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# ENUMERATION OF GRAPH COVERINGS*

By

JAEUN LEE

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**Abstract.** Enumerative research is presently a major center of interest in topological graph theory, as in the work of Archdeacon et al [2], Gross and Furst [4], Hofmeister [9]–[13], Kwak and Lee [20]–[24], Mizuno and Sato [25]–[28] and [32], Mohar [29], Mull, Rieper and White [30], Negami [31], etc. In this paper, we survey some results on the enumeration of graph coverings and introduce some unsolved problems.

## Introduction

Let $G$ be a connected finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to $v$. We use $|X|$ for the cardinality of a set $X$. The number $\beta(G) = |E(G)| - |V(G)| + 1$ is equal to the number of independent cycles in $G$ and it is referred to as the *Betti number* of $G$. Let $\mathrm{Aut}(G)$ be the group of all automorphisms of $G$.

A graph $\widetilde{G}$ is called a *covering* of $G$ with projection $p : \widetilde{G} \to G$ if there is a surjection $p : V(\widetilde{G}) \to V(G)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. We also say that the projection $p : \widetilde{G} \to G$ is an *n-fold covering* of $G$ if $p$ is $n$-to-one. A covering $p : \widetilde{G} \to G$ is said to be *regular* if there is a subgroup $\mathcal{A}$ of the automorphism group $\mathrm{Aut}(\widetilde{G})$ of $\widetilde{G}$ acting freely on $\widetilde{G}$ such that the quotient graph $\widetilde{G}/\mathcal{A}$ is isomorphic to $G$. The fiber of an edge or a vertex is its preimage under $p$.

Let $\Gamma$ be a group of automorphisms of the graph $G$. Two coverings $p_i : \widetilde{G}_i \to G$, $i = 1, 2$, are said to be *isomorphic with respect to* $\Gamma$ or $\Gamma$-*isomorphic* if there exist a graph isomorphism $\Phi : \widetilde{G}_1 \to \widetilde{G}_2$ and a graph automorphism $\gamma \in \Gamma$ such that the diagram

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$$\begin{tikzcd} \widetilde{G}_1 \arrow[r, "\Phi"] \arrow[d, "p_1"'] & \widetilde{G}_2 \arrow[d, "p_2"] \\ G \arrow[r, "\gamma"] & G \end{tikzcd}$$

commutes. Such a $\Phi$ is called a *covering isomorphism with respect to* $\Gamma$ or $\Gamma$-*isomorphism.* Note that for any group $\Gamma$ of automorphisms of $G$, the covering isomorphic relation with respect to $\Gamma$ on the coverings of $G$ is an equivalence relation.

Every edge of a graph $G$ gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse edge to a directed edge $e = uv$. We denote the set of directed edges of $G$ by $D(G)$. Following Gross and Tucker [7], a *permutation voltage assignment* $\phi$ of $G$ is a function $\phi : D(G) \to S_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$, where $S_n$ is the symmetric group on $n$ elements $\{1, \ldots, n\}$. The *permutation derived graph* $G^\phi$ is defined as follows: $V(G^\phi) = V(G) \times \{1, \ldots, n\}$, and for each edge $e = uv \in D(G)$ and $j \in \{1, \ldots, n\}$, let there be an edge $(e, j)$ in $D(G^\phi)$ joining a vertex $(u, j)$ and $(v, \phi(e)j)$. The first coordinate projection $p^\phi : G^\phi \to G$, called the natural projection, is a covering. Let $\mathcal{A}$ be a finite group. An *ordinary voltage assignment* (or, $\mathcal{A}$-*voltage assignment*) of $G$ is a function $\phi : D(G) \to \mathcal{A}$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The values of $\phi$ are called *voltages*, and $\mathcal{A}$ is called the *voltage group*. The *ordinary derived graph* $G \times_\phi \mathcal{A}$ derived from an ordinary voltage assignment $\phi : D(G) \to \mathcal{A}$ has as its vertex set $V(G) \times \mathcal{A}$ and as its edge set $E(G) \times \mathcal{A}$, so that an edge of $G \times_\phi \mathcal{A}$ joins a vertex $(u, g)$ to $(v, \phi(e)g)$ for $e = uv \in D(G)$ and $g \in \mathcal{A}$. The first coordinate projection $p_\phi : G \times_\phi \mathcal{A} \to G$, also called the natural projection, commutes with the left multiplication action of the $\phi(e)$ and the right action of $\mathcal{A}$ on the fibers, which is free and transitive, so that $p_\phi$ is an $|\mathcal{A}|$-fold regular covering, called simply an $\mathcal{A}$-*covering.*

