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GRAPH COVERING AND ITS GENERALIZATION

By

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Abstract. We present some topics on graph covering and its generalization. We survey some results on enumeration of isomorphism classes of coverings of a graph and g-cyclic A-covers of a symmetric digraph, where A is a finite group and $g \in A$. We also mention some related questions.

1. Introduction

The central concern of topological graph theory is the embedding of graphs on surfaces. One mathematical structure that permits an economical description of graphs and their embeddings is a covering space(graph covering) of a graph. The details for constructing a graph covering are efficiently encoded in what is called a voltage graph. Every covering of a graph aries from some permutation voltage graph. Furthermore, every regular covering of a graph is constructed by some ordinary voltage graph.

Recently, enumeration results are of particular interest in topological graph theory. For example, Mohar [39,40], used coverings of K_4 to enumerate the akempic triangulations of the 2-sphere with 4 vertices of degree 3. Negami [42] established a bijection between the equivalence classes of embeddings of a 3-connected nonplanar graph G into a projective plane and the isomorphism classes of planar 2-fold coverings of G. Mull, Rieper and White [41] enumerated 2-cell embeddings of connected graphs. Hofmeister [11-17], Kwak and Lee [5,7,19,22-28] enumerated several classes of graph coverings.

Enumeration of graph coverings started from classification of double coverings of a graph by Waller [46] in 1976. After about ten years, Hofmeister [11] enumerated the isomorphism (Γ-isomorphism) classes of double coverings (2-fold coverings or $\boldsymbol{Z}_2$-coverings) of a graph with respect to any group Γ of its automorphisms. Hofmeister [12] and, independently, Kwak and Lee [24] enumerated the I-isomorphism classes of n-fold coverings of a connected graph G, for any $n \in \boldsymbol{N}$, where I is the trivial automorphism group of G. The general problem of

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counting the Γ-isomorphism classes of all n-fold coverings of G is still unsolved except in the cases of $n = 2$ or $\Gamma = I$.

The enumeration of Γ-isomorphism classes of regular n-fold coverings of G is a weak version of the above problem, but is still unsolved except in the case of prime n. Sato[44] counted the Γ-isomorphism classes of regular p-fold coverings of G for any prime $p\,(> 2)$. Some enumeration of I-isomorphism classes of regular coverings of G were done by Hofmeister [15], Kwak and Lee [19,22,25].

Cheng and Wells [3] discussed isomorphism classes of cyclic triple covers (1-cyclic $\boldsymbol{Z}_3$-covers) of a complete symmetric digraph. Furthermore, Mizuno and Sato [33] presented cyclic p-tuple covers of a symmetric digraph D, and enumerated the number of Γ-isomorphism classes of cyclic p-tuple covers of D. For a symmetric digraph D, a finite group A and $g \in A$, Mizuno and Sato [34] introduced a g-cyclic A-cover of D as a generalization of regular coverings and cyclic p-tuple covers, and enumerated the number of I-isomorphism classes of g-cyclic $\boldsymbol{F}_p^r$-covers of a connected symmetric digraph D for any finite dimensional vector space $\boldsymbol{F}_p^r$ over the finite field $\boldsymbol{F}_p = GF(p)(p > 2)$. Mizuno and Sato [37] gave a necessary and sufficient condition for two cyclic A-covers of a connected symmetric digraph D to be Γ-isomorphic for any finite abelian group A with the isomorphism extension property, and enumerated the number of I-isomorphism classes of g-cyclic $\boldsymbol{Z}_{p^n}$-covers of D for any prime $(p > 2)$. Furthermore, Mizuno, Lee and Sato [30] enumerated the number of I-isomorphism classes of connected g-cycic $\boldsymbol{Z}_p^n$-covers and connected h-cycic $\boldsymbol{Z}_{p^n}$-covers of D, where $p\,(> 2)$ is prime, and the orders of g and h are odd.

In this article, we survey some results on enumeration of graph coverings, g-cyclic A-covers and mention a related topics. In Section 2, we give definition and notation of graph coverings. In Sections 3, we deal with enumeration of isomorphism classes of coverings of a graph. In Sections 4,5, we treat enumeration of isomorphism classes of g-cyclic A-covers and connected g-cyclic A-covers of a connected symmetric digraph. In Section 6, we describe decomposition formulas for the characteristic polynomials of regular coverings and g-cyclic A-covers. A general theory of graph coverings refer to Gross and Tucker [10].

2. Definition and notation

Graphs and digraphs treated here are finite and simple.

A graph H is called a *covering* of a graph G with projection $\pi : H \to G$ if there is a surjection $\pi : V(H) \to V(G)$ such that $\pi|_{N(v')} : N(v') \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. The projection $\pi : H \to G$ is an *n-fold covering* of G if π is n-to-one. A covering $\pi : H \to G$ is said to be *regular* if there is a subgroup B of the automorphism group $\mathrm{Aut}\, H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Permutation voltage assignments were introduced by Gross and Tucker [9]. For a graph G, let $D(G)$ be the arc set of the symmetric digraph corresponding to G. A *permutation voltage assignment* of G with voltages in the symmetric group S_r of degree r is a function $\phi : D(G) \to S_r$ such that inverse arcs have inverse assignments. The pair (D, ϕ) is called a *permutation voltage graph.* The *(permutation) derived graph* G^ϕ derived from a permutation voltage assignment ϕ is defined as follows:

$V(G^\phi) = V(G) \times \{1, \cdots, r\}$, and $((u,h),(v,k)) \in D(G^\phi)$ if and only if $(u,v) \in D(G)$ and $\phi(u,v)(h) = k$.

The *natural projection* $\pi : G^\phi \to G$ is a function from $V(G^\phi)$ onto $V(G)$ which erases the second coordinates. Gross and Tucker [9] showed that every covering of a given graph aries from some permutation voltage assignment in a symmetric group.

Ordinary voltage assignments were introduced by Gross [8]. Let A a finite group. Then a mapping $\alpha : D(G) \to A$ is called an *ordinary voltage assignment* if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in D(G)$. The *(ordinary) derived graph* G^α derived from an ordinary voltage assignment α is defined as follows:

$V(G^\alpha) = V(G) \times A$, and $((u,h),(v,k)) \in D(G^\alpha)$ if and only if $(u,v) \in D(G)$ and $k = h\alpha(u,v)$.

The *natural projection* $\pi : G^\alpha \to G$ is a function from $V(G^\alpha)$ onto $V(G)$ which erases the second coordinates. The graph G^α is called an *A-covering* of G . The A-covering G^α is an $|A|$-fold regular covering of G. Every regular covering of G is an A-covering of G for some group A (see [9]).

Let α and β be two permutation (ordinary) voltage assignments on G with voltages in $S_r(A)$, and let Γ be a group of automorphisms of G, denoted $\Gamma \leq \operatorname{Aut} G$. Two coverings G^α and G^β are called *Γ-isomorphic*, denoted $G^\alpha \cong_\Gamma G^\beta$, if there exist an isomorphism $\Phi : G^\alpha \to G^\beta$ and a $\gamma \in \Gamma$ such that $\pi\Phi = \gamma\pi$, i.e., the diagram

$$\begin{tikzcd} G^\alpha \arrow[r, "\Phi"] \arrow[d, "\pi"'] & G^\beta \arrow[d, "\pi"] \\ G \arrow[r, "\gamma"] & G \end{tikzcd}$$

commutes. Let $I = \{1\}$ be the trivial group of automorphisms. A general theory of graph coverings is developed in Gross and Tucker [10].

