YOKOHAMA MATHEMATICAL
JOURNAL VOL. 47, 1999
SPECIAL ISSUE

GEOMETRY OF FRAMEWORKS

By

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(Received November 15, 1998)

Abstract. This is an expository paper concerning geometry of frameworks. Many interesting results on frameworks including several recent works are presented together with some open problems. The contents are:

1. Introduction

By a *motion* of a graph with vertices in Euclidean space $\boldsymbol{R}^d$, we mean a continuous movement of the vertices in $\boldsymbol{R}^d$ with keeping the distances between adjacent vertices unchanged. When we are interested in motions of a graph with vertices in $\boldsymbol{R}^d$, we usually call the graph a *framework* in $\boldsymbol{R}^d$. The vertices and edges of a framework are then called the *joints* and *bars*. The distance between the two end-points of a bar is called the *length* of the bar. So, during a motion of a framework, the lengths of bars are all fixed. A continuous *deformation* of a framework F in $\boldsymbol{R}^d$ is a motion of F in $\boldsymbol{R}^d$ that changes the distance between a pair of non-adjacent joints. If a framework F in $\boldsymbol{R}^d$ admits a continuous deformation, then F is called *flexible*, otherwise, it is called *rigid*.

For example, a 4-cycle in $\boldsymbol{R}^2$ consisting of four vertices and four edges of a square is flexible. Indeed, it continuously deforms into a family of rhombi. On the other hand, every complete graph in $\boldsymbol{R}^d$ is rigid.

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The subjects of the present paper are mainly related to so-called bar-and-joint-frameworks. Some of them were themes of my talks in the series of meetings on Topological Graph Theory (organized by Prof. S. Negami) at Yokohama

1991 Mathematics Subject Classification: 52C25
Key words and phrases: framework, rigidity, flexibility

National University, during the past ten years. Other closely related things such as, tensegrity frameworks, bar-and-body-frameworks, scene analysis, matroid theory, are omitted here. For matroid and rigidity, see, e.g., [11], [32], [42].

1.1 Linkages

A planar (spatial) *linkage* is a framework in which a few joints are fixed in the plane (space). Usually, the interest in linkages is not in finding rigid ones, but in finding linkages one of whose joints will trace specific curves (surfaces).

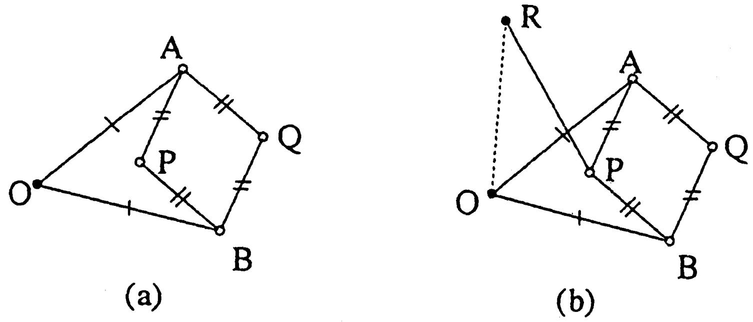

Figure 1. Peaucellier's linkages

Figure 1(a) shows a planar linkage called the *Peaucellier's inversor*. It consisits of six bars $OA, OB, AP,\ BP, AQ, BQ$ of two different lengths, and the joint O is fixed on the plane. Notice that the joint P can move freely on the plane to some extent without any constraint. It is not difficult to see that the three joints O, P, Q are always collinear, and $OP \cdot OQ = OA^2 - AP^2$. Hence, Q is the *inversion* of P with respect to the circle centered at O having radius $\sqrt{OA^2 - AP^2}$. Hence this linkage is called the *inversor*.

Since James Watt had completed the steam engine, it became an interesting problem to find a linkage in which a certain joint draws a straight line-segment. In 1784, James Watt found such a linkage that works approximately satisfactorily. An exact solution was found by Peacellier in 1864. In the Peaucellier's inversor (Figure 1(a)), if we move P along a circle that passes through O, then Q will draw a straight line. (Recall that the inversion of a circle that passes through the center of the inversion is a straight line.) Hence, in the linkage shown in Figure 1(b) (in which the joint R is also fixed (pinned) on the plane), Q draws a straight line-segment.

Hart also made an inversor in different way and made a linkage drawing a straight line-segment. Kempe devised a linkage drawing a straight line-segment without using 'inversion'. For the details on their linkages, see Rademacher-Toeplits [31] Chapter 16.

Modifying Peaucellier's inversor, we can make a spatial inversor. Figure 2 shows one. It consisits of 9 bars of two different lengths. The joint O is fixed in the space. As far as the three joints A, B, C take different positions, the three joints O, P, Q are collinear, and $OP \cdot OQ = OA^2 - AP^2$. Hence Q is the inversion of P with respect to the sphere centered at O with radius $\sqrt{OA^2 - AP^2}$. (If you worry about the case that two of A, B, C, say, A, B come to the same position by chance, you may add two bars AE, BE of different lengths. Then A, B cannot come to the same position.) If we add further a bar PR and fix the joint R at the position such that $OR = PR$, then Q will draw (a part of) a plane.

For another type of spatial linkage that draws a plane, see the book of Hilbert and Cohn-Vossen [15] Ch. V, Section 40.

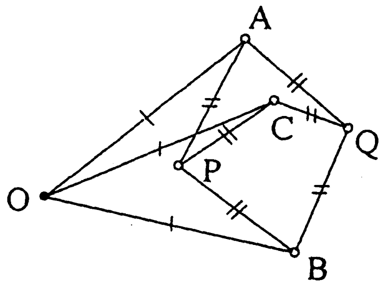

Figure 2. A spatial inversor

Kempe [18] proved the following amazing result.

THEOREM 1.1. (Kempe 1876) *For any polynomial $\varphi(x, y)$, there is a planar linkage that draws (a part of) the curve $\varphi(x, y) = 0$.*

Motivated by planning of motion in robotics, Hopcroft-Joseph-Whitesides [16] gave a modified version of Kempe's construction of a linkage that "solves" a multivariable polynomial equations. Using it, they gave, among others, a linkage that traces a 'triangle'.

1.2 Hinged-panel-surfaces

Let us mention here some results on the rigidity of a polyhedral surface. We regard a polyhedral surface as a *hinged-panel-surface*, that is, a 2-dimensional manifold in $\boldsymbol{R}^3$ obtained by attaching rigid panel-polygons along their edges with hinges. Its dihedral angles can be changed freely if no constraints come from other faces. In a polyhedral surface, if we can change continuously some dihedral angle of the surface, then the surface is called flexible, otherwise, it is called rigid.

As you know empirically, a box is rigid as a hinged-panel-manifold, even if one face of a box is removed. On the other hand, by removing a face from a cube, and by triangulating the remaining 5 faces by diagonals as illustrated in Figure 3, we have a flexible polyhedral surface. Actually, it can be folded flat.