In this paper, we mainly discuss the enumeration of the $I$-isomorphism classes of coverings of $G$, where $I$ is the trivial automorphism group $\{1\}$ of $G$. From now on, we shall assume that the isomorphism classes of coverings stand for the $I$-isomorphism classes of them. Waller [34] studied the structures of double coverings of $G$. After then, Hofmeister [9] enumerated the isomorphism classes of double coverings of $G$. Kwak and Lee [20, 23] enumerated the isomorphism classes of graph bundles and those of all finite-fold graph coverings. Hofmeister [10, 13] independently enumerated the isomorphism classes of all finite-fold graph coverings. But the enumeration of the isomorphism classes of regular graph coverings has not been answered completely. As its partial answers, Hofmeister [9] enumerated the isomorphism classes of regular double coverings of $G$, Kwak and

Lee [21], and Sato [32] did the same work for regular prime-fold coverings. Hong and Kwak [14] did it for regular four-fold coverings. Hofmeister [11] enumerated the isomorphism classes of $\bigoplus_{i=1}^{n} \mathbb{Z}_p$-coverings. Recently, Kwak et al [18] enumerated the isomorphism classes of $\mathcal{A}$-coverings when $\mathcal{A}$ is a finite abelian group or the dihedral group $\mathbb{D}_n$. Notice that there are also several interesting results on the enumeration of the $\Gamma$-isomorphism classes of graph coverings for any nontrivial subgroup $\Gamma$ of $\mathrm{Aut}(G)$. For example, Hofmeister [9] enumerated the $\Gamma$-isomorphism classes of double coverings and Sato [28, 32] did the same work for $\mathbb{Z}_2 \times \mathbb{Z}_2$- or $\mathbb{Z}_p$-coverings. When $\Gamma$ fixes a spanning tree of $G$, some results on the enumeration of the $\Gamma$-isomorphism classes of graph coverings can be found in [15] and [20]. It can be also found in [3], [12], [26] and [27] that several generalized notion of graph coverings and some interesting results on the enumeration of the isomorphism classes of them.

This paper is organized as follows. In section 1, we deal with some algebraic characterizations when given two coverings are isomorphic and derive some new enumeration formulas for the $\Gamma$-isomorphism classes of regular coverings. In section 2, we treat some explicit enumeration formulas for graph coverings. In section 3, we conclude with some problems.

## 1. Algebraic characterizations

Gross and Tucker [7] showed that every covering $\widetilde{G}$ (resp. regular covering) of $G$ can be described by a permutation (resp. ordinary) voltage assignment. By the virtue of their work, we may enumerate the isomorphism classes of coverings of $G$ by enumerating some equivalence classes of voltage assignments of $G$.

**THEOREM 1.** ([20]) *Let $\Gamma$ be a subgroup of* $\mathrm{Aut}(G)$ *and $\phi, \psi : D(G) \to S_n$ permutation voltage assignments. Then two $n$-fold coverings $G^{\phi}$ and $G^{\psi}$ are isomorphic with respect to $\Gamma$ if and only if there exist a function $f : V(G) \to S_n$ and $\gamma \in \Gamma$ such that*

$$\psi(\gamma u\, \gamma v) = f(v)\phi(uv)f(u)^{-1}$$

*for each $uv \in D(G)$.*

Notice that an equivalent form of Theorem 1 can also be found in [10]. For a finite group $\mathcal{A}$, let $S_{\mathcal{A}}$ denote the set of all bijective functions on the underlying set $\mathcal{A}$ and $\mathcal{L}_{\mathcal{A}}$ denote the left translation group of $\mathcal{A}$, *i.e.*, $\mathcal{L}_{\mathcal{A}}$ is the group of left translations $\mathcal{L}_g$ for $g \in \mathcal{A}$, where $\mathcal{L}_g(g') = gg'$ for all $g' \in \mathcal{A}$. Then Theorem 1 can be rephrased for regular coverings as follows:

