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A NOTE ON CYCLIC COLORATIONS AND MINIMAL TRIANGULATIONS

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Abstract. Let M be a closed 2-manifold. A face coloring of a triangulation T of M is called a cyclic coloration if, for any vertex, the incident faces have different colors. Let $V(T)$ denote the vertex set of T. We conjecture that there will be found a constant $C(M)$ so that $|V(T)|+C(M)$ colors are enough for cyclic coloration of any triangulation T of M. When M is not the projective plane, we conjecture that $|V(T)|$ colors will suffice, whenever T is minimal with respect to the number of vertices. If this conjecture is true, the formula for the minimum number of vertices in a triangulation of a given 2-manifold with boundary is determined to have at most one gap.

1. Introduction

In this note we consider only connected, compact 2-manifolds, with or without boundary. Every 2-manifold is uniquely determined, up to homeomorphisms, by this triple: {orientability class, Euler characteristic, number of boundary components}. Every orientable 2-manifold $M_\sigma = M_{\chi,\sigma}$ with boundary is obtained from a closed orientable 2-manifold $M = M_\chi$ with Euler characteristic χ by deleting some σ disjoint open 2-disks. The deleted disks are the "holes" of $M_{\chi,\sigma}$. We sometimes omit "χ" in indices. The nonorientable antipode of an orientable 2-manifold M will be differentiated by a tilde in notation, $\tilde{M}$. As matter of notation, $M_\chi = M_{\chi,0}$ is a closed 2-manifold.

We first introduce the concept of a "cyclic coloration of a triangulation" T of a fixed closed 2-manifold M. A face coloring of T is called a *cyclic coloration* if, for any vertex, the incident faces have different colors. By $V(T)$ we denote the vertex set of T. We conjecture that there will exist a constant $C(M)$ such that $|V(T)| + C(M)$ colors suffice for cyclic coloration of any triangulation T of M. By $V_{\min}(M)$ we denote the minimum number of vertices that a triangulation of a given 2-manifold M can have. Then a triangulation T of M is called a *minimal triangulation* if $|V(T)| = V_{\min}(M)$. We conjecture that $|V(T)|$ colors will suffice

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for cyclic coloration of any minimal triangulation T of any closed 2-manifold M, not the projective plane. This conjecture would imply the completeness of the formula, Eq.(7) obtained in Section 3, for the minimum number of vertices which a triangulation of a given 2-manifold with boundary can have.

2. Cyclic colorations

The *cyclic chromatic number* of a triangulation T of a given 2-manifold M is denoted by $\xi(T)$ and is defined to be the smallest number of colors that suffice for cyclic coloration of T.

CONJECTURE 1. *Let $M = M_\chi$ be a closed 2-manifold (orientable or nonorientable), not the projective plane, and let T be a minimal triangulation of M_χ. Then*

$$\xi(T) = |V(T)| = V_{\min}(M_\chi). \tag{1}$$

For example, consider the complete graph K_7 whose vertices are labeled by 0, 1, 2, 3, 4, 5 and 6. Let us interpret these labels as the elements of the cyclic group $\boldsymbol{Z}_7$, the additive group of integers modulo 7. It is well-known that K_7 admits a triangulation of the torus M_0 which is minimal. One such triangulation is induced by the following triangular rotation scheme, taken from Ringel's book [5]:

0.	1	3	2	6	4	5
1.	2	4	3	0	5	6
2.	3	5	4	1	6	0
3.	4	6	5	2	0	1
4.	5	0	6	3	1	2
5.	6	1	0	4	2	3
6.	0	2	1	5	3	4

Each row indicates the cyclic ordering of the vertices around the corresponding vertex. This means, for instance, that row 0 induces the faces 0 1 3, 0 3 2, 0 2 6, 0 6 4, 0 4 5 and 0 5 1. Let us color the faces of this triangulation algebraically; associate the colors with the elements of $\boldsymbol{Z}_7$ and color face uvw with $u + v + w$ modulo 7. The so-obtained face coloring is a cyclic coloration, which is easily seen from the following two observations.

(i) Row 0 of the scheme is *precise*, i.e., the sums of the neighboring elements, in pairs in cyclic order around 0, are all different (addition is always carried out in the group $\boldsymbol{Z}_7$).

(ii) The scheme is of *index* 1, i.e., row i is obtained from row 0 by adding i to each element.

It would be tempting to apply the same algebraic method for cyclic coloration of the other minimal triangulations of the Map Color Theorem presented in [5], since they are usually of index 1 (in particular, in the regular orientable Cases 7 and 10). Unfortunately, except the above example, we have failed to identify a scheme in [5] with precise row 0.