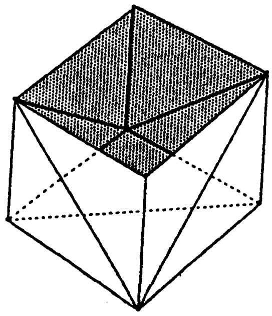

Figure 3. A folding container

From a cube, remove the top face and the bottom face, and triangulate the remaining 4 faces by pairs of diagonals crossing in the shape 'X', see Figure 4. Then we have a polyhedral 'tube' consisiting of $4 \times 4 = 16$ right-angled isosceles triangles of the same size. This tube is called a *flexagon*. As a hinged-panel-surface, the flexagon is not only flexible. It is reversible! (Paint the inside of the tube red, outside blue. Then it is possible to deform the tube to a one with outside color red.) The process of turning the flexagon inside out is complicated and intriguing, and hence the flexagon works as a nice toy. The flexagon was discovered by Arthor Stone in 1939, see Bolt [5] p. 22. For other types of flexagons, see Gardner [10].

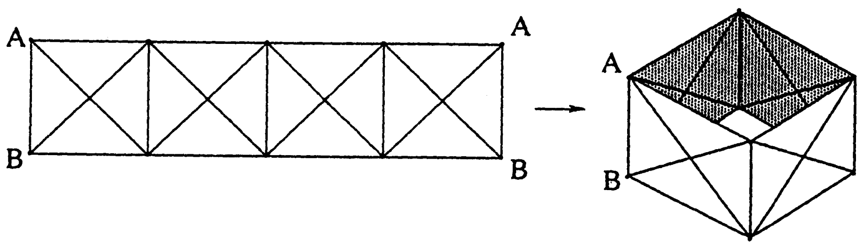

Figure 4. A flexagon

1.3 A flexible closed surface

A flexagon and a folding container (Figure 3) are flexible, but they are polyhedral surfaces *with* boundaries. How about *closed* polyhedral surfaces? Are all closed polyhedral surface rigid? Euler thought so, and conjectured in 1766 "a closed spatial figure allows no changes, as long as it is not ripped apart". Cauchy proved that Euler's conjecture is true for convex polyhedral surfaces.

THEOREM 1.2. (Cauchy 1813) *Every closed convex polyhedral surface is rigid.*

In his proof, Cauchy overlooked an exceptional possibility in the auxiliary lemma. This minor flaw was amended by Stoker [37].

Since a triangle is rigid as a framework, it follows from Cauchy's rigidity theorem that the graph (1-skeleton) of a convex polyhedron consisting of only triangular faces is rigid.

Let $G = (V, E)$ be an abstract graph. Then, any injection $f : V \to \boldsymbol{R}^d$ define a framework $f(G)$ in $\boldsymbol{R}^d$ with joints $f(i)$ $(i \in V)$ and bars $f(i)f(j)$ $(ij \in E)$. The framework $f(G)$ in $\boldsymbol{R}^d$ is called a *representation* of G in $\boldsymbol{R}^d$.

THEOREM 1.3. (Gluck 1975) *Let $G = (V, E)$ be a maximal planar graph. Then an arbitrary representation of G in $\boldsymbol{R}^3$ is rigid almost surely.*

What do we mean by almost surely? Suppose that G has n vertices. Then an injection $f : V \to \boldsymbol{R}^3$ is represented by a point in $\boldsymbol{R}^{3n}$. The theorem implies that the set of points of $\boldsymbol{R}^{3n}$ corresponding to rigid representations of G, is an open dense subset of $\boldsymbol{R}^{3n}$.

Since a 1-skeleton of a triangulation of a 2-sphere is a maximal planar graph, the above theorem implies that almost all simply connected, triangulated polyhedral surfaces in $\boldsymbol{R}^3$ are rigid. Thus, we may say Euler's conjecture is *statistically* true.

In 1976, however, Connelly [7], [8] proved the following startling result.

THEOREM 1.4. (Connelly 1976) *There is a closed polyhedral surface in $\boldsymbol{R}^3$ with faces all triangles, homeomorphic to a 2-dimesional sphere, and yet flexible.*

His flexible surface preseves its volume (content) under any continuous deformation, that is, the volume of the polyhedron remains constant during continuous deformation. This fact lead him to the *Bellow Conjecture*. It asserts that each flexible closed surface in $\boldsymbol{R}^3$ conserves its volume during continuous deformation. Recently, the affirmative answer to the Bellow Conjecture was obtained for all flexible polyhedra in $\boldsymbol{R}^3$, see [9], [34], [35]. On the other hand, the Bellow Conjecture is no longer true in the *spherical space* S^3; Alexandrov [1] presented a flexible polyhedron in S^3 which changes its volume during deformation.

2. Rigidity and infinitesimal rigidity

Here, we recall some fundamentals concerning rigidity and infinitesimal rigidity of frameworks.

2.1 Infinitesimal deformations

A *vector field* f on $X \subset \boldsymbol{R}^d$ is a map $f : X \to \boldsymbol{R}^d$. When we want to show the domain of f explicitly, we use the notation $f|X$. If the values of f are obtained as the velocity vectors of a smooth 'rigid motion' of X in $\boldsymbol{R}^d$, then f is called *trivial*. An *infinitesimal motion* of a framework F with the vertex-set $X \subset \boldsymbol{R}^d$ is a vector field $f|X$ that satisfies

$$(f(x) - f(y)) \cdot (x - y) = 0$$

for all bars xy of F, where $\cdot$ denotes the inner product. Consider, for example, a smooth motion of F in $\boldsymbol{R}^d$. Since, for any bar xy of F,

$$\| x - y \|^2 = \text{ const.}$$

holds during the motion, by differentiating with the time parameter t, we have $(dx/dt - dy/dt) \cdot (x - y) = 0$ for all bars xy of F. Hence the velocity vectors (at any instant) of a smooth motion of F in $\boldsymbol{R}^d$ is an infinitesimal motion of F. A *nontrivial* infinitesimal motion of F is called an *infinitesimal deformation* of F. If F admits an infinitesimal deformation in $\boldsymbol{R}^d$, then F is called *infinitesimally flexible* in $\boldsymbol{R}^d$, otherwise, F is called *infinitesinally rigid* in $\boldsymbol{R}^d$.

The infinitesimal motions of F in $\boldsymbol{R}^d$ form a vector space W with respect to joint-wise addition and multiplication of scalars. The trivial motions of F yield a $\binom{d+1}{2}$-dimensional subspace of this vector space W, where

$$\binom{d+1}{2} = \dim \mathrm{SO}(d) + d$$

($\mathrm{SO}(d)$ is the rotation group of $\boldsymbol{R}^d$ fixing the origin). Hence

$$\dim W \geq \binom{d+1}{2}, \quad \text{and}$$

$$\dim W = \binom{d+1}{2} \iff F \text{ is infinitesimally rigid.}$$

If a framework in $\boldsymbol{R}^d$ admits a continous deformation, then it admits a *smooth* deformation, see, e.g., Asimov-Roth [2]. If a framework F in $\boldsymbol{R}^d$ admits a smooth deformation, then the velocity vectors of the joints at some instant constitute an infinitesimal deformation of F. Hence we have

$$\text{flexible} \Longrightarrow \text{infinitesimally flexible}$$

and hence

$$\text{infinitesimally rigid} \Longrightarrow \text{rigid}.$$

Note that a rigid framework is not always infinitesimally rigid. For example, the framework in $\boldsymbol{R}^2$ shown in Figure 5(a) is rigid but not infinitesimally rigid. (By assigning the dotted vector to the joint x, and zero vectors to all other joints, we get an infinitesimal deformation.)