3. Enumeration of graph coverings

3.1 Characterization

Enumeration of graph coverings is to cout the isomorphism classes of coverings of a graph with respect to a group Γ of its automorphisms.

A characterizations for isomorphic graph coverings were given by Hofmeister [12], Kwak and Lee [24].

THEOREM 1. (Hofmeister; Kwak and Lee) *Let G be a graph and $\Gamma \leq \mathrm{Aut}\, G$. For two permutation voltage assignments $\alpha : D(G) \to S_r$ and $\beta : D(G) \to S_r$, the following are equivalent:*

1. $G^{\alpha} \cong_{\Gamma} G^{\beta}$.
2. *There exist a family $(\pi_u)_{u \in V(G)} \in S_r^{V(G)}$ and $\gamma \in \Gamma$ such that*

$$\beta^{\gamma}(u,v) = \pi_v \alpha(u,v) \pi_u^{-1} \text{ for each } (u,v) \in D(G),$$

where the multiplication of permutations is carried out from right to left.

3.2 N-fold coverings

We state one problem on enumeration of graph coverings.

PROBLEM 1. *For any natural number n, enumerate the Γ-isomorphism classes of n-fold coverings of a graph G.*

Problem 1 is still unsolved except in the cases of $n = 2$ or $\Gamma = I$. Two-fold coverings (or double coverings) of graphs are regular, and were dealed in Hofmeister [11] and Waller [46]. The Γ-isomorphism classes of 2-fold coverings of a graph G was counted by Hofmeister [11], where the enumeration was done by commutative algebra arguments.

For $\gamma \in \Gamma$, a $\langle\gamma\rangle$-orbit σ of length k on $E(G)$ is called *diagonal* if $\sigma = \langle\gamma\rangle\{x, \gamma^k(x)\}$ for some $x \in V(G)$. The vertex orbit $\langle\gamma\rangle x$ and the arc orbit $\langle\gamma\rangle(x, \gamma^k(x))$ are also called *diagonal*. For $\gamma \in \Gamma$, let $G(\gamma)$ be a simple graph whose vertices are the $\langle\gamma\rangle$-orbits on $V(G)$, with two vertices adjacent in $G(\gamma)$ if and only if some two of their representatives are adjacent in G. The kth *2-level* of $G(\gamma)$ is the induced subgraph of $G(\gamma)$ on the vertices ω such that $\theta_2(|\omega|) = 2^k$, where $\theta_2(i)$ is the largest power of 2 dividing i. A *2-level component* of $G(\gamma)$ is a connected component of some 2-level of $G(\gamma)$. A 2-level component H is called *favorable* if there exists a vertex σ of H which is diagonal or adjacent in $G(\gamma)$ to a vertex ω such that $\theta_2(|\sigma|) > \theta_2(|\omega|)$. Otherwise H is called *defective* (see [47]).

THEOREM 2. (Hofmeister) *The number of Γ-isomorphism classes of double coverings ($\boldsymbol{Z}_2$-coverings) of a graph G is*

$$\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} 2^{\epsilon(\gamma)-\nu(\gamma)+\omega(\gamma)},$$

where $\epsilon(\gamma)$ and $\nu(\gamma)$ is the number of $\langle\gamma\rangle$-orbits on $E(G)$ and $V(G)$, respectively, and $\omega(\gamma)$ is the number of defective 2-level components in $G(\gamma)$.

Hofmeister [12] and, independently, Kwak and Lee [24] enumerated the I-isomorphism classes of n-fold coverings of a graph, for any $n \in \boldsymbol{N}$.

THEOREM 3. (Hofmeister; Kwak and Lee) *The number of I-isomorphism classes of n-fold coverings of a connected graph G is*

$$\sum_{k_1+2k_2+\cdots+nk_n=n} (k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!)^{\beta(G)-1},$$

where $\beta(G) = |E(G)| - |V(G)| + 1$ is the Betti number of G.

3.3 Regular n-fold coverings

A weak version of Problem 1 is

PROBLEM 2. *For any natural number n, enumerate the Γ-isomorphism classes of regular n-fold coverings of a graph G.*

Problem 2 is solved for any prime number p. Since any regular 2-fold coverings are double coverings, the case of $p = 2$ is given in Theorem 2.

Sato [44] counted the Γ-isomorphism classes of regular p-fold coverings of a connected graph G for any prime $p (> 2)$.

Let $\gamma \in \Gamma, \lambda \in \boldsymbol{Z}_p^*$ and $\mathrm{ord}(\lambda) = m$. A diagonal arc orbit of length $2k$ (the corresponding edge orbit of length k and the corresponding vertex orbit of length $2k$) is called *type-1* if $\lambda^k = -1$, and *type-2* otherwise. A $\langle\gamma\rangle$-orbit σ on $V(G), E(G)$ or $D(G)$ is called *m-divisible* if $|\sigma| \equiv 0 \pmod m$. A m-divisible $\langle\gamma\rangle$-orbit σ on $V(G)$ ia called *strongly m-divisible* if σ satisfies the following condition:

If $\Omega = \langle\gamma\rangle(x,y)$ is any not diagonal $\langle\gamma\rangle$-orbit on $D(G)$, and $y = \gamma^j(x)$, $x, y \in \sigma$, then $j \equiv 0 \pmod m$.

The kth *p-level* and *p-level components* of $G(\gamma)$ are defined similarly to the kth 2-level and 2-level components of $G(\gamma)$. Let $G_\lambda(\gamma)$ be the subgraph of $G(\gamma)$

induced by the set of m-divisible $\langle\gamma\rangle$-orbits on $V(G)$. The kth *p-level* and *p-level components* of $G_\lambda(\gamma)$ are defined similarly to the case of $G(\gamma)$. A p-level component K of $G_\lambda(\gamma)$ is called *defective* if each vertex σ of H is strongly m-divisible, not type-1 diagonal, and satisfies $\theta_p(|\omega|) > \theta_p(|\sigma|)$ whenever $\omega \notin V(H)$ and $\sigma\omega \in E(G(\gamma))$. Otherwise H is called *favorable.*

THEOREM 4. (Sato) *Let G be a connected graph, $p\,(>2)$ prime and $\Gamma \leq \mathrm{Aut}\,G$. The number of Γ-isomorphism classes of regular p-fold coverings of a connected graph G is*

$$\frac{1}{|\Gamma|(p-1)}\sum_{\gamma\in\Gamma}\sum_{\lambda\in \mathbf{Z}_p^*} p^{\epsilon(\gamma)-\nu(\gamma)+\nu_0(\gamma,\lambda)-\kappa(\gamma,\lambda)-\mu(\gamma,\lambda)+d(\gamma,\lambda)},$$

where $\nu_0(\gamma,\lambda), \mu(\gamma,\lambda)$ and $d(\gamma,\lambda)$ is the number of not m-divisible $\langle\gamma\rangle$-orbits on $V(G)$, type-2 diagonal $\langle\gamma\rangle$-orbits on $E(G)$ and defective p-level components in $G_\lambda(\gamma)$, respectively, where $\mathrm{ord}\,(\lambda) = m$, *and $\kappa(\gamma,\lambda)$ is the number of not m-divisible $\langle\gamma\rangle$-orbits on $E(G)$ which are not diagonal.*

3.4 *A*-coverings

PROBLEM 3. *Enumerate the Γ-isomorphism classes of A-coverings of a graph for any finite group A.*

Problem 3 is still unsolved except in the case that A is any cyclic group of prime order or $A = \mathbf{Z}_2 \times \mathbf{Z}_2$. $\mathbf{Z}_p$-coverings are regular p-fold coverings for any prime p.

