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SPECIAL ISSUE

PROBLEMS IN TOPOLOGICAL GRAPH THEORY
— QUESTIONS I CAN'T ANSWER —

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Abstract. This paper describes my Problems in Topological Graph Theory, which can be accessed through the world-wide-web at

`http://www.emba.uvm.edu/~archdeac/problems/problems.html.`

This list of problems is constantly being revised; the interested reader is encouraged to submit additions and updates.

1. Introduction

I have collected a number of open problems in topological graph theory. The problems are placed on the world-wide web at

`http://www.emba.uvm.edu/~archdeac/problems/problems.html.`

Each problem is a self-contained module with some of the basic definitions, the statement of the problem, and some citations into the literature. The intent is to spread interest in what I consider to be a very interesting field.

I toyed with the idea of writing a one-line paper: namely, giving only the web address. In the end I decided to provide three sample problems to give the reader an idea of what to expect.

Enjoy my problems—I do!

2. How often must curves touch?

Suppose that we are given n simple closed curves in the plane such that each curve shares a point with every other curve, but no three curves share a common point. Two curves that share a point need not cross transversally—tangential intersections are allowed. Let $f(n)$ denote the minimum number of intersections over all such families of curves.

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It is obvious that $f(n) \geq \binom{n}{2}$. It is also easy to construct examples that show $f(n) \leq 2\binom{n}{2}$: take n circles C_i of the same radius with centers perturbed a small distance ϵ_i from the origin. It is also known that $f(4) = 6$, $f(5) = 12$, $f(6) = 20$.

PROBLEM. *Determine* $f(7)$.

Asymptotically:

CONJECTURE. $\lim_{n\to\infty} \frac{f(n)}{\binom{n}{2}} = 2.$

It is not too hard to show that $f(n+1) \geq \lceil \frac{n+1}{n-1} f(n) \rceil$: in any realization of $n+1$ curves, count the number of induced intersections over all sub-realizations of n curves. In combination with $f(6) = 20$, this implies that the limit in the conjecture is at least $3/2$.

I believe that these problems were first posed by Bruce Richter and Carsten Thomassen in [2]. A partial result is given by Gelasio Salazar [3], who showed that if any two such curves intersect in at most k points (for a fixed value of k), then the above limit is 2. This includes many special cases; for example, when each curve is an ellipse, or when each curve is a piecewise-linear p-gon for fixed p.

3. Drawing rotations in the plane

Suppose that we are given a drawing of the complete graph K_n in the plane (in a drawing nonadjacent edges are allowed to cross at most once, and this crossing must be transversal). This drawing determines a *local rotation* at each vertex: the (clockwise) cyclic permutation of the edges incident with each vertex from 1 to n. This collection of local rotations is called a *rotation*. It is well-known that rotations are in a bijective correspondence with embeddings of K_n in an oriented surface. Less well-known is their use in studying drawings in the plane.

For example, there are 16 possible rotations on K_4. Two of these are planar corresponding to the two unique embeddings of K_4 into the oriented plane. Six more correspond to planar drawings of K_4 (there are three ways to choose two nonadjacent crossing edges and two mirror images for each choice). The remaining eight rotations on K_4 cannot be achieved by planar drawings. For example, $\{0 : 132,\ 1 : 023,\ 2 : 031,\ 3 : 012\}$ denotes a planar embedding of the complete graph on $\{0, 1, 2, 3\}$ where $a : bcd$ represents the permutation of edges around vertex a incident with vertices (b, c, d) respectively. Similarly $\{0 : 132,$

1 : 023, 2 : 031, 3 : 021} represents a permutation which cannot arise in any planar drawing.

PROBLEM. *Determine those rotations on K_n which can arise from a planar drawing.*

It is tempting to conjecture that a rotation can arise from a planar drawing of K_n if and only if the rotation induced on every K_4 is achievable. However, this conjecture is false. The rotation {0 : 1243, 1 : 2430, 2 : 3041, 3 : 0412, 4 : 1032} on K_5 cannot be realized by a planar drawing, but every induced K_4 rotation can be so realized.

This problem arose when trying to find a lower bound for the crossing number of the complete graph (see the related problem *The Crossing Number of the Complete Graph* from this problem list). Knowing the rotation is sufficient to count the number of crossings in a drawing. Hence a solution to the above problem would reduce finding the minimum crossing number of the complete graph to a purely combinatorial problem.

I note that the problem naturally suggests itself to finding the rotations corresponding to drawings of a arbitrary graph G, although without the applications to finding the minimum crossing number of G. Another related problem would be to find the minimum genus g so that a given rotation ρ could be realized by a drawing on the sphere with g handles—the *crossing genus* of ρ.

PROBLEM. *Find the maximum crossing genus over all rotations on K_n.*

4. Defective-list-coloring planar graphs

Perhaps the most famous result in graph theory is the *Four-Color Theorem*: every planar graph can be vertex-4-colored such that no two vertices with the same color are adjacent. The problem posed here combines two variations on this theme.

The first variation is *list colorings*. Suppose that each vertex of a graph is given a list of k colors. The goal is to assign to each vertex a color from its list such that adjacent vertices receive distinct colors. The usual concept of k-coloring corresponds to the special case when each vertex has the same list of colors. If a graph G can always list colored for every possible k-list assignment to the vertices, then we say that G is *k-list-colorable*.

At first glance, coloring from lists appears less difficult than coloring from a fixed set of colors: if the lists on adjacent vertices are not identical, selecting distinct colors should be easier. However, Voigt showed [5] that there are planar graphs with lists of 4 colors on each vertex that cannot be properly list-colored. Thomassen showed [4] that every planar graph is 5-list-colorable.

The second variation involves *defective colorings.* A k-coloring has *defect* d if each vertex is adjacent to at most d other vertices with the same color. The usual concept of coloring corresponds to the case $d = 0$. A graph is (k, d)*-colorable* if it can be colored with k colors with defect d. It is known that every planar graph can be (4,0)-colored and (3,2)-colored. See e.g. [1] for an entry into the literature.

Now combine the two variations on the theme. Suppose that each vertex is given a list of k colors, and we want to assign each vertex a color from its list such that it is adjacent to at most d vertices with the same color. If this can be done for every possible k-list assignment, then we say that the graph is (k, d)-*list-colorable.* It is natural to ask for which pairs (k, d) is every planar graph (k, d)-list-colorable. I believe that the answer is known for all pairs except (4,1), although I do not have the reference.

PROBLEM. *Is every planar graph (4,1)-list-colorable?*

5. Conclusion

I urge the interested reader to visit the web site. I also urge the reader to do two things: 1) If you have an interesting problem in Topological Graph Theory, submit it to me for addition to the list. 2) If you have any additional information on a problem, especially a solution, please let me know so that the list can be updated.

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