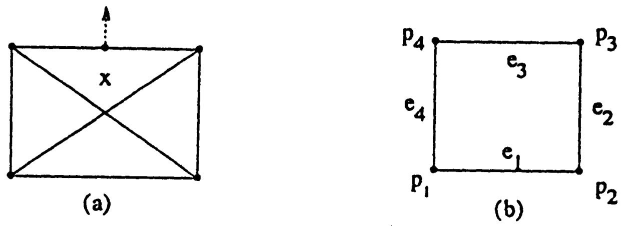

Figure 5.

2.2 The rigidity matrix

Let F be a framework in $\boldsymbol{R}^d$ with joints $p_1, \dots, p_n$ and bars $e_1, \dots, e_m$. Consider an infinitesimal motion f of F, and put $\boldsymbol{x}_i = f(p_i)$. Then for each bar $e_k = p_i p_j$ of F, we have $(p_i - p_j) \cdot (\boldsymbol{x}_i - \boldsymbol{x}_j) = 0$, that is,

$$(p_i - p_j) \cdot \boldsymbol{x}_i + (p_j - p_i) \cdot \boldsymbol{x}_j = 0.$$

Collecting such equations for the bars $e_1, \dots, e_m$, we have a linear equation for the 'unknown' vector $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in \boldsymbol{R}^{dn}$. The coefficient matrix of this linear equation is called the *rigidity matrix* of F, and it is denoted by M_F. Note that if $e_k = p_i p_j$ $(i < j)$, then, the kth row of M_F is

$$(0, \dots, 0, p_i - p_j, 0, \dots, 0, p_j - p_i, 0, \dots, 0) \in \boldsymbol{R}^d \times \dots \times \boldsymbol{R}^d = \boldsymbol{R}^{dn}.$$

EXAMPLE 2.1. The rigidity matrix of the framework in $\boldsymbol{R}^2$ illustrated in Figure 5(b) is

$$\begin{pmatrix} p_1 - p_2 & p_2 - p_1 & 0 & 0 \\ 0 & p_2 - p_3 & p_3 - p_2 & 0 \\ 0 & 0 & p_3 - p_4 & p_4 - p_3 \\ p_1 - p_4 & 0 & 0 & p_4 - p_1 \end{pmatrix}$$

(with all entries row-vectors in $\boldsymbol{R}^2$) which is 4×8 matrix as a matrix with number-entries.

Let W denote the vector space of all infinitesimal motions of F, and φ : $\boldsymbol{R}^{dn} \to \boldsymbol{R}^m$ be the linear map defined by M_F. Then we have $W = \ker \varphi$, and $\dim W + \text{rank}(M_F) = dn$. Hence the next theorem follows.

THEOREM 2.1. *Let F be a framework with $n(> d)$ joints in $\boldsymbol{R}^d$, and M_F be its rigidity matrix. Then*
(1) $\text{rank}(M_F) \leq dn - \binom{d+1}{2}$,
(2) *F is infinitesimally rigid* $\Longleftrightarrow \text{rank}(M_F) = dn - \binom{d+1}{2}$.

Since $\text{rank}(M_F) \leq m$ (m is the number of bars of F), the next follows.

COROLLARY 2.1. *Let F be a framework in $\boldsymbol{R}^d$ with $n(> d)$ joints and m bars. If $m < nd - \binom{d+1}{2}$, then F is infinitesimally flexible.*

2.3 Regular points

Let G be an abstract graph with vertices $1, 2, \ldots, n$. By taking n distinct points $p_1, p_2, \ldots, p_n$ in $\boldsymbol{R}^d$, and defining p_i, p_j to be adjacent whenever $ij \in E$, we have a framework in $\boldsymbol{R}^d$. This frmework is called a *representation* of G and denoted by

$$G(P) \quad \text{or} \quad G(p_1, p_2, \ldots, p_n),$$

where $P = (p_1, p_2, \ldots, p_n)$ is a point of $\boldsymbol{R}^d \times \boldsymbol{R}^d \times \ldots \times \boldsymbol{R}^d = \boldsymbol{R}^{nd}$. Let $M_{G(P)}$ be the rigidity matrix of $G(P)$. Regarding $\text{rank}(M_{G(P)})$ as a function of $P \in \boldsymbol{R}^{dn}$, a point P where $\text{rank}(M_{G(P)})$ takes its maximum value is called a *regular point* of G. Then the next theorem holds.

THEOREM 2.2. *Let G be a graph with $n(> d)$ vertices. Then,*
(1) *the set of regular points of G in $\boldsymbol{R}^{dn}$ is an open dense subset of $\boldsymbol{R}^{dn}$, and*
(2) *for any regular point $P \in \boldsymbol{R}^{dn}$ of G, the following three are equivalent:*
1. *$G(P)$ is rigid in $\boldsymbol{R}^d$,*
2. *$G(P)$ is infinitesimally rigid in $\boldsymbol{R}^d$,*
3. $\text{rank}(M_{G(P)}) = dn - \binom{d+1}{2}$.

Thus, there is no big difference between the rigidity and the infinitesimal rigidity. For more information on rigidity or flexibility, see, e.g., [2], [3], [33].

3. Bipartite frameworks in the plane

Suppose you have to construct a rigid framework in the plane. Then, usually you will use triangles (3-cycles) to make the framework rigid. Since $K(3,3)$

contains no 3-cycle, it would be an interesting fact that most representations of $K(3,3)$ in the plane are rigid.

3.1 Conics

For two disjoint, nonempty (possibly infinite) sets $X, Y \subset \boldsymbol{R}^2$, let $K(X,Y)$ denote the complete bipartite graph with partite sets X and Y. The size (cardinality) of X is denoted by $|X|$. It is not hard to see that if $|X| \leq 2$, then $K(X,Y)$ is always infinitesimally flexible.