Mizuno and Sato [38] enumerated the number of Γ-isomorphism classes of $\mathbf{Z}_2 \times \mathbf{Z}_2$-coverings of a connected graph G for any group Γ of automorphisms of G. This enumeration is a unique result for Problem 3 except in the case of $A = \mathbf{Z}_p$ (p: prime).

For $\gamma \in \Gamma$, a $\langle\gamma\rangle$-orbit σ on $V(G), E(G)$ or $D(G)$ is called *3-divisible* if $|\sigma| \equiv 0 \pmod 3$. A 3-divisible $\langle\gamma\rangle$-orbit σ on $V(G)$ is called *strongly 3-divisible* if σ satisfies the following condition:

If $\Omega = \langle\gamma\rangle(x,y)$ is any not diagonal $\langle\gamma\rangle$-orbit on $D(G)$, and $y = \gamma^j(x)$, $x, y \in \sigma$, then $j \equiv 0 \pmod 3$.

Let $G_3(\gamma)$ be the subgraph of $G(\gamma)$ induced by the set of 3-divisible $\langle\gamma\rangle$-orbits on $V(G)$. The kth *2-level* and *2-level components* of $G_3(\gamma)$ are defined similarly to the case of $G(\gamma)$. A 2-level component K of $G_3(\gamma)$ is called *strongly favorable* if some vertex σ of H is not strongly 3-divisible, diagonal or adjacent in $G(\gamma)$ to a vertex ω such that $\theta_2(|\sigma|) > \theta_2(|\omega|)$. Otherwise H is called *strongly defective.*

THEOREM 5. (Mizuno and Sato) *Let G be a connected graph and $\Gamma \leq \operatorname{Aut} G$. For $\gamma \in \Gamma$, let $\epsilon_0(\gamma)$ and $\nu_0(\gamma)$ be the number of not 3-divisible $\langle\gamma\rangle$-orbits on $E(G)$ and $V(G)$, respectively. Furthermore, let $d(\gamma)$ be the number of strongly defective 2-level components in $G_3(\gamma)$. Then the number of Γ-isomorphism classes of $\boldsymbol{Z}_2^2$-coverings of G is*

$$\frac{1}{6|\Gamma|}\sum_{\gamma\in\Gamma}\{4^{\epsilon(\gamma)-\nu(\gamma)+\omega(\gamma)}+3\cdot 2^{\epsilon(\gamma^2)-\nu(\gamma^2)+\omega(\gamma^2)}+2\cdot 4^{\epsilon(\gamma)-\nu(\gamma)+\nu_0(\gamma)+d(\gamma)-\epsilon_0(\gamma)}\}.$$

The number of I-isomorphism classes of regular fourfold coverings of graphs were enumerated by Hong and Kwak [18].

THEOREM 6. (Hong and Kwak) *Let G be a connected graph. Then the number of I-isomorphism classes of regular fourfold coverings of G is*

$$\frac{1}{3}(2^{2\beta(G)+1}+1),$$

where $\beta(G)$ is the Betti number of G.

A regular 4-fold covering of G is either a $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$-covering or $\boldsymbol{Z}_4$-covering of G. If the enumeration of Γ-isomorphism classes of $\boldsymbol{Z}_4$-coverings of G is established, then we might be able to count the number of Γ-isomorphism classes of regular 4-fold coverings of G. This will be a unique result for Problem 2 except in the case that n is prime.

PROBLEM 4. *Enumerate the Γ-isomorphism classes of regular 4-fold coverings of a graph G for any $\Gamma \leq \operatorname{Aut} G$.*

In general, it is natural to ask

PROBLEM 5. *Enumerate the Γ-isomorphism classes of regular p^2-fold coverings of G for any prime $p\,(>2)$ and any $\Gamma \leq \operatorname{Aut} G$.*

It seems that Problem 5 is very hard to answer, because the parameters in counting formula might be more complicated than Theorem 4.

Some reults for the enumeration of I-isomorphism classes of of A-coverings of a connected graph G were known. Kwak and Lee [25] did it for $\boldsymbol{Z}_p \bigoplus \boldsymbol{Z}_q$ ($p \neq q$: prime) or $\boldsymbol{Z}_{p^2}$-coverings of G. The I-isomorphism classes of regular coverings of graphs with voltages in finite dimensional vector spaces over finite fields were enumerated by Hofmeister [15]. Hong, Kwak and Lee [19] gave the number of I-isomorphism classes of $\boldsymbol{Z}_{p^m}$-coverings, $\boldsymbol{Z}_p \bigoplus \boldsymbol{Z}_p$-coverings and D_n-coverings, n: odd, of graphs, respectively.

THEOREM 7. (Hofmeister) *The number of I-isomorphism classes of $m\mathbf{Z}_p$-coverings of G is*

$$1+\sum_{h=1}^{m}\frac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)\cdots(p^{\beta(G)-h+1}-1)}{(p^h-1)(p^{h-1}-1)\cdots(p-1)}.$$

THEOREM 8. (Hong, Kwak and Lee) *The number of I-isomorphism classes of $\mathbf{Z}_{p^m}$-coverings of G is*

$$\begin{cases} m+1 & \text{if } \beta(G)=1\ , \\ \dfrac{p^{m(\beta(G)-1)+1}-1}{p-1}+\dfrac{p^{m(\beta(G)-1)}-1}{p^{\beta(G)-1}-1} & \text{otherwise.} \end{cases}$$

For a connected graph G and $n \in \mathbf{N}$, let $\mathrm{Iso}_{\Gamma}(G;n)$ ($\mathrm{Iso}_{\Gamma}^{R}(G;n)$, $\mathrm{Isoc}_{\Gamma}^{R}(G;n)$) be the number of Γ-isomorphism classes of (regular, connected regular) n-fold coverings of G. For a finite group A, let $\mathrm{Iso}_{\Gamma}(G;A)$ ($\mathrm{Isoc}_{\Gamma}(G;A)$) be the number of Γ-isomorphism classes of (connected) A-coverings of G.

Kwak and Lee [27] enumerated the I-isomorphism classes of connected n-fold coverings of a connected graph. Kwak, Chun and Lee [22] investigated the I-isomorphisms of connected A-coverings of a connected graph G, and gave a decomposition formula for them.

