLE.** Let $\Gamma$ be a graph consisting of two vertices $u$, $v$ and three parallel edges joining the two vertices. For definiteness, let $x$, $y$ and $z$ be the corresponding three parallel arcs of $\Gamma$ emanating from $u$ and terminating at $v$; for the reverse of an arc $a \in \{x, y, z\}$ we will now use the symbol $\bar{a}$ rather than $a^{-1}$. Let a voltage assignment $\alpha$ on the graph $\Gamma$ in the (additive) group $H = \mathcal{Z}_3$ be defined by letting $\alpha(x) = 0$, $\alpha(y) = 1$, and $\alpha(z) = 2$; in agreement with the general definition, reverse arcs receive (additively) inverse voltages. Thus, according to the general formula $(a_h)^{-1} = (a^{-1})_{h+\alpha(a)}$ for reverses of arcs in the lift $\Gamma^\alpha$, we have

$$\bar{x}_i = (x_i)^{-1},\ \bar{y}_i = (y_{i-1})^{-1},\ \text{and}\ \bar{z}_i = (z_{i+1})^{-1}\ \text{for any}\ i \in \mathcal{Z}_3\ . \tag{16}$$

Let $M$ be the toroidal map of Fig. 1 whose underlying graph is $\Gamma$. It is easy to check that the permutation $A = (x\bar{y}z\bar{x}y\bar{z})$ of $D(\Gamma)$ is an orientation-preserving automorphism of the map $M$. Observe that, geometrically, $A$ rotates the map about the centre of its unique hexagonal face $F$ in the counterclockwise direction. The group $G = \langle A \rangle$ acts regularly on $D(\Gamma)$ and thus $M$ is, in fact, a *regular map*. Also, it is not hard to verify that the voltage assignment $\alpha$ is $G$-compatible, and so all automorphisms in $G$ (that is, the powers of $A$) have a lift.

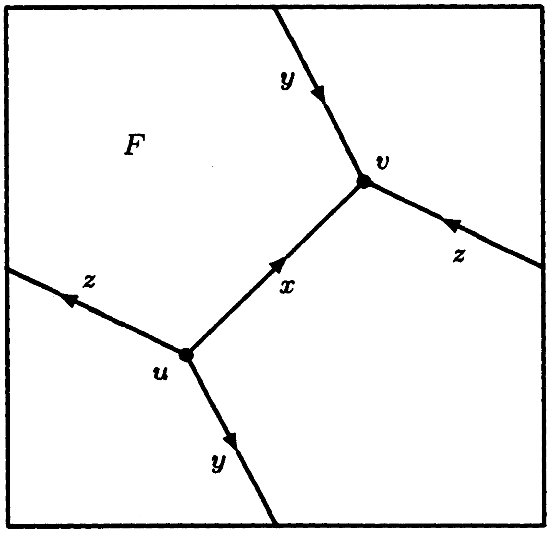


**Figure 1.** The toroidal regular map $M$.

The lift $\Gamma^\alpha$ is isomorphic to the complete bipartite graph $K_{3,3}$, and the lift of the map $M$ is isomorphic to the toroidal embedding $M^\alpha$ of $K_{3,3}$ as depicted in Fig. 2. Note that the single face $F$ of $M$ with the (counterclockwise) boundary walk $W = x\bar{y}z\bar{x}y\bar{z}$ of net voltage $\alpha(W) = 0$ lifts to three faces $F_i$ of $M^\alpha$, $i \in \mathcal{Z}_3$, bounded by the walks $W_i = x_i\bar{y}_i z_{i+2}\bar{x}_{i+1}y_{i+1}\bar{z}_{i+2}$.

In all computations that follow we fix the vertex $u$. Now, evaluating the three lifts $A_i = A_{i;u}$ ($i \in \mathcal{Z}_3$) of the automorphism $A$ gives, according to the formula (2) of Section 3 (and taking (16) into account),

$$A_0 = (x_0\bar{y}_0 z_2\bar{x}_1 y_1\bar{z}_2)(x_1\bar{y}_2 z_0\bar{x}_0 y_2\bar{z}_1)(x_2\bar{y}_1 z_1\bar{x}_2 y_0\bar{z}_0)\ ,$$

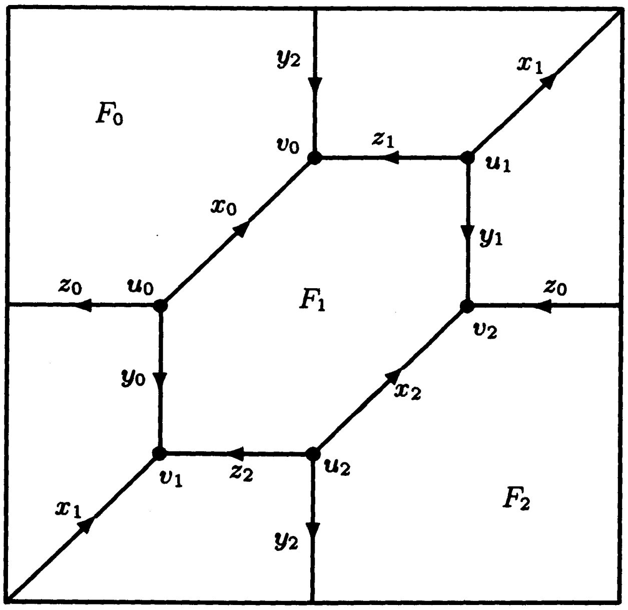


**Figure 2.** The lifted map $M^\alpha$ on a torus.

$$A_1 = (x_0\bar{y}_1 z_2\bar{x}_2 y_1\bar{z}_0)(x_1\bar{y}_0 z_0\bar{x}_1 y_2\bar{z}_2)(x_2\bar{y}_2 z_1\bar{x}_0 y_0\bar{z}_1) ,$$
$$A_2 = (x_0\bar{y}_2 z_2\bar{x}_0 y_1\bar{z}_1)(x_1\bar{y}_1 z_0\bar{x}_2 y_2\bar{z}_0)(x_2\bar{y}_0 z_1\bar{x}_1 y_0\bar{z}_2) .$$