Note that in Conjecture 1 we have to exclude the case of the projective plane. For, the familiar embedding of the complete graph K_6 in the projective plane is a triangulation with ten faces, by Euler's formula. Since any two faces have at least one vertex in common, the cyclic chromatic number of this triangulation is 10.

The minimality assumption is essential, too. For instance, consider K_5 without one edge in the 2-sphere M_2.

In the general case, we obviously have $\xi(T) \geq \Delta(T)$, where $\Delta(T)$ denotes the maximum degree of a vertex of T. It is a sensible task to estimate $\xi(T)$ above by an expression of the form $\Delta(T) + C(M)$, where $C(M)$ is some (yet unknown) constant.

Plummer and Toft [3] studied the dual of this problem. They colored the *vertices* of an embedding $\Psi : G \to M_2$ of a planar graph G in the 2-sphere M_2 *cyclically*, i.e., so that vertices on the same face receive different colors.

THEOREM 1. (Plummer and Toft [3]) *For every spherical embedding $\Psi : G \to M_2$ of a 3-connected graph G, we have*

$$\xi^*(\Psi) \leq \Delta^*(\Psi) + 9, \tag{2}$$

where $\xi^*(\Psi)$ denotes the minimum number of colors that suffice to color the vertices of Ψ cyclically and $\Delta^*(\Psi)$ denotes the maximum size of a face of Ψ. ■

We do not know any results dealing with cyclic coloration of graphs embedded in 2-manifolds other than the 2-sphere.

Now we return to the cyclic (face) colorations of triangulations. As the extremal possibility, we have $\Delta(T) = |V(T)| - 1$. This extreme is the case in *pyramidal* triangulations, i.e., triangulations having a vertex adjacent to all other vertices; in particular, minimal triangulations are normally pyramidal (see [5]). Therefore, it is also a sensible task to establish upper bounds of the form $|V(T)| + C(M)$.

CONJECTURE 2. *Let M be a closed 2-manifold. Then there will exist a constant $C(M)$ such that*

$$\xi(T) \leq |V(T)| + C(M) \tag{3}$$

for any triangulation T of M, and therefore

$$\limsup_{|V(T)|\to\infty} \frac{\xi(T)}{|V(T)|} = 1 \tag{4}$$

over triangulations T of M.

3. Application to minimal triangulations

The number $V_{\min}(M)$ has been determined by Ringel [4] for each nonorientable closed 2-manifold $M = \tilde{M}_\chi$, and by Jungerman and Ringel [2] for each orientable closed 2-manifold $M = M_\chi$:

$$V_{\min}(M) = \left\lceil \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rceil, \tag{5}$$

except the 2-manifolds $\tilde{M}_0$, $\tilde{M}_{-1}$, and M_{-2}, for which $V_{\min}(M) = 8$, 9 and 10, respectively.

For 2-manifolds M_σ with boundary, the problem of determining $V_{\min}(M_\sigma)$ is open, except several small cases. Here we look at the problem of determining $V_{\min}(M_\sigma)$, restricting ourselves to the triangulations in which every boundary component is a cycle of length 3. [*Note*: under this restriction $V_{\min}(M_{0,1}) = V_{\min}(M_0) = 7$, while in general $V_{\min}(M_{0,1}) = 6$.] It has turned out that the so-restricted problem is closely related to the problem of determining the maximum number, denoted by $I(M)$, of (pairwise) independent faces in a minimal triangulation of closed 2-manifold M, where two faces are regarded as independent if they are disjoint in the form of closures.

CONJECTURE 3. (Archdeacon [1]) *Let M be a closed 2-manifold, not the projective plane. Then*

$$I(M) = \left\lfloor \frac{V_{\min}(M)}{3} \right\rfloor. \tag{6}$$

The following is a formula for the minimum number of vertices in a triangulation of a given 2-manifold with boundary.