THEOREM 9. (Kwak, Chun and Lee)

1. *For any $n \in \mathbf{N}$, $\mathrm{Iso}_{I}^{R}(G;n)=\sum_{d|n}\mathrm{Isoc}_{I}^{R}(G;d)$.*
2. *For any finite group A, $\mathrm{Iso}_{I}(G;A)=\sum_{S}\mathrm{Isoc}_{I}(G;S)$, where S runs over all representatives of isomorphism classes of subgroups of A.*
3. *For any $n \in \mathbf{N}$, $\mathrm{Isoc}_{I}^{R}(G;n)=\sum_{A}\mathrm{Isoc}_{I}(G;A)$, where A runs over all representatives of isomorphism classes of groups of order n.*

Thus, they enumerated the I-isomorphism classes of connected A-coverings of G when A is a finite abelian group or the dihedral group $\mathbf{D}_n$.

4. A generalization of graph coverings

4.1 Background

Let G be a graph with vertex set V and X a subset of V. Then the operation of *switching* at X replaces all edges between X and $V \setminus X$ with nonedges and nonedges with edges, leaving edges and nonedges within each part unaltered. We say that H is *switching equivalent* to G if H is obtained from G by switching at X for some $X \subseteq V$. The equivalence classes in graphs with vertex set V are called *switching classes* of graphs on V. Mallows and Sloane [29] showed that two-graphs, Euler graphs and switching classes of graphs on n vertices have the same number of isomorphism classes. Cameron [2] stated the "equivalence" of switching classes of graphs on V and double coverings of the complete graph on V.

Wells [47] defined signed switching classes of a graph. Given a graph G, let $C^0(G;\boldsymbol{Z}_2)$ and $C^1(G;\boldsymbol{Z}_2)$ be the set of all functions $s : V(G) \to \boldsymbol{Z}_2$ and all ordinary voltage assignments $\alpha : D(G) \to \boldsymbol{Z}_2$, respectively. The coboundary operator $\delta : C^0(G;\boldsymbol{Z}_2) \to C^1(G;\boldsymbol{Z}_2)$ is defined by $(\delta s)(x,y) = s(x) - s(y)$ for $s \in C^0(G;\boldsymbol{Z}_2)$ and $(x,y) \in D(G)$. Two elements α, β in $C^1(G;\boldsymbol{Z}_2)$ are called switching equivalent if $\beta = \alpha + \delta s$ for some $s \in C^0(G;\boldsymbol{Z}_2)$. The equivalence classes are called *signed switching classes* of G. Zaslavsky [48] showed that there is a one-to-one correspondence between I-isomorphism classes of double coverings and signed switching classes of G. Wells [47] enumerated the number of Γ-isomorphism classes of signed switching classes of G, which is equal to that of Γ-isomorphism classes of double coverings ($\boldsymbol{Z}_2$-coverings) of G by Hofmeister [11] (see Theorem 2).

Cheng and Wells [3] presented the switching classes of digraphs and a cyclic triple cover of a complete symmetric digraph. Given a finite set X, let V^0 and V^1 be the set of all functions $s : X \to \boldsymbol{Z}_3$ and all alternating functions $\alpha : X \times X \to \boldsymbol{Z}_3$, respectively. For the coboundary operator $\delta : V^0 \to V^1$, the cosets of $\mathrm{Im}\,\delta$ in V^1 are called *switching classes of digraphs* on X. Let KD be the complete symmetric digraph with vertex set X. For $\alpha \in V^1$, the *cyclic triple cover* $D(\alpha)$ of KD is given by

$$V(D(\alpha)) = X \times \boldsymbol{Z}_3 \text{ and } ((x,i),(y,j)) \in A(D(\alpha)) \text{ if and only if } \\ x \neq y \text{ and } j = \alpha(x,y) + i - 1.$$

Cheng and Wells [3] led the fact that there exists a one-to-one correspondence between I-isomorphism classes of cyclic triple covers of KD and switching classes of digraphs on X.

Replacing $\boldsymbol{Z}_3$ and KD with $\boldsymbol{Z}_p$ (p: prime) and a symmetric digraph D in the definition of switching classes of digraphs, Mizuno and Sato [31] introduced *switching classes of alternating functions* on $A(D)$, and counted the Γ-isomorphism classes of them. Furthermore, Mizuno and Sato [33] presented cyclic p-tuple covers of D. For an alternating function $\alpha : A(D) \to \boldsymbol{Z}_p$, the *cyclic p-tuple cover* $D(\alpha)$ of D is defined by

$$V(D(\alpha)) = V(D) \times \boldsymbol{Z}_p \text{ and } ((x,i),(y,j)) \in A(D(\alpha)) \text{ if and only}$$

if $(x,y) \in A(D)$ and $j = \alpha(x,y) + i - 1$.

They [33] showed that the number of Γ-isomorphism classes of switching classes of alternating functions on $A(D)$ is equal to that of Γ-isomorphism classes of cyclic p-tuple covers of D, and enumerated them.

For a graph G and a finite field $\boldsymbol{F}_q$ $(q = p^n)$, let $C^0(G;\boldsymbol{F}_q)$ and $C^1(G;\boldsymbol{F}_q)$ be the set of all functions $s : V(G) \to \boldsymbol{F}_q$ and all ordinary voltage assignments $\alpha : D(G) \to \boldsymbol{F}_q$, respectively. For the coboundary operator $\delta : C^0(G;\boldsymbol{F}_q) \to C^1(G;\boldsymbol{F}_q)$, the cosets of $\operatorname{Im}\delta$ in $C^1(G;\boldsymbol{F}_q)$ are called *switching equivalence classes.*

Hofmeister [17] defined the above switching equivalence classes and gave a counting formula for the number of Γ-isomorphism classes of switching equivalence classes.

Let $\Gamma \leq \operatorname{Aut} D$ and $\gamma \in \Gamma$. A p-level component H of $G(\gamma)$ is called *minimal* if there exists no vertex σ of H which is adjacent to a vertex ω such that $\theta_p(|\sigma|) > \theta_p(|\omega|)$ (see [17, 47]).

THEOREM 10. (Hofmeister) *The number of Γ-isomorphism classes of switching equivalence classes is*

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} q^{\epsilon(\gamma) - \nu(\gamma) + \xi(\gamma) - \rho(\gamma)},$$

where $\xi(\gamma)$ and $\rho(\gamma)$ is the number of minimal p-level components on $G(\gamma)$, and diagonal $\langle\gamma\rangle$-obits on $E(G)$, respectively.

Let D be a symmetric digraph, A a finite group and $g \in A$. Mizuno and Sato [34] introduced a g-cyclic A-cover of D as a generalization of regular coverings and cyclic p-tuple covers, and discussed the number of Γ-isomorphism classes of g-cyclic $\boldsymbol{Z}_p^r$-covers of a connected symmetric digraph D for any finite dimensional vector space $\boldsymbol{Z}_p^r$ over the finite field $\boldsymbol{Z}_p = GF(p)$ $(p > 2)$. Thus, they enumerated the number of I-isomorphism classes of g-cyclic $\boldsymbol{Z}_p^r$-covers of D.