THEOREM 3.1. *Suppose that $X, Y \subset \boldsymbol{R}^2$ are two disjoint sets of size ≥ 3 such that no three points in $X \cup Y$ are collinear. If $K(X,Y)$ admits an infinitesimal deformation, then $X \cup Y$ lies on a conic.*

Proof. Suppose that $f : X \cup Y \to \boldsymbol{R}^2$ is an infinitesimal deformation of $K(X,Y)$, and let p_1, p_2, p_3 be any three points in X. Then for any $q \in Y$,

$$(q - p_i) \cdot (f(q) - f(p_i)) = 0 \quad (i = 1,2,3). \tag{1}$$

Hence for each $q \in Y$, the value $f(q)$ is uniquely determined by the two values $f(p_1), f(p_2)$, and similarly $f(p)$ $(p \in X - \{p_1, p_2\})$ is also determined by $f(p_1), f(p_2)$ via some two values $f(q), f(q')$ $(q, q' \in Y)$. Therefore $f|\{p_1, p_2\}$ must be an infinitesimal deformation, i.e., $(p_1 - p_2) \cdot (f(p_1) - f(p_2)) \neq 0$. (For otherwise, $f : X \cup Y \to \boldsymbol{R}^2$ becomes a trivial motion.) Now, letting $f(q) = (u, v)$, we have

$$\begin{aligned}
&(1) \iff (q - p_i) \cdot f(q) - (q - p_i) \cdot f(p_i) = 0 \quad (i = 1,2,3) \\
&\iff \begin{pmatrix} q - p_1 & (q - p_1) \cdot f(p_1) \\ q - p_2 & (q - p_2) \cdot f(p_2) \\ q - p_3 & (q - p_3) \cdot f(p_3) \end{pmatrix} \begin{pmatrix} u \\ v \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{2}$$

Let $p_i = (a_i, b_i)$ $(i = 1,2,3)$, and define a polynomial $\varphi(x, y)$ of x, y by

$$\varphi(\boldsymbol{x}) = \varphi(x, y) = \begin{vmatrix} x - a_1 & y - b_1 & (x - a_1, y - b_1) \cdot f(p_1) \\ x - a_2 & y - b_2 & (x - a_2, y - b_2) \cdot f(p_2) \\ x - a_3 & y - b_3 & (x - a_3, y - b_3) \cdot f(p_3) \end{vmatrix}$$

Then, it follows easily from $(p_1 - p_2) \cdot (f(p_1) - f(p_2)) \neq 0$ that $\varphi((p_1 + p_2)/2) \neq 0$. Hence $\varphi(x, y)$ is a nontrivial polynomial of x, y with degree at most 2. Since $\varphi(q) = 0$ for all $q \in Y$ by (2) and $\varphi(p_i) = 0$ $(i = 1,2,3)$ as verified easily, the set $\{p_1, p_2, p_3\} \cup Y$ lies on the conic $\varphi(x, y) = 0$. Similarly, for any $p \in X - \{p_1, p_2\}$, the set $\{p_1, p_2, p\} \cup Y$ lies on a conic. Since a *proper* conic is determined by five points on it, we can conclude that $X \cup Y$ lies on a conic. ∎

The following general result was essentially proved in Bolker-Roth [4]. The precise form was given by Whiteley [41].

THEOREM 3.2. *Let $X, Y \subset \boldsymbol{R}^d$ be two disjoint sets such that $|X|, |Y| \geq d+1$. Then $K(X,Y)$ is infinitesimally flexible in $\boldsymbol{R}^d$ if and only if one of the following holds:*

(1) *X and a point of Y lie on a hyperplane.*
(2) *Y and a point of X lie on a hyperplane.*
(3) *$X \cup Y$ lies on a quadratic hypersurface.*

3.2 Flexible representations

Now a question: When does a representation of $K(m,n)$, $m,n \geq 3$, in $\boldsymbol{R}^2$ admit a continuous deformation? Recently Maehara-Tokushige [30] proved the following result.

THEOREM 3.3. *Let $X, Y \subset \boldsymbol{R}^2$ be two disjoint finite-sets such that $|X| \geq 3, |Y| \geq 5$. Then $K(X,Y)$ admits a continuous deformation if and only if X lies on a line L and Y lies on a line perpendicular to L.*

The *if* part of the theorem is easy. To see this, suppose that $X = \{p_1, p_2, p_3, \ldots\}$ lies on the x-axis and $Y = \{q_1, q_2, q_3, \ldots\}$ lies on the y-axis, with no q_j at the origin. Then we can put

$$p_i = (\sigma_i \sqrt{a_i - t}, 0), \quad i = 1, 2, \ldots,$$
$$q_j = (0, \varepsilon_j \sqrt{b_j + t}), \quad j = 1, 2, \ldots,$$

where $\sigma_i, \varepsilon_j = \pm 1$ and $a_i \geq t > 0$, $b_j > 0$. Then the length of the bar $p_i q_j$ is equal to $\sqrt{a_i + b_j}$, which is irrelevant to t. Hence by varying t, we can deform $K(X,Y)$. The *only if* part of the proof is not easy.

3.3 Bottema's example

In Theorem 3.3, we cannot relax the condition $|Y| \geq 5$ to $|Y| \geq 4$, as seen in the following result due to Bottema (see Wunderlich [44]).

THEOREM 3.4. *There is a flexible representation $K(X,Y)$ of $K(4,4)$ in the plane such that the convex hulls of X and Y are both rectangles.*

Proof. Consider the simultaneous equation on x, y, z containing the parameter t:

$$\begin{cases} (x-t)^2 + (y-z)^2 &= a \\ (x-t)^2 + (y+z)^2 &= b \\ (x+t)^2 + (y-z)^2 &= c \\ (x+t)^2 + (y+z)^2 &= d \end{cases}$$

where a, b, c, d are positive constants such that $a + d = b + c$. We can choose a, b, c, d suitably so that the above equation has real solutions for some range of t. For example, letting $a = 4, b = 6, c = 8, d = 10$, we have the solutions

$$x = 1/t, \ y = \frac{\pm\sqrt{8 - f(t)} \pm \sqrt{6 - f(t)}}{2}, \ z = \frac{1}{2y}, \tag{3}$$

where $f(t) = (t^4 + 1)/t^2$, which take real values for $\sqrt{2} - 1 \leq t \leq \sqrt{2} + 1$. Now choosing, say, $+$ signs in (3), let

$$\begin{array}{llll} p_1 = (t, z), & p_2 = (-t, z), & p_3 = (-t, -z), & p_4 = (t, -z), \\ q_1 = (x, y), & q_2 = (-x, y), & q_3 = (-x, -y), & q_4 = (x, -y), \end{array}$$

and put $X = \{p_1, p_2, p_3, p_4\}$, $Y = \{q_1, q_2, q_3, q_4\}$. Then, varying t from $2 - \sqrt{3}$ to $2 + \sqrt{3}$, we have a continuous deformation of $K(X, Y)$. ■

PROBLEM 3.1. *Characterize the flexible representations of*

$$K(3,3), \ K(3,4), \ K(4,4)$$

in the plane.

Let us call an (abstract) graph *absolutely* 2*-rigid* if it admits no flexible representation in $\boldsymbol{R}^2$.

PROBLEM 3.2. *Characterize absolutely* 2*-rigid graphs.*

I conjecture that a graph G of order > 2 is absolutely 2-rigid if and only if G can be obtained from K_2 by repeating the following operations: (1) attaching a vertex of degree 2, and (2) adding an edge.

4. Unit-bar-frameworks

A framework F in $\boldsymbol{R}^d$ is called a *unit-bar-framework* if its all bars have the unit length.

4.1 Triangle-free frameworks

As seen in Theorem 3.1, there is a 'bipartite' framework that is rigid in $\boldsymbol{R}^2$. How about unit-bar-frameworks? Can you think of a (nontrivial) rigid unit-bar-framework in the plane that has no 3-cycle? Maehara [24] presented such a one.