The reader may check that for each $i \in \mathcal{Z}_3$ the lifted automorphism $A_i$ of $\Gamma^\alpha$ is, at the same time, an automorphism of the lifted *map* $M^\alpha$, counterclockwise rotating the face $F_i$ about its centre (compare with Fig. 2). It readily follows that $M^\alpha$ is a regular map (which is a well known fact); moreover, just by looking at the cycle structure of the $A_i$'s we see that the lifted group $G^\alpha = (Aut(M))^\alpha = Aut(M^\alpha)$ is a split extension of the voltage group $H = \mathcal{Z}_3$ by a subgroup $G'$ isomorphic to $G = Aut(M)$.

In order to identify the extension more exactly, we explicitly compute $G'$ together with the function $l$. Of course $G'$ (and hence $l$) need not be determined uniquely (and in our case they are *not*); we choose *the* subgroup $G'$ which is obtained by considering powers of, say, $A_1$. It can be checked that

$$(A_1)^2 = (A^2)_2, \quad (A_1)^3 = (A^3)_2, \quad (A_1)^4 = (A^4)_1, \quad \text{and} \quad (A_1)^5 = (A^5)_0 ,$$

and so $G' = \langle A_1 \rangle$ is a subgroup of $G^\alpha$ isomorphic to $G$. For the corresponding function $l : G \to H$ introduced in (9) we have $l(id) = l(A^5) = 0$, $l(A) = l(A^4) = 1$, and $l(A^2) = l(A^3) = 2$. (The fact that the condition (11) is satisfied for the mapping $l$ can be checked directly but the computation is a bit time consuming.) The associated homomorphism $\Psi : G \to Aut(H)$ referred to in (13) is given by $\Psi_A(i) = 2i$ where $i \in H = \mathcal{Z}_3$.

Thus, Theorem 3 implies that the lifted group $G^\alpha$ is indeed isomorphic to the semidirect product $H \times_\Psi G = \mathcal{Z}_3 \times_\Psi \mathcal{Z}_6$. However, an easy inspection shows that our voltage assignment $\alpha$ is *not* equivalent to any $G$-supercompatible voltage assignment.

Finally, we note that commutativity of both $G$ and $H$ does not guarantee that the lifted group $G^\alpha$ is isomorphic to a split extension of $H$ by $G$. For example, let $\Gamma$ be a graph consisting of two vertices $u$, $v$ and two arcs $x$, $y$ from $u$ to $v$. Let $\alpha$ be a voltage assignment on $\Gamma$ in the group $\mathcal{Z}_2$ such that $\alpha(x) = 1$ and $\alpha(y) = 0$. Consider the automorphism $A = (x\bar{y})(\bar{x}y)$ and let $G = \langle A \rangle \simeq \mathcal{Z}_2$. It is easy to check that $G^\alpha \simeq \mathcal{Z}_4$ and this is clearly not a split extension of $\mathcal{Z}_2$ by $\mathcal{Z}_2$.

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## References

[1] D. Archdeacon, P. Gvozdjak, J. Širáň, Constructing and forbidding automorphisms in lifted maps, *Mathematica Slovaca*, **47** (1997) No. 2, 113–129.

[2] N. L. Biggs, "Algebraic Graph Theory", 2nd Ed., Cambridge Univ. Press, 1993.

[3] P. Gvozdjak, J. Širáň, Regular maps from voltage assignments, in: "Graph Structure Theory" (*Contemporary Mathematics*, AMS Series) **147** (1993), 441–454.

[4] J. L. Gross, T. W. Tucker, "Topological Graph Theory", Wiley, 1987.

[5] J. H. Kwak, J.-H. Chun, J. Lee, Enumeration of regular graph coverings having finite Abelian covering transformation groups, *SIAM J. Discrete Math.*, **11** (1998) No. 2, 273–285.

[6] A. Malnič, Group actions, coverings and lifts of automorphisms, *Discrete Math.*, **182** (1998), 203–218.

[7] B. D. McKay, M. Miller, J. Širáň, A note on large graphs of diameter two and given maximum degree, *J. Combin. Theory, Ser.B*, **74** (1998) No. 1, 110–118.

[8] R. Nedela, M. Škoviera, Regular maps from voltage assignments and exponent groups, *Europ. J. Combin.*, **18** (1997), 807–823.

[9] J. Rotman, "The Theory of Groups - An Introduction", Allyn and Bacon, 1973.

Department of Mathematics,
SvF, Slovak University of Technology
813 68 Bratislava,
SLOVAKIA
E-mail: `siran@lux.svf.stuba.sk`