THEOREM 11. (Mizuno and Sato) *Let $g \neq \boldsymbol{0}$. Then the number of I-isomorphism classes of g-cyclic $\boldsymbol{Z}_p^r$-covers of D is*

$$\frac{1}{|GL_r(\boldsymbol{Z}_p)|} \sum_{m=1}^{r} \begin{bmatrix} r-1 \\ m-1 \end{bmatrix}_p \alpha(p^r, m) p^{m\beta(D)},$$

where $GL_r(\boldsymbol{Z}_p)$ is the general linear group, $\alpha(p^r, m)(0 \leq m \leq r)$ is the number of $\boldsymbol{A} \in GL_r(\boldsymbol{Z}_p)$ such that a given m-dimensional subspace of $\boldsymbol{Z}_p^r$ is the eigenspace of $\boldsymbol{A}$ belonging to the eigenvalue 1 and $[{l \atop k}]_p$ is the p-binomial number.

The proof is made by counting some isomorphism classes of switching equivalence classes.

4.2 Cyclic A-covers of symmetric digraphs

Let D be a symmetric digraph and A a finite group. A function $\alpha : A(D) \to A$ is called *alternating* if $\alpha(y,x) = \alpha(x,y)^{-1}$ for each $(x,y) \in A(D)$. For $g \in A$, a *g-cyclic A-cover* $D_g(\alpha)$ of D is the digraph as follows:

$V(D_g(\alpha)) = V(D) \times A$, and $((u,h),(v,k)) \in A(D_g(\alpha))$ if and only if $(u,v) \in A(D)$ and $k^{-1}h\alpha(u,v) = g$.

The *natural projection* $\pi : D_g(\alpha) \to D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a *cyclic A-cover* of D if D' is a g-cyclic A-cover of D for some $g \in A$. In the case that A is abelian, then $D_g(\alpha)$ is simply called a *cyclic abelian cover.* Furthermore the 1-cyclic A-cover $D_1(\alpha)$ of a symmetric digraph D can be considered as the A-covering $\widetilde{D}^\alpha$ of the underlying graph $\widetilde{D}$ of D.

Let α and β be two alternating functions from $A(D)$ into A, and let Γ be a subgroup of the automorphism group $\operatorname{Aut} D$ of D, denoted $\Gamma \le \operatorname{Aut} D$. Let $g, h \in A$. Then two cyclic A-covers $D_g(\alpha)$ and $D_h(\beta)$ are called *Γ-isomorphic,* denoted $D_g(\alpha) \cong_\Gamma D_h(\beta)$, if there exist an isomorphism $\Phi : D_g(\alpha) \to D_h(\beta)$ and a $\gamma \in \Gamma$ such that $\pi\Phi = \gamma\pi$, i.e., the diagram

$$\begin{tikzcd} D_g(\alpha) \arrow[r, "\Phi"] \arrow[d, "\pi"'] & D_h(\beta) \arrow[d, "\pi"] \\ D \arrow[r, "\gamma"] & D \end{tikzcd}$$

commutes. Let $I = \{1\}$ be the trivial group of automorphisms.

The group Γ of automorphisms of D acts on the set $C(D)$ of alternating functions from $A(D)$ into A as follows:

$$\alpha^\gamma(x,y) = \alpha(\gamma(x),\gamma(y)) \text{ for all } (x,y) \in A(D),$$

where $\alpha \in C(D)$ and $\gamma \in \Gamma$. Any voltage $g \in A$ determines a permutation $\rho(g)$ of the symmetric group S_A on A which is given by $\rho(g)(h) = hg,\ h \in A$.

Mizuno and Sato [34] gave a characterization for two cyclic A-covers of D to be Γ-isomorphic.

THEOREM 12. (Mizuno and Sato) *Let D be a symmetric digraph, A a finite group, $g, h \in A$, $\alpha, \beta \in C(D)$ and $\Gamma \le \operatorname{Aut} D$. Then the following are equivalent:*

1. $D_g(\alpha) \cong_\Gamma D_h(\beta)$.
2. *There exist a family $(\pi_u)_{u \in V(D)} \in S_A^{V(D)}$ and $\gamma \in \Gamma$ such that*

$$\rho(\beta^\gamma(u,v)h^{-1}) = \pi_v \rho(\alpha(u,v)g^{-1})\pi_u^{-1} \textit{ for each } (u,v) \in A(D),$$

where the multiplication of permutations is carried out from right to left.

4.3 Isomorphisms of cyclic abelian covers

Let D be a connected symmetric digraph and A a finite abelian group. Let G be the underlying graph, T be a spanning tree of G and w a root of T. For any $\alpha \in C(D)$ and any walk W in G, the *net α-voltage* of W, denoted $\alpha(W)$, is the sum of the voltages of the edges of W. Then the *T-voltage* α_T of α is defined as follows:

$$\alpha_T(u,v) = \alpha(P_u) + \alpha(u,v) - \alpha(P_v) \ \textit{for each} \ (u,v) \in D(G) = A(D),$$

where P_u and P_v denote the unique path from w to u and v in T, respectively. For a function $f : C(D) \to A$, the *net f-value* $f(W)$ of any walk W is defined as the net α-voltage of W. For a function $f : C(D) \to A$, let the *pseudolocal voltage group* $A_f(v)$ of f at v denote the subgroup of A generated by all net f-values of the closed walk based at $v \in V(D)$. Let $\operatorname{ord}(g)$ be the order of $g \in A$.

THEOREM 13. (37, Theorem 2) *Let D be a connected symmetric digraph, A a finite abelian group $g, h \in A$ and $\alpha, \beta \in C(D)$. Furthermore, let G be the underlying graph of D, T a spanning tree of G and $\Gamma \leq \operatorname{Aut} G$. Assume that the orders of g and h are equal and odd. Then the following are equivalent:*

1. $D_g(\alpha) \cong_\Gamma D_h(\beta)$.
2. *There exist $\gamma \in \Gamma$ and an isomorphism $\sigma : A_{\alpha_T - g}(w) \to A_{\beta_{\gamma T} - h}(\gamma(w))$ such that*

$$\beta^\gamma_{\gamma T}(u,v) - h = \sigma(\alpha_T(u,v) - g) \ \textit{for each} \ (u,v) \in A(D),$$

where $(\alpha_T - g)(u,v) = \alpha_T(u,v) - g, \ (u,v) \in A(D)$ *and* $w \in V(D)$.

An finite group $\mathcal{B}$ is said to have the *isomorphism extension property* (IEP), if every isomorphism between any two isomorphic subgroups $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{B}$ can be extented to an automorphism of $\mathcal{B}$ (see [19]). For example, the cyclic group $\boldsymbol{Z}_n$ for any $n \in \boldsymbol{N}$, the dihedral group $\boldsymbol{D}_n$ for odd $n \geq 3$, and the direct sum of m copies of $\boldsymbol{Z}_p$ have the IEP.

Let $\operatorname{Iso}(D, A, g, \Gamma)$ be the number of Γ-isomorphism classes of g-cyclic A-covers of D.