THEOREM 4.1. *There is a rigid unit-bar-framework in the plane that is a bipartite graph.*

Since a bipartite graph contains no odd-cycle, it contains no 3-cycle. Unfortunately, the rigid framework presented in [24] is not infinitesimally rigid. An infinitesimally rigid unit-bar-framework in the plane that has no 3-cycle is constructed by Maehara-Chinen [28]. Figure 6 shows their framework.

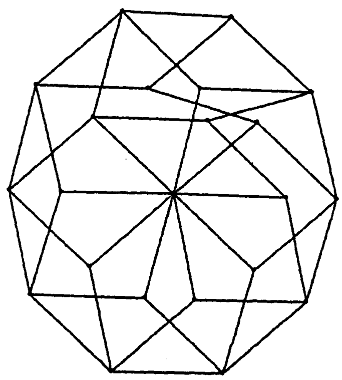

Figure 6. A rigid unit-bar-framework in the plane

It is an easy exercise of elementary geometry to verify that the unit-bar-framework in Figure 6 is rigid. The infinitesimal rigidity of this framework is shown by calculating the rank of its rigidity matrix.

Maehara-Tokushige [29] constructed a rigid unit-bar-framework in $\boldsymbol{R}^3$ that contains no 3-cycle. It consists of 26 joints and 78 bars (unit-bars). Its infinitesimal rigidity was checked by calculating the rank of rigidity matrix.

PROBLEM 4.1. *Find an infinitesimally rigid bipartite unit-bar-framework in the plane.*

PROBLEM 4.2. *Find a general construction of triangle-free, infinitesimally rigid unit-bar-framework in $\boldsymbol{R}^d$.*

4.2 Extensions, distances, constructibility

Let F be a flexible unit-bar-framework in $\boldsymbol{R}^d$. Then, by adding some bars of appropriate lengths, we can always extend G to a rigid framework in $\boldsymbol{R}^d$. But, how about when only unit-bars are available? Can you always extend F to a rigid unit-bar-framework in $\boldsymbol{R}^d$? If necessary, you may continuously deform F as far as no two distinct joints come to the same position.

The following result was proved by Maehara [25].

THEOREM 4.2. *Any unit-bar-framework in $\boldsymbol{R}^d$ can be extended to a rigid unit-bar-framework in $\boldsymbol{R}^d$.*

The next theorem was proved in Maehara [23] by following the idea of Kempe [18].

THEOREM 4.3. *For any positive algebraic number α, there is a rigid unit-bar-framework F in $\boldsymbol{R}^2$ that contains two joints exactly distance α apart.*

This theorem was also extended to arbitrary dimension $d > 2$ by [25].

Given a line-segment of unit length in the plane, a line-segment of, say, length $2^{1/3}$ cannot be constructed by *ruler and compass*. Hence, by Theorem 4.3, there is a rigid framework that cannot be constructed by ruler and compass from the data of bar-lengths, and its graph-structure. It is known that 6 is the minimum order of a rigid (not necessarily unit-bar-) framewrok that cannot be constructed from its bar-lengths and graph-structure. Figure 7 is such a framework obtained in [26].

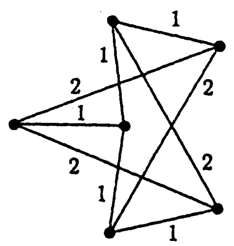

Figure 7. A nonconstructible framework

5. Generic rigidity of graphs

Let G be an abstract graph with n vertices. If a representation $G(P_0)$ in $\boldsymbol{R}^d$ for a regular point $P_0 \in \boldsymbol{R}^{dn}$ is rigid, then $G(P)$ is rigid for any regular point $P \in \boldsymbol{R}^{dn}$, by Theorem 2.2. Let us call G *generically d-rigid* (simply *d-rigid*) when $G(P)$ in $\boldsymbol{R}^d$ is rigid for a regular point P. If G is not d-rigid, then G is called *generically d-flexible* (simply *d-flexible*).

5.1 Generic rigidity of $K(m,n)$

Let X, Y be disjoint subsets of $\boldsymbol{R}^d$. It is not hard to see that if one of $|X|, |Y|$ is less than $d+1$, then $K(X,Y)$ is infinitesimally flexible in $\boldsymbol{R}^d$. Suppose that $|X| > d, |Y| > d$ and neither X nor Y lie on a hyperplane. Then it follows from Theorem 3.2 that $K(X,Y)$ is infinitesimally flexible in $\boldsymbol{R}^d$ if and only if $X \cup Y$ lies on a quadratic surface in $\boldsymbol{R}^d$.

Since a quadratic polynomial on d variables has $d + \binom{d}{2} + d + 1 = \binom{d+2}{2}$ coefficients, a point-set $Z \subset \boldsymbol{R}^d$ of size $|Z| < \binom{d+2}{2}$ lies on a quadratic surface almost surely, and a point-set of size $\geq \binom{d+2}{2}$ does not lie on a quadratic surface almost surely. Hence the next result follows from Theorem 3.2.

THEOREM 5.1. *$K(m,n)$ is d-rigid $\Longleftrightarrow$ $m, n \geq d+1$ and $m+n \geq \binom{d+2}{2}$.*

5.2 The Henneberg operations

It is clear that for any $d > 0$ and for any $n > 0$, the complete graph K_n is d-rigid. We introduce here two operations $\mathcal{A}_d$ and $\mathcal{B}_d$ which will be applied to a d-rigid graph to produce another 'bigger' d-rigid graph. Let G be an abstract graph.

1. Operation $\mathcal{A}_d$:
   Choose d distinct vertices $v_1, v_2, \ldots, v_d$ of G and add a new vertex v_0 together with d edges $v_0v_1, \ldots, v_0v_d$ to G. The resulting graph is denoted by $\mathcal{A}_dG$.
2. Operation $\mathcal{B}_d$:
   Remove an edge $e = v_1v_2$ from G and then add a new vertex v_0 together with $d+1$ edges $v_0v_1, v_0v_2, \ldots, v_0v_{d+1}$, where $v_1, \ldots, v_{d+1}$ are distinct vertices of G. The resulting graph is denoted by $\mathcal{B}_dG$.

Note that $\mathcal{A}_d$ is applicable only to a graph with at least d vertices, $\mathcal{B}_d$ is applicable to a graph with at least $d+1$ vertices and an edge. Further, $\mathcal{A}_dG, \mathcal{B}_dG$ depend on the choice of the vertices (and an edge) of G. These operations are known as generalized *Henneberg operations*, see Tay-Whiteley [39], Maehara [21].

THEOREM 5.2. (Tay-Whiteley 1985).
(1) *G is d-rigid $\Longleftrightarrow$ $\mathcal{A}_dG$ is d-rigid*
(2) *G is d-rigid $\Longrightarrow$ $\mathcal{B}_dG$ is d-rigid.*

EXAMPLE 5.1. *$K(3,3)$ is 2-rigid, see Figure* 8.