THEOREM 2. *For a given 2-manifold M_σ with σ, $\sigma \geq 1$, boundary components, there exists an integral constant $J(M)$ such that:*

$$V_{\min}(M_\sigma) = \begin{cases} V_{\min}(M) & \text{if } \sigma \leq I(M), \\ 3\sigma & \text{if } \sigma > J(M). \end{cases} \tag{7}$$

Proof. The part $\sigma \leq I(M)$ is obvious. To prove the existence of $J(M)$, let T_0 be a triangulation of the closed 2-manifold M with a collection of independent faces removed. Assume T_0 has a vertex, v, not on the boundary. It can be easily seen that by suitably triangulating some face of T_0 incident to v and then removing one of the newly-formed faces, again incident to v, one can diminish the number of interior (nonboundary) vertices, still having a 2-manifold (with boundary). Repeatedly applying this process to T_0, one can finally make all the vertices be on the boundary of the resulting 2-manifold. We may take $J(M)$ equal to the number of holes (or boundary components) in the resulting 2-manifold. For, once all the vertices become on the boundary, we can continue to increase repeatedly the number of vertices by three, and the number of holes by one at the same time, by adding an octahedron to a suitable face. On the other hand, it is clear that $|V(T)| \geq 3\sigma$ for any triangulation T of M_σ. The result follows. ∎

As matter of notation, hereafter by $J(M)$ we will denote the minimum value of $J(M)$ for which the formula of Eq.(7) holds. The following conjecture in fact says that there is no gap in that formula:

CONJECTURE 4. *Let M be a closed 2-manifold, not the projective plane. Then*

$$J(M) = I(M). \tag{8}$$

EXAMPLE 1. (i) For the orientable part, consider the triangulation of the torus M_0 with the complete graph K_7 given by the rotation scheme in the preceding section. We have $I(M_0) = 2$, $V_{\min}(M_{0,3}) = 9$, and $J(M_0) = 2$.

(ii) For the nonorientable part, consider the triangulation of the projective plane $\tilde{M}_1$ with the complete graph K_6. We have $I(\tilde{M}_1) = 1$, $V_{\min}(\tilde{M}_{1,2}) = 7$, and $J(\tilde{M}_1) = 2$.

THEOREM 3. *In the case of orientable $M = M_\chi$, if Conjecture 1 is true, then Conjecture 4 is true, with "$I(M)$" possibly replaced by "$I(M) + 1$;" in other words, $J(M) = I(M)$ or $I(M) + 1$.*

Proof. We first prove that if Conjecture 3 is true, then Conjecture 4 is true. The notation s is used to denote the quantity $\lfloor V_{\min}(M_\chi)/3 \rfloor$. Let T_0 be a minimal triangulation of M_χ with some s independent faces deleted. When $V_{\min}(M_\chi) \equiv 0 \pmod 3$, all the vertices of T_0 are on the boundary. Then we can increase repeatedly the number of vertices by 3 and the number of holes by 1 as in the proof of Theorem 2. When $V_{\min}(M_\chi) \equiv 1 \pmod 3$, we can make all the vertices be on the boundary by accommodating an $(s + 1)$st hole as described in the proof of Theorem 2, and then proceed as in that proof. Finally, when $V_{\min}(M_\chi) \equiv 2 \pmod 3$, it may require already $(s+1)$st and $(s+2)$nd holes to make all the vertices be on the boundary, in which event we would have to replace "$I(M)$" by "$I(M)+1$." [*Note*: the $(s+2)$nd hole would be required in the apparently rare case where the two interior vertices of the 2-manifold $M_{\chi,\sigma}$, constructed as in the proof of Theorem 2, are nonadjacent.]

Now we prove that if Conjecture 1 is true then Conjecture 3 is true. Let T be a minimal triangulation of M_χ. It is obvious that $I(M_\chi) \le \lfloor V_{\min}(M_\chi)/3 \rfloor$. On the other hand, by Euler formula,

$$|F(T)| = 2|V(T)| - 2\chi, \tag{9}$$

and we have

$$I(M_\chi) \ge \left\lceil \frac{|F(T)|}{\xi(T)} \right\rceil = \left\lceil \frac{2V_{\min}(M_\chi) - 2\chi}{V_{\min}(M_\chi)} \right\rceil \ge \left\lfloor \frac{V_{\min}(M_\chi)}{3} \right\rfloor. \tag{10}$$

The last inequality in this sequence follows from the following inequalities:

$$0 < \frac{V_{\min}(M_\chi)}{3} - \frac{2V_{\min}(M_\chi) - 2\chi}{V_{\min}(M_\chi)} < 1. \tag{11}$$

These inequalities are equivalent to the following ones (respectively),

$$V_{\min}(M_\chi)^2 - 6V_{\min}(M_\chi) + 6\chi > 0 > V_{\min}(M_\chi)^2 - 9V_{\min}(M_\chi) + 6\chi, \tag{12}$$

which are obvious, since $V_{\min}(M_\chi)$ given by Eq.(5) is strictly between the larger roots of the quadric polynomials $x^2 - 6x + 6\chi$ and $x^2 - 9x + 6\chi$, respectively. The exceptional case of M_{-2} is settled by a straightforward verification. The proof is complete. ∎

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