THEOREM 14. (37, Theorem 3) *Let D be a connected symmetric digraph, G its underlying graph, A a finite abelian group with the IEP, $g, h \in A$ and $\Gamma \leq \operatorname{Aut} D$.*

Assume that the orders of g and h are odd, and $\rho(g) = h$ for some $\rho \in \operatorname{Aut} A$. Then

$$\operatorname{Iso}(D, A, g, \Gamma) = \operatorname{Iso}(D, A, h, \Gamma).$$

4.4 Isomorphisms of orbit-cyclic abelian covers

Let D be a connected symmetric digraph, A a finite abelian group with the IEP and $\Pi = \operatorname{Aut} A$. For an element g of A with odd order, the Π-orbit on A containing g is denoted by $\Pi(g)$. A cyclic A-cover $D_h(\alpha)$ of D is called $\Pi(g)$-*cyclic* if $h \in \Pi(g)$. Let $\mathcal{D}_k$ be the set of all k-cyclic A-covers of D for any $k \in A$, and let $\mathcal{D} = \bigcup_{h \in \Pi(g)} \mathcal{D}_h$. Then $\mathcal{D}$ is the set of all $\Pi(g)$-cyclic A-covers of D. Let $\mathcal{D}/\cong_\Gamma$ and $\mathcal{D}_h/\cong_\Gamma$ be the set of all Γ-isomorphism classes over $\mathcal{D}$ and $\mathcal{D}_h$, respectively.

THEOREM 15. (37, Theorem 4) *Let D be a connected symmetric digraph, A a finite abelian group with the IEP, $\Gamma \leq \operatorname{Aut} D$ and $\Pi = \operatorname{Aut} A$. Furthermore, let g be an element of A with odd order. Then*

$$|\mathcal{D}/\cong_\Gamma| = \operatorname{Iso}(D, A, h, \Gamma) \ \textit{for each}\ h \in \Pi(g).$$

Now, we state the structure of Γ-isomorphism classes of $\Pi(g)$-cyclic A-covers of D.

The set of ordinary voltage assignments of G with voltages in A is denoted by $C^1(G; A)$. Note that $C(D) = C^1(G; A)$. Furthremore, let $C^0(G; A)$ be the set of functions from $V(G)$ into A. We consider $C^0(G; A)$ and $C^1(G; A)$ as additive groups. The homomorphism $\delta : C^0(G; A) \to C^1(G; A)$ is defined by $(\delta s)(x, y) = s(x) - s(y)$ for $s \in C^0(G; A)$ and $(x, y) \in A(D)$. For each $\alpha \in C^1(G; A)$, let $[\alpha]$ be the element of $C^1(G; A)/\operatorname{Im}\delta$ which contains α.

The automorphism group $\operatorname{Aut} A$ acts on $C^1(G; A)$ as follows:

$$(\sigma\alpha)(x, y) = \sigma(\alpha(x, y)) \ \textit{for}\ (x, y) \in A(D),$$

where $\alpha \in C^1(G; A)$ and $\sigma \in \operatorname{Aut} A$.

THEOREM 16. (37, Theorem 5) *Let D be a connected symmetric digraph, A a finite abelian group with the IEP, $\Gamma \leq \operatorname{Aut} D$ and $\Pi = \operatorname{Aut} A$. Suppose that $g \in A$ has odd order. Let σ_h be a fixed automorphism of A such that $\sigma_h(g) = h$ for $h \in \Pi(g)$. Then any Γ-isomorphism class of $\Pi(g)$-cyclic A-covers of D is of the form*

$$\bigcup_{h \in \Pi(g)} \{D_h(\sigma_h\beta) \mid \beta = \sigma\alpha^\gamma + \delta s,\ \sigma \in \Pi_g,\ \gamma \in \Gamma,\ s \in C^0(G; A)\},$$

where $\alpha \in C(D)$ and G is the underlying graph of D.

4.5 Isomorphisms of cyclic Z_n-covers

Mizuno and Sato [37] enumerated the number of I-isomorphism classes of g-cyclic $\boldsymbol{Z}_{p^m}$-covers of D, for any $g \in \boldsymbol{Z}_{p^m}$. Let $\beta(D) = m - n + 1$ be the Betti-number of D, where $m = |A(D)|/2$ and $n = |V(D)|$.

THEOREM 17. (Mizuno and Sato) *Let D be a connected symmetric digraph and $p\,(> 2)$ prime. Let $g \in \boldsymbol{Z}_{p^m}$ and* $\mathrm{ord}\,(g) = p^{m-\mu}$ *the order of g. Set $\beta = \beta(D)$. Then the number of I-isomorphism classes of g-cyclic $\boldsymbol{Z}_{p^m}$-covers of D is*

$$\mathrm{Iso}\,(D, \boldsymbol{Z}_{p^m}, g, I)$$
$$= \begin{cases} p^{m\beta-\mu} + p^{(m-\mu)\beta-1}(p-1)(p^{\mu(\beta-1)}-1)/(p^{\beta-1}-1) & \textit{if } \mu \neq m \textit{ and } \beta > 1, \\ p^{m-\mu-1}\{(\mu+1)p-\mu\} & \textit{if } \mu \neq m \textit{ and } \beta = 1, \\ (p^{m(\beta-1)+1}-1)/(p-1) + (p^{m(\beta-1)}-1)/(p^{\beta-1}-1) & \textit{if } \mu = m \textit{ and } \beta > 1, \\ m+1 & \textit{otherwise.} \end{cases}$$

In Table 1, we give some values of $isc(D, \boldsymbol{Z}_{3^6}, g, I)$.

Table 1.

$\mu \setminus B$	1	2	3	4	5
1	405	216513	138706101	96467701761	69195236437845
2	189	76545	46589661	32184598401	23067403335549
3	81	26001	15543009	10728553761	7689144011121
4	33	8721	5181489	3576188961	2563048043073
5	13	2913	1727181	1192063041	854349347853

Mizuno and Sato [37] showed that the number of Γ-isomorphism classes of g-cyclic $\boldsymbol{Z}_p$-covers of a connected symmetric digraph is equal to that of nonisomorphic switching equivalence classes of its underlying graph for each $g \in \boldsymbol{Z}_p^*$.

THEOREM 18. (Mizuno and Sato) *Let D be a connected symmetric digraph, $p\,(> 2)$ prime, $g \in \boldsymbol{Z}_p \setminus \{0\}$ and $\Gamma \leq$* $\mathrm{Aut}\,D$. *Then the number of Γ-isomorphism classes of g-cyclic $\boldsymbol{Z}_p$-covers of D is*

$$\mathrm{Iso}\,(D, \boldsymbol{Z}_p, g, \Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} p^{\epsilon(\gamma)-\nu(\gamma)+\xi(\gamma)-\rho(\gamma)}.$$

4.6 Further remarks

PROBLEM 6. *Let A be any finite abelian group and $g \in A$ of odd order. Then, enumerate the Γ-isomorphism classes of g-cyclic A-covers of a connected symmetric digraph D.*

It seems that this problem is a difficult problem, because Problem 3 is very hard.

In the case that A is a finite group and the order of $g \in A$ is even, we have no information about the isomorphisms of g-cyclic A-covers of D. We propose the following problem.