**THEOREM 2.** ([15]) *Let $\Gamma$ be a subgroup of* $\mathrm{Aut}(G)$ *and $\phi, \psi$ $\mathcal{A}$-voltage assignments of $G$. Then two $\mathcal{A}$-coverings $G \times_\phi \mathcal{A}$ and $G \times_\psi \mathcal{A}$ are isomorphic with respect to $\Gamma$ if and only if there exist a graph automorphism $\gamma \in \Gamma$ and a function $f : V(G) \to S_{\mathcal{A}}$ such that*

$$\mathcal{L}_{\psi(\gamma u\,\gamma v)} = f(v) \circ \mathcal{L}_{\phi(uv)} \circ f(u)^{-1}$$

*for each $uv \in D(G)$.*

A finite group $\mathcal{A}$ is said to have the *isomorphism extension property* (IEP) if every isomorphism between any two isomorphic subgroups $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}$. For example, the cyclic group $\mathbb{Z}_n$ for any natural number $n$, the dihedral group $\mathbb{D}_n$ for odd $n \geq 3$, and the direct sum of $m$ copies of $\mathbb{Z}_p$ for a prime number $p$ have the IEP. Moreover, the direct sum of two groups whose orders are relatively prime has the IEP if each direct summand has it. However, neither $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ nor $\mathbb{D}_n$, $n =$ even has the IEP.

**THEOREM 3.** ([15]) *Let $\Gamma$ be a subgroup of* $\mathrm{Aut}(G)$ *and $\phi, \psi$ $\mathcal{A}$-voltage assignments of $G$. If either $\mathcal{A}$ has the* IEP *or both $\phi$ and $\psi$ derive connected coverings, then the following are equivalent:*

(a) *Two $\mathcal{A}$-coverings $G \times_\phi \mathcal{A}$ and $G \times_\psi \mathcal{A}$ are isomorphic with respect to $\Gamma$.*

(b) *There exist a graph automorphism $\gamma \in \Gamma$, a group automorphism $\sigma \in$* $\mathrm{Aut}(\mathcal{A})$ *and a function $f : V(G) \to \mathcal{A}$ such that*

$$\psi(\gamma u\,\gamma v) = f(v)\,\sigma(\phi(uv))\,f(u)^{-1}$$

*for each $uv \in D(G)$.*

*Moreover, if $\mathcal{A}$ is abelian, then each of these two conditions is also equivalent to the following condition:*

(c) *There exist a graph automorphism $\gamma \in \Gamma$, a group automorphism $\sigma \in$* $\mathrm{Aut}(\mathcal{A})$, *and a function $f : V(G) \to \mathcal{A}$ such that*

$$\psi(\gamma u\,\gamma v) = \sigma(\phi(uv)) f(v) f(u)^{-1}$$

*for each $uv \in D(G)$.*

Let $\mathcal{A}$ be an abelian group and let $H^1(G;\mathcal{A})$ be the first cohomology group of $G$. We define a $\Gamma \times \mathrm{Aut}(\mathcal{A})$ action on $H^1(G;\mathcal{A})$ by

$$(\gamma,\sigma)\cdot[\phi] = [(\gamma,\sigma)\cdot\phi], \quad \text{where } ((\gamma,\sigma)\cdot\phi)(uv) = \sigma\left(\phi(\gamma^{-1}u\,\gamma^{-1}v)\right)$$

for $(\gamma, \sigma) \in \Gamma \times \mathrm{Aut}(\mathcal{A})$, $[\phi] \in H^1(G;\mathcal{A})$, and $uv \in D(G)$. If $\mathcal{A}$ has the IEP, then, by Theorem 3, two $\mathcal{A}$-coverings $G \times_\phi \mathcal{A}$ and $G \times_\psi \mathcal{A}$ are isomorphic with respect to $\Gamma$ if and only if $[\phi]$ and $[\psi]$ are in the same orbit of the $\Gamma \times \mathrm{Aut}(\mathcal{A})$ action on $H^1(G;\mathcal{A})$. Hence, the number of the $\Gamma$-isomorphism classes of the $\mathcal{A}$-coverings of $G$ is equal to $|H^1(G;\mathcal{A})/(\Gamma \times \mathrm{Aut}(\mathcal{A}))|$.