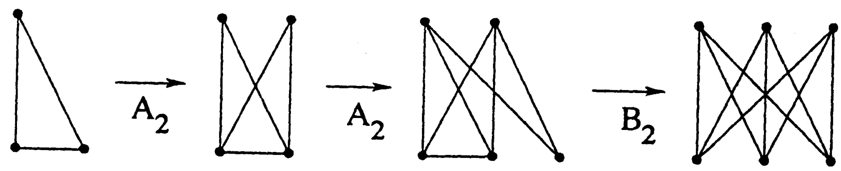

Figure 8.

EXAMPLE 5.2. *$K(2,2,2)$ is 3-rigid, see Figure* 9.

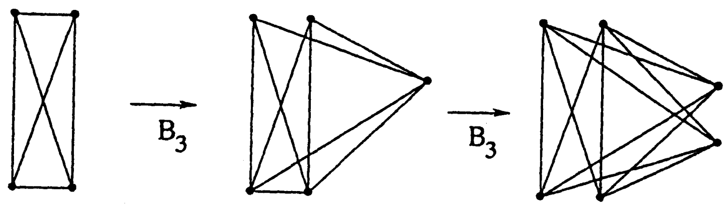

Figure 9.

5.3 Laman's theorem

A d-rigid graph G is called a *minimal d-rigid* graph if for any edge e of G, $G-e$ (the removal of e) is d-flexible. It is obvious that a minimal d-rigid graph with at most $d+1$ vertices is complete. The next theorem follows from Theorem 2.1.

THEOREM 5.3. *A minimal d-rigid graph with $n(>d)$ vertices has $dn-\binom{d+1}{2}$ edges.*

For exmple, a minimal 2-rigid graph with $n(\geq 3)$ vertices has $2n-3$ edges. The next result was obtained by Laman [19], see also Tay [38]. We denote the set of vertices and the set of edges of G by $V(G)$ and $E(G)$, respectively.

THEOREM 5.4. (Laman 1970) *A graph G with $n(\geq 2)$ vertices is a minimal 2-rigid graph if and only if $|E(G)|=2n-3$, and, for any subgraph H of G, $|E(H)|\leq 2|V(H)|-3$.*

Similarly, the next holds.

THEOREM 5.5. *Let G be a minimal 2-rigid graph with at least 3 vertices. Then G can be obtained from K_2 by applying Henneberg operations.*

5.4 Connectivity and rigidity

It will be clear that a graph with a cut point is not 2-rigid. Similarly the following holds.

THEOREM 5.6. *If a graph G is d-rigid, then G is d-connected.*

Since an n-cycle ($n > 3$) is not 2-rigid, the converse of this theorem is not true.

THEOREM 5.7. (Lovász-Yenemi 1982) *Any 6-connected graph is 2-rigid.*

Lovász and Yenemi [20] also constructed infinitely many 5-connected graphs that are 2-flexible.

5.5 3D replacement conjecture

Generally, a minimal 3-rigid graph cannot be constructed by Henneberg operations. For example, the graph G of a regular icosahedron is 3-rigid by Cauchy's rigidity thorem or Gluck's theorem. It has 12 vertices and $30 = 3 \times 12 - 6$ edges. Hence it is a minimal 3-rigid graph. However, since G is 5-regular, it is impossible to get G from a 3-rigid graph by Henneberg operation. Is there any other valid operation such as $\mathcal{A}_d$, $\mathcal{B}_d$? The following would be a natural candidate of such an operation we can think of.

- Operation $\mathcal{C}_d$:
  remove two edges v_1v_2, v_3v_4 from a graph G and add a new vertex v_0 together with $d+2$ edges $v_0v_1, \ldots, v_0v_{d+2}$, where $v_0, v_1, \ldots, v_{d+2}$ are all distinct vertices of G.

This operation is, however, not valid for $d = 4$ [22]. To see this, consider the graph $G_0 = K_6 - (an\ edge)$. G_0 is 4-rigid, since it can be obtained from K_4 by applying $\mathcal{A}_4$ two times. The graph G_0 has 14 edges and they can be partitioned into seven non-adjacent edge-pairs, see Figure 10. Let $v_1, v_2, \ldots, v_6$ be the vertices of G_0. Consider the sequence of graphs $G_0, G_1, \ldots, G_7$, where G_i is obtained from G_{i-1} by applying the operation $\mathcal{C}_4$: remove the i-th edge-pair from G_{i-1} and add a new vertex together with 6 edges connecting the new vertex to $v_1, \ldots, v_6$. Then G_7 becomes the graph $K(6,7)$, which is 4-flexible by

Theorem 5.1. Hence $\mathcal{C}_4$ is not a valid operation. Similarly, it can be seen that $\mathcal{C}_d$ is not valid for $d > 4$. Only the case $d = 3$ remains unknown. The assertion that $\mathcal{C}_3$ is a valid operation is (a part of) the 3D replacement conjecture, see [11], [39], [43].

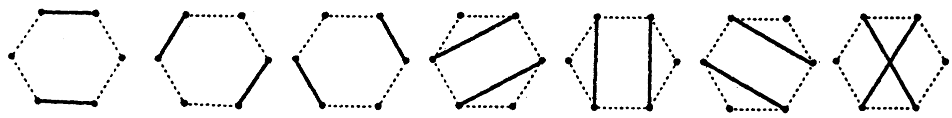

Figure 10.

5.6 Vertex splitting

Let G be a graph, x be a vertex of G such that $\deg x \geq d$. The neighborhood of x (the set of vertices adjacent to x) is denoted by $N(x)$. Take two disjoint sets of vertices A, B such that

$$A \cup B \subset N(x) \cup \{x\}, \ |A| = d, \ x \notin B$$

and add a new vertex y to G together with d edges yu ($u \in A$), and, for each $v \in B$, replace the edge xv by yv. Such an operation is called a *vertex-split with d feet*. Figure 11 shows two types of vertex-splits with 3 feet.

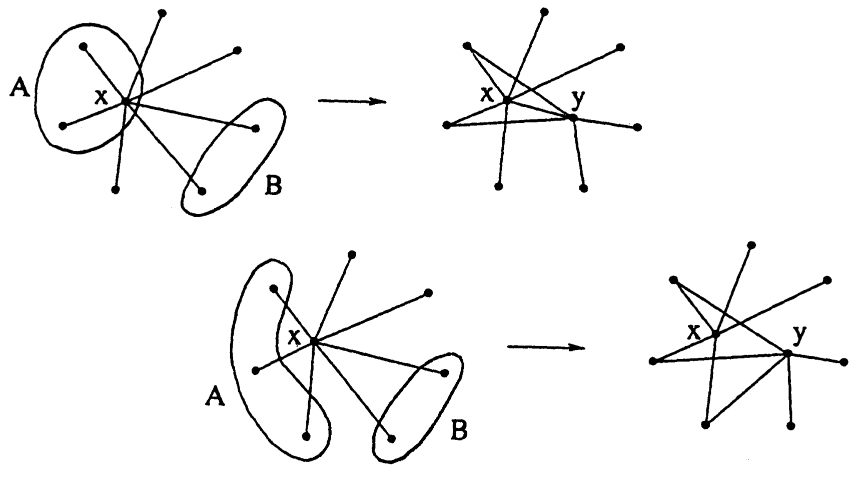

Figure 11. Two vertex-splits with 3 feet

The next theorem was proved by Whiteley (see [42], [43]) by considering the rank of rigidity matrix.