PROBLEM 7. *Let g be the element of even order in a finite abelian group A. Then, what is an algebraic condition for two g-cyclic A-covers of D to be Γ-isomorphic (c.f., Theorem 13).*

In general, we ask

PROBLEM 8. *For any finite group A and $g \in A$, what is an algebraic condition for two g-cyclic A-covers of D to be Γ-isomorphic.*

5. Connected cyclic abelian covers

5.1 Isomorphisms of connected cyclic abelian covers

Mizuno, Lee and Sato [30] considered the number of Γ-isomorphism classes of connected g-cycic A-covers of D for a finite abelian group A and $g \in A$ of odd order.

Let D be a connected symmetric digraph, G the underlying graph of D, T a spanning tree of G and A a finite abelian groups. If $g \in A$ with odd order, then the pseudolocal voltage groups $A_{\alpha - g}$ and $A_{\alpha_T - g}$ are equal to the group generated by g and $A_\alpha = A_{\alpha_T}$. Moreover, $D_g(\alpha)$ is connected if and only if $A_{\alpha - g}$ is the full group A.

The following result is given by a method similar to the proof of Theorem 13.

THEOREM 19. (30, Theorem 1) *Let D be a connected symmetric digraph, G the underlying graph of D, T a spanning tree of G and $\Gamma \leq Aut(G)$. Let A, B be two finite abelian groups, $g \in A$ and $h \in B$. Let $\alpha \in C^1(G; A)$ and $\beta \in C^1(G; B)$. Assume that the orders of g and h are odd. Then the following are equivalent:*

1. $D_g(\alpha) \cong_\Gamma D_h(\beta)$.
2. *There exist* $\gamma \in \Gamma$ *and an isomorphism* $\sigma : A_{\alpha_T - g} \to B_{\beta^\gamma_{\gamma T} - h}$ *such that*

$$\beta^\gamma_{\gamma T}(u,v) = \sigma(\alpha_T(u,v)) \text{ for each } (u,v) \in A(D) \quad \text{and} \quad \sigma(g) = h.$$

Furthermore, if both α *and* β *derive connected cyclic abelian covers, then the above statement* 1 *is also equivalent to:*

There exist $\gamma \in \Gamma$ *and a group isomorphism* $\sigma : A \to B$ *such that*

$$\beta^\gamma_{\gamma T}(u,v) = \sigma(\alpha_T(u,v)) \text{ for each } (u,v) \in A(D) \quad \text{and} \quad \sigma(g) = h.$$

Let $\alpha \in C(D)$ which assigns identity for each arc in a spanning tree T of the underlying graph G of D. Let $v \in V(D)$ be fixed. Then the component of g-cyclic A-cover $D_g(\alpha)$ containing $(v, 0)$ is called the *identity component* of $D_g(\alpha)$. By the definition of cyclic A-covers, it is not hard to show that each component of $D_g(\alpha)$ is isomorphic to the identity component and two cyclic abelian coverings of D are I-isomorphic if and only if their identity components are I-isomorphic. Furthermore, the identity component of g-cyclic A-cover $D_g(\alpha)$ is just a g-cyclic $A_{\alpha-g}$-cover if g is of odd order.

For a finite abelian group A and $g \in A$, let $\mathrm{Isoc}\,(D, A, g, I)$ be the number of I-isomorphism classes of connected g-cyclic A-covers of D. Let S_1 and S_2 be two subgroups of A containing g. We say that S_1 and S_2 are *isomorphic* with respect to g or *g-equivalent* if there exists an isomorphism $\sigma : S_1 \to S_2$ such that $\sigma(g) = g$.

THEOREM 20. (30, Theorem 2) *Let D be a connected symmetric digraph, A a finite abelian group and $g \in A$. Assume that the order of g is odd. Then*

$$\mathrm{Iso}\,(D, A, g, I) = \sum_S \mathrm{Isoc}\,(D, S, g, I),$$

where S runs over all representatives of g-equivalence classes of subgroups of A which contain g.

PROBLEM 9. *For any* $\Gamma \leq \mathrm{Aut}\, D$, *is there a decomposition formula for the number of* Γ*-isomorphism classes of g-cyclic A-covers of D.*

If Problem 9 is affirmative, then we guess that an approach for Problem 6 is obtained. Furthermore, Problem 9 is related to

PROBLEM 10. *Enumerate the* Γ*-isomorphism classes of connected g-cyclic A-covers of D.*

This problem is a difficult problem, and so it is interesting to count the I-isomorphism classes of connected g-cyclic A-covers of D.

Let D be a connected symmetric digraph and A a finite abelian group. For A and a natural number n, let

$$F_g(A;n) = \{(g_1, \cdots, g_n) \in A^n \mid \{g\} \cup \{g_1, \cdots, g_n\} \text{ generates } A\}.$$

We give a formula on the number of I-isomorphism classes of connected g-cyclic A-covers of D for an element g of odd order.

THEOREM 21. (30, Theorem 3) *Let D be a connected symmetric digraph, A a finite abelian group and $g \in A$. Furthermore, assume that the order of g is odd. Then*

$$\mathrm{Isoc}\,(D, A, g, I) = |F_g(A; \beta(D))| / |(\mathrm{Aut}\, A)_g|,$$

where $\beta(D)$ is the Betti number of D and $(\mathrm{Aut}\, A)_g = \{\sigma \in \mathrm{Aut}\, A \mid \sigma(g) = g\}$.

5.2 Connected cyclic $\boldsymbol{Z}_p^n$-covers and cyclic $\boldsymbol{Z}_{p^n}$-covers

Mizuno, Lee and Sato [30] counted the number of I-isomorphism classes of connected g-cyclic A-covers of D, when A is the cyclic group $\boldsymbol{Z}_{p^n}$ and the direct sum $\boldsymbol{Z}_p^n$ of n copies of $\boldsymbol{Z}_p$ for any prime number $p\,(> 2)$.

THEOREM 22. (Mizuno, Lee and Sato) *Let D be a connected symmetric digraph and $g \in \boldsymbol{Z}_p^n \setminus \{0\}$. Then the number of I-isomorphism classes of connected g-cyclic $\boldsymbol{Z}_p^n$-covers of D is*

$$\mathrm{Isoc}\,(D, \boldsymbol{Z}_p^n, g, I) = \frac{p^{\beta-n+1}(p^{\beta}-1)\cdots(p^{\beta-n+2}-1)}{(p^{n-1}-1)(p^{n-2}-1)\cdots(p-1)},$$

where $\beta = \beta(D)$.

The following formula is an explicit form of the formula in Theorem 11.

COROLLARY 1. *Let D be a connected symmetric digraph and $g \in \boldsymbol{Z}_p^n \setminus \{0\}$. Then the number of I-isomorphism classes of g-cyclic $\boldsymbol{Z}_p^n$-covers of D is*

$$\mathrm{Iso}\,(D, \boldsymbol{Z}_p^n, g, I) = \sum_{k=1}^{n} \frac{p^{\beta-k+1}(p^{\beta}-1)\cdots(p^{\beta-k+2}-1)}{(p^{k-1}-1)(p^{k-2}-1)\cdots(p-1)}.$$

THEOREM 23. (Mizuno, Lee and Sato) *Let D be a connected symmetric digraph, $\boldsymbol{Z}_{p^n}$ the cyclic group of order $p^n\,(p\,(> 2)$: prime) and $g \in \boldsymbol{Z}_{p^n} \setminus \{0\}$. Furthermore,* *let* $\operatorname{ord}(g) = p^{n-\mu}$ $(\mu < n)$ *be the order of* g. *Then the number of I-isomorphism classes of connected* g*-cyclic* $\boldsymbol{Z}_{p^n}$*-covers of* D *is*

$$\operatorname{Isoc}(D, \boldsymbol{Z}_{p^n}, g, I) = \begin{cases} p^{(n-1)\beta-\mu}(p^\beta - 1) & \text{if } \mu \geq 1, \\ p^{n\beta} & \text{otherwise.} \end{cases}$$

The following formula is an alternative form of the formula in Theorem 17.