For a finite group $\mathcal{A}$ and a subgroup $\Gamma$ of $\mathrm{Aut}(G)$, let $\mathrm{Iso}_\Gamma(G;\mathcal{A})$ (resp., $\mathrm{Isoc}_\Gamma(G;\mathcal{A})$) denote the number of the $\Gamma$-isomorphism classes of $\mathcal{A}$-coverings (resp., connected $\mathcal{A}$-coverings) of $G$. Let $\mathrm{Iso}_\Gamma^R(G;n)$ (resp., $\mathrm{Isoc}_\Gamma^R(G;n)$) denote the number of the $\Gamma$-isomorphism classes of *regular* $n$-fold coverings (resp., connected regular $n$-fold coverings) of $G$. When $\Gamma$ is the trivial group, they are simply denoted by $\mathrm{Iso}(G;\mathcal{A})$, $\mathrm{Isoc}(G;\mathcal{A})$, $\mathrm{Iso}^R(G;n)$ and $\mathrm{Isoc}^R(G;n)$.

By using Theorem 3, Kwak et al [18] derived some formulas for enumerating the isomorphism classes of regular coverings. Now, we aim to derive some enumeration formulas for the $\Gamma$-isomorphism classes of regular coverings. It comes from the definition of a regular covering that each component of a regular $n$-fold covering of $G$ is also a connected regular $d$-fold covering of $G$ for a divisor $d$ of $n$, and any two components are isomorphic with respect to the trivial automorphism group of $G$. Hence, the number of the $\Gamma$-isomorphism classes of regular $n$-fold coverings with $n/d$ components is equal to the number of the $\Gamma$-isomorphism classes of connected regular $d$-fold coverings. Moreover, each component of an $\mathcal{A}$-covering of $G$ is $I$-isomorphic to a connected $\mathcal{S}$-covering of $G$ for a subgroup $\mathcal{S}$ of $\mathcal{A}$. Let $\mathcal{A}$ and $\mathcal{B}$ be two non-isomorphic groups such that $|\mathcal{A}| = |\mathcal{B}|$. Then, by a method similar to the proof of Theorem 3, we can have that, for each subgroup $\Gamma$ of $\mathrm{Aut}(G)$, every connected $\mathcal{A}$-covering of $G$ can not be $\Gamma$-isomorphic to any $\mathcal{B}$-covering of $G$. Hence, we have the following theorems.

**THEOREM 4.** *For any natural number $n$ and any subgroup $\Gamma$ of* $\mathrm{Aut}(G)$,

$$\mathrm{Iso}_\Gamma^R(G;n) = \sum_{d|n} \mathrm{Isoc}_\Gamma^R(G;d) \quad \textit{and} \quad \mathrm{Isoc}_\Gamma^R(G;n) = \sum_{\mathcal{A}} \mathrm{Isoc}_\Gamma(G;\mathcal{A}),$$

*where $\mathcal{A}$ runs over all representatives of the isomorphism classes of groups of order $n$.*

**THEOREM 5.** *For any finite group $\mathcal{A}$ and any subgroup $\Gamma$ of* $\mathrm{Aut}(G)$,

$$\mathrm{Iso}_\Gamma(G;\mathcal{A}) = \sum_{\mathcal{S}} \mathrm{Isoc}_\Gamma(G;\mathcal{S}),$$

*where $\mathcal{S}$ runs over all representatives of the isomorphism classes of subgroups of $\mathcal{A}$*

Theorems 4 and 5 say that the computation of the number $\mathrm{Iso}_\Gamma^R(G;n)$ may be completed if one can compute the number $\mathrm{Isoc}_\Gamma(G;\mathcal{A})$ for each group $\mathcal{A}$ whose order is a divisor of $n$. In general, for a non-trivial subgroup $\Gamma$ of $\mathrm{Aut}(G)$, it is very hard to derive a formula for computing the number $\mathrm{Isoc}_\Gamma(G;\mathcal{A})$. The computation of the number $\mathrm{Isoc}_\Gamma(G;\mathcal{A})$ has been done for only few cases (see the table in Section 3). When $\Gamma = I$, Kwak et al [18] computed this number as follows.