THEOREM 5.8. (Vertex-splitting) *If G is d-rigid then a graph obtained from G by applying a vertex-split with d feet is also d-rigid.*

Applying this theorem, let us show Gluck's theorem: every maximal planar graph G is 3-rigid.

The proof is by induction on the number n of the vertices of G. If $n \leq 4$ then G is a complete graph, and hence it is 3-rigid. Suppose that maximal planar graphs with $n(\geq 4)$ vertices are all 3-rigid, and let G be a maximal planar graph having $n+1$ vertices, embedded in the plane. Recall that any maximal planar graph has a vertex whose degree is at most 5. If G has a vertex x of degree 3, then $G - x$ is a maximal planar graph with n vertices, which is 3-rigid by inductive hypothesis. Since G is obtained from $G - x$ by applyng the Henneberg operation $\mathcal{A}_3$, G is 3-rigid.

If G has a vertex y of degree 4, then (recalling G is embedded in the plane) $G - y$ has a quadrilateral face. By adding a diagonal to this quadrilateral, we get a maximal planar graph G' with n vertices, which is 3-rigid by the inductive assumption. Since G is obtained from G' by applying Henneberg operation $\mathcal{B}_3$, G is also 3-rigid.

Finally, suppose G has no vertex of degree ≤ 4. Then G has a vertex z of degree 5, and $G - z$ has a pentagonal face. By adding two diagonals to this pentagonal face, we get a maximal planar graph G'' with n vertices, which is 3-rigid by the inductive assumption. Since G can be obtained from G'' by applying a vertex-split with 3 feet (see Figure 12), G is 3-rigid.

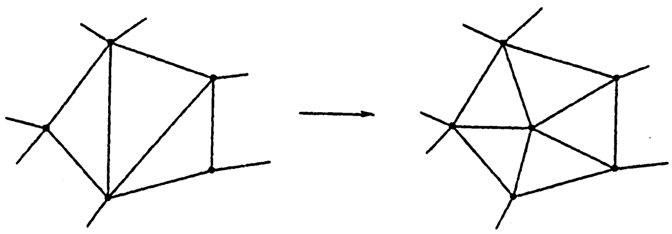

Figure 12. A vertex-split with 3 feet

6. Configuration spaces of frameworks

A *configuration* of n points in $\boldsymbol{R}^d$ is an ordered n-tuple $P = (p_1, \ldots, p_n)$ of n points $p_i \in \boldsymbol{R}^d$. We may regard P as a point in $\boldsymbol{R}^{dn}$. Two configurations $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ in $\boldsymbol{R}^d$ are said to be *isometric* (denoted by $P \simeq Q$) if there is an isometry $f : \boldsymbol{R}^d \to \boldsymbol{R}^d$ such that $f(p_i) = q_i$ $(i = 1, 2, \ldots, n)$. If the isometry f preserves the orientation of $\boldsymbol{R}^d$, then the two configurations are called *congruent*, and denoted by $P \cong Q$.

Let F be a connected, flexible framework in $\boldsymbol{R}^d$ with n joints indexed in some order. Then, n joints of F define a configuration in $\boldsymbol{R}^d$, which is regarded as a point of $\boldsymbol{R}^{dn}$. Let X be the set of points $P \in \boldsymbol{R}^{dn}$ obtained as configurations of F by continuously deforming it in $\boldsymbol{R}^d$ in all possible ways. The quotient space $X/\cong$ obtained from X by identifying congruent configurations is called the (*oriented*) *configuration space* of F in $\boldsymbol{R}^d$, and it is denoted by $\Gamma(F, d)$. It is regarded as the space of 'shapes' obtained by deforming F in $\boldsymbol{R}^d$. If F is a connected framework, then its configuration space is clealy compact.

Thurston-Weeks [40] gave a few examples of configuration spaces of frameworks in the plane.

6.1 Pentagonal frameworks

By a *pentagonal framework*, we mean a framework that is a 5-cycle. Let $F(a, b, c, d, e)$ denote a pentagonal framework in $\boldsymbol{R}^2$ whose consecutive bar-lengths are a, b, c, d, e.

The following result was proved by Havel [13]. He presented this result as an application of distance geometry in computer-aided proofs of theorems.

THEOREM 6.1. *Let $F = F(1,1,1,1,1)$ be an equilateral pentagonal framework in the plane. Then the configuration space of F is a connencted orientable 2-dimensional manifold with genus 4.*

Outline of the proof. Let $p_1, p_2, \ldots, p_5$ be the five joints of F, $p_i p_{i+1}$ $(i = 1, 2, 3, 4, 5)$ be the unit-bars of F, where $p_{5+1} = p_1$. It will be clear that the configuration space of F can be identified with the set M of points $(p_1, \ldots, p_5) \in \boldsymbol{R}^{10}$ such that

$$p_5 = (0,0), p_1 = (1,0), \text{and} \quad \| p_i - p_{i+1} \| = 1 \quad (i = 1, 2, 3, 4).$$

For $i = 1, 2, 3, 4, 5$, let

$$x_i = \| p_i - p_{i+2} \|^2$$
$$y_i = \| p_i - p_{i+3} \|^2,$$

where the index sums are computed mod 5. Then for any point $P \in M$, some (x_i, y_i) serves as the local coordinates in a neighborhood of P, and M turns out to be a 2-dimensional smooth manifold without boundary. It is clear that M is compact and connected.

Let $\alpha = \angle p_1 p_5 p_4$, $\beta = \angle p_5 p_4 p_3$, $\gamma = \angle p_4 p_3 p_2$, where angles are always measured in counterclockwise way. Consider the map

$$M \ni (p_1, \ldots, p_5) \mapsto (\alpha, \beta, \sin \gamma) \in S^1 \times S^1 \times [-1, 1],$$

where S^1 denotes the unit circle and α, β, γ are taken mod 2π. It is not difficult to see that this map is an embedding of M into $S^1 \times S^1 \times [-1,1]$. Since $S^1 \times S^1 \times [-1,1]$ is embeddable in $\boldsymbol{R}^3$, M is also embeddable in $\boldsymbol{R}^3$. Since a closed surface embeddable in $\boldsymbol{R}^3$ is orientable (for an intuitive proof of this fact, see [14]), we can deduce that M is orientable. Therefore, M is characterized by its Euler characteristic $\chi(M)$.