COROLLARY 2. *Let* D *be a connected symmetric digraph and* $g \in \boldsymbol{Z}_{p^n} \setminus \{0\}$. *Furthermore, let* $\operatorname{ord}(g) = p^{n-\mu}$ $(0 < \mu < n)$ *be the order of* g. *Then the number of I-isomorphism classes of* g*-cyclic* $\boldsymbol{Z}_{p^n}$*-covers of* D *is*

$$\operatorname{Iso}(D, \boldsymbol{Z}_{p^n}, g, I) = p^{(n-\mu-1)\beta} + p^{(n-\mu-1)\beta}(p^\beta - 1)\frac{p^{(\mu+1)(\beta-1)} - 1}{(p^{\beta-1} - 1)}.$$

6. Characteristic polynomials

6.1 Characteristic polynomials of cyclic A-covers

Let G be a graph or a digraph. Two vertices are *adjacent* if they are joined by an edge (arc). The *adjacency matrix* $\boldsymbol{A}(G)$ of a graph (digraph) G whose vertex set is $\{v_1, \cdots, v_n\}$ is a square matrix of order n, whose entry a_{ij} at the place (i, j) is equal to 1 if there exists an edge (arc) starting at the vertex v_i and terminating at the vertex v_j, and 0 otherwise. Then the *characteristic polynomial* $\Phi(G; \lambda)$ of G is defined by $\Phi(G; \lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}(G))$.

Schwenk [43] studied relations between the characteristic polynomials of some related graphs. Kitamura and Nihei [21] discussed the structure of regular double coverings of graphs by using their eigenvalues. Chae, Kwak and Lee [5] gave the complete computation of the characteristic polynomials of K_2 (or $\overline{K}_2$)-bundles over graphs. Kwak and Lee [26] obtained a formula for the chracteristic polynomial of a graph bundle when its voltage assignment takes in an abelian group. Sohn and Lee [45] showed that the characteristic polynomial of a weighted K_2 (or $\overline{K}_2$)-bundles over a weighted graph of G can be expressed as a product of characteristic polynomials of two weighted graphs whose underlying graphs are G. Mizuno and Sato [35] established an explicit decomposition formula for the characteristic polynomial of a regular covering of a graph.

Mizuno and Sato [36] gave a decomposition formula for the characteristic polynomial of a g-cyclic A-cover of a symmetric digraph D for any finite group A and any $g \in A$. As a corollary, we obtained the above formula for the characteristic polynomial of a regular covering of a graph.

THEOREM 24. (Mizuno and Sato) *Let D be a symmetric digraph, A a finite group, $g \in A$ and $\alpha : A(D) \to A$ an alternating function. Furthermore, let $\rho_1 = 1, \rho_2, \cdots, \rho_t$ be the irreducible representations of A, and f_i the degree of ρ_i for each i, where $f_1 = 1$. For $h \in A$, the matrix $\mathbf{A}_h = (a_{uv}^{(h)})$ is defined as follows:*

$$a_{uv}^{(h)} := \begin{cases} 1 & \text{if } \alpha(u,v) = h \text{ and } (u,v) \in A(D), \\ 0 & \text{otherwise.} \end{cases}$$

Then the characteristic polynomial of the g-cyclic A-cover $D_g(\alpha)$ of D is

$$\Phi(D_g(\alpha);\lambda) = \Phi(D;\lambda) \cdot \prod_{j=2}^{t} \{\Phi(\sum_{h \in A} \rho_j(h) \otimes \mathbf{A}_{hg};\lambda)\}^{f_j},$$

where $\otimes$ is the Kronecker product of matrices.

COROLLARY 3. $\Phi(D;\lambda) \mid \Phi(D_g(\alpha);\lambda)$.

Let D be the symmetric digraph corresponding to a graph G. Then, note that $\mathbf{A}(D) = \mathbf{A}(G)$.

COROLLARY 4. (Mizuno and Sato) *Let G be a graph, A a finite group and $\alpha : D(G) \to A$ an ordinary voltage assignment. Let ρ_i, f_i be as in Theorem 24. Then the characteristic polynomial of the A-covering G^{α} of G is*

$$\Phi(G^{\alpha};\lambda) = \Phi(G;\lambda) \cdot \prod_{j=2}^{t} \{\Phi(\sum_{h} \rho_j(h) \otimes \mathbf{A}_h;\lambda)\}^{f_j}.$$

6.2 Characteristic polynomials of cyclic abelian covers

Mizuno and Sato [36] presented two formulas for the characteristic polynomial of a cyclic abelian cover.

Let D be a symmetric digraph, A a finite abelian group and A^* the character group of A. For a mapping $f : A(D) \to A$, a pair $D_f = (D, f)$ is called a *weighted symmetric digraph*. Given any weighted symmetric digraph D_f, the adjacency matrix $\mathbf{A}(D_f) = (a_{f,uv})$ of D_f is the square matrix of order $|V(D)|$ defined by

$$a_{f,uv} = a_{uv} \cdot f(u,v).$$

The characteristic polynomial of D_f is that of its adjacency matrix, and is denoted $\Phi(D_f;\lambda)$ (see [45]).

COROLLARY 5. (Mizuno and Sato) *Let D be a symmetric digraph, α an alternating function from $A(D)$ to a finite abelian group A, and $g \in A$. Then we have*

$$\Phi(D_g(\alpha);\lambda) = \prod_{\chi \in A^*} \Phi(D_{\chi(g)^{-1}(\chi\circ\alpha)};\lambda).$$

Another formula for the characteristic polynomial of a cyclic abelian cover is obtained by considering its structure.

COROLLARY 6. (Mizuno and Sato) *Let D be a symmetric digraph, A a finite abelian group, $g \neq 1 \in A$ and $\alpha : A(D) \to A$ an alternating function. Set $|V(D)| = t$, $\operatorname{ord}(g) = n$, $|A| = nq$ and $H = \langle g \rangle$. Furthermore, let $\beta : A(D) \to A/H$ be the alternating function such that $\beta(x,y) = \alpha(x,y)H$ for each $(x,y) \in A(D)$. Then the characteristic polynomial of the g-cyclic A-cover $D_g(\alpha)$ of D is*

$$\Phi(D_g(\alpha);\lambda) = \zeta^{-qtn(n-1)/2} \prod_{k=0}^{n-1} \prod_{\chi \in (A/H)^*} \Phi(D_{\chi\circ\beta};\zeta^k\lambda),$$

where $(A/H)^*$ is the character group of A/H and $\zeta = \exp(2\pi i/n)$.

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