**THEOREM 6.** *For any finite group $\mathcal{A}$,*

$$\mathrm{Isoc}(G;\mathcal{A}) = \frac{|\mathfrak{G}(\mathcal{A};\beta(G))|}{|\mathrm{Aut}(\mathcal{A})|},$$

*where* $\mathfrak{G}(\mathcal{A};n) = \{\,(g_1,g_2,\cdots,g_n) \in \mathcal{A}^n \;:\; \{g_1,g_2,\cdots,g_n\}$ generates $\mathcal{A}\,\}$.

**THEOREM 7.** *For any two finite groups $\mathcal{A}$ and $\mathcal{B}$ with* $(|\mathcal{A}|,|\mathcal{B}|) = 1$,

$$\mathrm{Isoc}(G;\mathcal{A}\oplus\mathcal{B}) = \mathrm{Isoc}(G;\mathcal{A})\ \mathrm{Isoc}(G;\mathcal{B}),$$

*and*

$$\mathrm{Iso}(G;\mathcal{A}\oplus\mathcal{B}) = \mathrm{Iso}(G;\mathcal{A})\ \mathrm{Iso}(G;\mathcal{B}).$$

Note that, even though $\mathrm{Iso}(G;\mathcal{A}\oplus\mathcal{B}) = \mathrm{Iso}(G;\mathcal{A})\ \mathrm{Iso}(G;\mathcal{B})$ for $(|\mathcal{A}|,|\mathcal{B}|) = 1$, it does not hold that $\mathrm{Iso}^R(G;mn) = \mathrm{Iso}^R(G;m)\,\mathrm{Iso}^R(G;n)$ for two relatively prime numbers $m$ and $n$, because there may be a group of order $mn$ which can not be expressed as a direct sum of two groups of order $m$ and $n$. For example, the dihedral group $\mathbb{D}_3$ of order 6 is not isomorphic to the direct sum of the two cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

## 2. Some explicit formulas

In this section, we introduce some known explicit enumeration formulas for the isomorphism classes of graph coverings.

The enumeration of the isomorphism classes of $n$-fold coverings of $G$ was completely done by Kwak and Lee [20], and independently Hofmeister [10].

**THEOREM 8.** ([10], [20]) *The number* $\mathrm{Iso}(G;n)$ *of the isomorphism classes of $n$-fold coverings of $G$ is*

$$\mathrm{Iso}(G;n) = \sum_{\ell_1+2\ell_2+\cdots+n\ell_n=n} (\ell_1!\,2^{\ell_2}\ell_2!\cdots n^{\ell_n}\ell_n!)^{\beta(G)-1}.$$

For convenience, let $\mathfrak{P}(n)$ denote the set of all partitions of the natural number $n$. For a partition $\mathfrak{p}$ of $n$, let $j_k(\mathfrak{p})$ denote the multiplicity of $k$ in the partition $\mathfrak{p}$, so that $j_1(\mathfrak{p})+2j_2(\mathfrak{p})+\cdots+nj_n(\mathfrak{p}) = n$. By using this terminology, Kwak and Lee [23] enumerate the number of the isomorphism classes of connected $n$-fold coverings as follows. (also see Hofmeister [13]).

**THEOREM 9.** ([23]) *For $n \geq 2$, the number of the isomorphism classes of connected $n$-fold coverings of $G$ is*

$$\begin{aligned}
&\mathrm{Isoc}(G;n)\\
&= \sum_{\ell_1+2\ell_2\cdots+(n-1)\ell_{n-1}=n-1} \left((\ell_1+1)^{\beta(G)-1}-1\right)\\
&\qquad\qquad \times \left(\ell_1!2^{\ell_2}\ell_2!\cdots(n-1)^{\ell_{n-1}}\ell_{n-1}!\right)^{\beta(G)-1}\\
&\quad + \sum_{2\ell_2+3\ell_3+\cdots+n\ell_n=n} \left(2^{\ell_2}\ell_2!\,3^{\ell_3}\ell_3!\cdots n^{\ell_n}\ell_n!\right)^{\beta(G)-1}\\
&\quad - \sum_{\substack{\mathfrak{p}\in\mathfrak{P}(n)-\{[[n;1]]\}\\ j_1(\mathfrak{p})=0}} \prod_{j_k(\mathfrak{p})\neq 0} \left(\frac{1}{j_k(\mathfrak{p})!}\prod_{\ell=0}^{j_k(\mathfrak{p})-1}(\mathrm{Isoc}(G;k)+\ell)\right),
\end{aligned}$$

*where the summation over the empty index set is defined to be* 0.

Notice that the number of the isomorphism classes of connected $n$-fold coverings of $G$ is equal to the total number of the conjugacy classes of subgroups of index $n$ of the free group generated by $\beta(G)$ elements.

For any natural number $n$, the enumeration of the isomorphism classes of regular $n$-fold coverings is much harder than that of all coverings. It has been done for only few cases.

**THEOREM 10.** ([9], [20], [32])

(a) *The number of the isomorphism classes of double coverings (resp. connected double coverings) of $G$ is*

$$\mathrm{Iso}^R(G;2) = 2^{\beta(G)} \quad \textit{and} \quad (\textit{resp. } \mathrm{Isoc}^R(G;2) = 2^{\beta(G)}-1).$$

(b) *For any prime number $p$, the number of the isomorphism classes of regular $p$-fold coverings (resp. connected regular $p$-fold coverings) of $G$ is*

$$\mathrm{Iso}^R(G;p) = \frac{p^{\beta(G)}+p-2}{p-1} \quad \textit{and} \quad \left(\textit{resp. } \mathrm{Isoc}^R(G;p) = \frac{p^{\beta(G)}-1}{p-1}\right).$$

The number of the isomorphism classes of $\mathcal{A}$-coverings was computed when $\mathcal{A} = \mathbb{Z}_p$([20, 32]), $\mathcal{A} = \mathbb{Z}_{p^2}$([21]), $\bigoplus_{i=1}^n \mathbb{Z}_p$([11]), etc. Almost all of these results can be contained in the following.

**THEOREM 11.** ([18]) *Let $m_1, \dots, m_\ell$ and $s_1, \dots, s_\ell$ be natural numbers with $s_\ell < \cdots < s_1$. Then the number of the isomorphism classes of connected $\oplus_{h=1}^{\ell} m_h \mathbb{Z}_{p^{s_h}}$-coverings of $G$ is*

$$\mathrm{Isoc}(G; \oplus_{h=1}^{\ell} m_h \mathbb{Z}_{p^{s_h}}) = p^{f(\beta(G), m_i, s_i)} \frac{\prod_{i=1}^{m} p^{\beta(G)-i+1} - 1}{\prod_{j=1}^{\ell} \prod_{h=1}^{m_j} p^{m_j - h+1} - 1},$$

*where $m = m_1 + \cdots + m_\ell$, $p$ is prime and*

$$f(\beta(G), m_i, s_i) = (\beta(G) - m)\left(\sum_{i=1}^{\ell} m_i(s_i - 1)\right) + \sum_{i=1}^{\ell-1} m_i \left(\sum_{j=i+1}^{\ell} m_j(s_i - s_j - 1)\right).$$

Now, we can calculate the number $\mathrm{Iso}(G; \mathcal{A})$ or $\mathrm{Isoc}(G; \mathcal{A})$ for any finite abelian group $\mathcal{A}$ by using Theorems 5, 7 and 11 repeatedly if necessary. As a consequence, we can obtain a formula to calculate the number of the subgroups of a given index of any finitely generated free abelian group.

For a non-abelian group $\mathcal{A}$, there is no explicit formulas for computing the number $\mathrm{Iso}(G; \mathcal{A})$ except that $\mathcal{A}$ is the dihedral group $\mathbb{D}_n$ of order $2n$.

**THEOREM 12.** ([18]) *For any $n \geq 3$, the number of the isomorphism classes of connected $\mathbb{D}_n$-coverings of $G$ is*

$$\mathrm{Isoc}(G; \mathbb{D}_n) = \left(2^{\beta(G)} - 1\right) \prod_{i=1}^{\ell} p_i^{(m_i - 1)(\beta(G)-2)} \frac{p_i^{\beta(G)-1} - 1}{p_i - 1},$$

*where $p_1^{m_1} \cdots p_\ell^{m_\ell}$ is the prime decomposition of $n$.*

Note that, by using the enumeration formulas discussed in Section 1 and Section 2, we can obtain explicit enumeration formulas for the isomorphism classes of (connected) regular $n$-fold coverings of $G$ when $n = p$, $n = 2p$, $n = p^2$ or $n = p^3$, where $p$ is a prime number.

## 3. Further remarks

In this section, we discuss some unsolved problems related to the enumeration of graph coverings. The known results on the enumeration of graph coverings are listed in following table.

| | $\Gamma$ is the trivial group | $\Gamma$ fixes a spanning tree | $\Gamma$ does not fix any spanning tree |
|---|---|---|---|
| $\mathrm{Iso}_\Gamma(G;n)$ | all $n$ [10, 20] | all $n$ [20] | $n=2$ [9] |
| $\mathrm{Isoc}_\Gamma(G;n)$ | all $n$ [13, 23] | $n=2$ [20] | $n=2$ [9] |
| $\mathrm{Iso}^R_\Gamma(G;n)$<br>$\mathrm{Isoc}^R_\Gamma(G;n)$ | $n=p,\ 2p,\ p^2,\ p^3$<br>($p$ : prime) [15, 18] | $n$ is prime [20] | $n$ is prime [32] |
| $\mathrm{Iso}_\Gamma(G;\mathcal{A})$<br>$\mathrm{Isoc}_\Gamma(G;\mathcal{A})$ | $\mathcal{A}=$ a finite abelian group or the dihedral group $\mathbb{D}_n$ [18] | $\mathcal{A}$ has the IEP [15] | $\mathcal{A}=\mathbb{Z}_2\oplus\mathbb{Z}_2$ [28],<br>$\mathbb{Z}_p$ ($p$ : prime) [32] |

From this table, we can ask the following problems.

**PROBLEM 1.** *For any natural number n and any group $\Gamma$ of automorphisms of $G$, enumerate the $\Gamma$-isomorphism classes of (connected) n-fold coverings of $G$.*

**PROBLEM 2.** *For any natural number n and any group $\Gamma$ of automorphisms of $G$, enumerate the $\Gamma$-isomorphism classes of (connected) regular n-fold coverings of $G$.*

**PROBLEM 3.** *For any finite group $\mathcal{A}$ and any group $\Gamma$ of automorphisms of $G$, enumerate the $\Gamma$-isomorphism classes of (connected) $\mathcal{A}$-coverings of $G$.*

Note that if one can solve Problem 3 for any group of order $n$, then the numbers $\mathrm{Iso}^R_\Gamma(G;n)$ or $\mathrm{Isoc}^R_\Gamma(G;n)$ can be computed. A weak version of Problem 3 is the following.

**PROBLEM 4.** *For any finite non-abelian group $\mathcal{A}$, enumerate the $I$-isomorphism classes of (connected) $\mathcal{A}$-coverings of $G$.*

Recently, Mizuno and Sato introduced the notion of a $g$-cyclic cover of a symmetric digraph as a generalization of a regular covering ([27], [28]). Sato [33] mentioned some open problems for the enumeration of the isomorphism classes of $g$-cyclic covers. Anyone interested in this topic is suggested to read his paper.

An enumeration of covering graphs can be applied to classify surface branched coverings (see [8], [19]).

**PROBLEM 5.** *For any pair of surfaces* $\mathbb{S}$ *and* $\widetilde{\mathbb{S}}$, *and natural number* $n$, *enumerate the equivalence classes of the (regular) branched* $n$*-fold coverings* $p : \widetilde{\mathbb{S}} \to \mathbb{S}$.

**PROBLEM 6.** *For any pair of surfaces* $\mathbb{S}$ *and* $\widetilde{\mathbb{S}}$, *and finite group* $\mathcal{A}$, *enumerate the equivalence classes of the regular branched coverings* $p : \widetilde{\mathbb{S}} \to \mathbb{S}$ *whose covering transformation groups are* $\mathcal{A}$.

If Problem 6 is solved for a finite group $\mathcal{A}$, then one can deduce some interesting results about $\mathcal{A}$-actions on surfaces. Problem 6 was solved in [19] and [24] when the covering transformation group $\mathcal{A}$ is $\mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_q$ or the dihedral group $\mathbb{D}_p$, where $p$ and $q$ are distinct primes.

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Department of Mathematics,
Yeungnam University,
Kyongsan, 712–749,
KOREA
E-mail: `julee@ynucc.yeungnam.ac.kr`