To calculate $\chi(M)$, we apply the 2-dimensional version of Morse theory. If we can choose a 'height function' on M that has a finite number of critical points consisting of only maximal points, minimal points, and saddle points, then $\chi(M)$ is computed by

$$\chi(M) = \#(\text{minimal points}) - \#(\text{saddle points}) + \#(\text{maximal points}).$$

Let $f(P)$ $(P = (p_1, \ldots, p_5) \in M)$ be the 4 times the oriented area of the (possibly self-intersecting) pentagon $p_1p_2p_3p_4p_5$, that is,

$$f(P) = 2(\overrightarrow{p_ip_{i+1}} \times \overrightarrow{p_ip_{i+2}} + \overrightarrow{p_ip_{i+2}} \times \overrightarrow{p_ip_{i+3}} + \overrightarrow{p_ip_{i+3}} \times \overrightarrow{p_ip_{i+4}}),$$

where $\times$ denotes the vector product. (The RHS is independent of the choice of i.) Using a suitable local coordinates $(x, y) = (x_i, y_i)$ and applying Helon's formula, $f(P) = f(x, y)$ can be expressed as

$$f(x,y) = \varepsilon_1\sqrt{4x - x^2} + \varepsilon_2\sqrt{2(x+y) - (x-y)^2 - 1} + \varepsilon_3\sqrt{4y - y^2},$$

where $\varepsilon_k = 1$ or -1 accordingly as $p_i \to p_{i+k} \to p_{i+k+1} \to p_i$ is counterclockwise or not. Using this expression, we can find all the critical points of f. There are exactly 14 critical points of f; 2 maximal points, 10 saddle points, and 2 minimal points. Hence we have

$$\chi(M) = 2 - 10 + 2 = -6.$$

Therefore, M is an orientable closed surface of genus $g = (2 - \chi(M))/2 = 4$. ∎

A framework F in $\boldsymbol{R}^d$ is called *reversible* in $\boldsymbol{R}^d$ if F can be continuously deformed in $\boldsymbol{R}^d$ to its mirror image (with respect to a hyperplane in $\boldsymbol{R}^d$). For example, consider a quadrilateral framework F in $\boldsymbol{R}^2$ with 4 joints x, y, z, w and 4 bars xy, yz, zw, wx such that

$$\| x - y \| = \| y - z \| = \| z - w \| = 3, \ \| w - x \| = 1.$$

Then it is impossible to deform F so that x, y, z become collinear. Hence $x \to y \to z \to x$ is either always clockwise, or always counterclockwise. Therefore F is *not* reversible.

Let us call $F(a,b,c,d,e)$ a *general* pentagonal framework if $F(a,b,c,d,e)$ cannot be folded into a line, that is,

$$a \pm b \pm c \pm d \pm e \neq 0$$

for any choice of signs.

The next result is obtained by Maehara [27].

THEOREM 6.2. *Let F be a general pentagonal framework in $\boldsymbol{R}^2$ with bar-lengths $a_1 < a_2 < a_3 < a_4 < a_5$. Let*

$$s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{2}.$$

(1) *If $a_1 + a_2 + a_5 < s$, then F is not reversible, and $\Gamma(F, 2)$ is (homeomorphic to) a 2-dimensional torus.*

(2) *If $s < a_1 + a_2 + a_5$, then F is reversible and $\Gamma(F, 2)$ is an orientable closed 2-dimensional manifold of genus g, where*

$$g = \begin{cases} 0 & (a_5 < s < a_1 + a_5) \\ 1 & (a_1 + a_5 < s < a_2 + a_5) \\ 2 & (a_2 + a_5 < s < a_3 + a_5) \\ 3 & (a_3 + a_5 < s < a_4 + a_5) \\ 4 & (a_4 + a_5 < s) \end{cases}$$

EXAMPLE 6.1. For $F = F(1,2,3,4,5)$, we have $s = 7.5$, and $\Gamma(F,2)$ is a double torus (an orientable closed surface of genus 2).

The configuration space of a flexible framework is not necessarily a manifold. However, for a framework with $n+1$ joints in n-dimensions, the next result holds.

THEOREM 6.3. *Let F be a connected flexible framework in $\boldsymbol{R}^n$ with $n+1$ joints and $\binom{n+1}{2} - k$ bars. Suppose that F can be deformed so that the $n+1$ joints do not lie on a hyperplane. Then $\Gamma(F, n)$ is homeomorphic to a k-dimensional sphere.*

This result follows easily from a theorem of Schoenberg, which we are going to recall next.

6.2 Schoenberg's theorem

Let F be a connected flexible framework in $\mathbf{R}^n$ with $n+1$ joints and $\binom{n+1}{2}-k$ bars. Let $\sqrt{x_1},\ldots,\sqrt{x_k}$ be the distances between k nonadjacent pair of joints in some order, and let $\mathbf{x}=(x_1,x_2,\ldots,x_k)$. Let $\Omega(F)$ denote the set of those points $\mathbf{x}=(x_1,\ldots x_k)\in\mathbf{R}^k$ obtained by continuously deforming F in $\mathbf{R}^n$ in all possible ways, and let $\check{\Omega}(F)$ be the subset of $\Omega(F)$ consisting of the points $\mathbf{x}$ corresponding to those deformed F that lie on a hyperplane in $\mathbf{R}^n$.

Schoenberg [36] proved the following result.

THEOREM 6.4. (Schoenberg 1969) *Let F be a connected flexible framework in $\mathbf{R}^n$ with $n+1$ joints and $\binom{n+1}{2}-k$ bars. Suppose that F can be deformed so that the $n+1$ joints do not lie on a hyperplane. Then $\Omega(F)$ is a compact convex set with interior points in $\mathbf{R}^k$, and $\check{\Omega}(F)$ is the boundary of $\Omega(F)$.*

Instead of the proof, let us observe an example. Let F be a quadrilateral framework in $\mathbf{R}^3$ with 4 joints p_1,p_2,p_3,p_4 and 4 bars p_ip_{i+1} $(i=1,2,3,4)$ with $p_5\equiv p_1$, such that

$$\| p_1-p_2 \|=\| p_3-p_4 \|=1,$$
$$\| p_2-p_3 \|=\| p_4-p_1 \|=2.$$

Let $x=\| p_1-p_3 \|^2, y=\| p_2-p_4 \|^2$. What are $\Omega(F)$ and $\check{\Omega}(F)$?

With fixing x and fixing the three joints p_1,p_2,p_3 in a plane H, the joint p_4 can move with drawing a circle in the plane perpendicular to the line p_1p_3. This circle intersects the plane H at two points, one is at the nearest position to p_2, the other is at the farthest position from p_2. When p_4 comes to the nearest position to p_2, F forms a *contra-parallelogram*, and in this case, we have

$$\sqrt{x}\sqrt{y}+1\cdot 1=2\cdot 2$$

by Ptolemy's theorem. When p_4 comes to the farthest position from p_2, F forms a parallelogram, and we have

$$\sqrt{x}^2+\sqrt{y}^2=1^2+1^2+2^2+2^2=10$$

by the parallelogram theorem. Hence we have

$$9\le xy \text{ gand } x+y\le 10.$$

Thus $\Omega(F)$ is a compact convex region in $\mathbf{R}^2$ and $\check{\Omega}(F)$ is its boundary.

Proof of Theorem. Let $p_0,p_1,\ldots,p_n$ denote the vertices of F. Since every configuration $P=(p_0,\ldots,p_n)$ determines a point $\mathbf{x}\in\Omega(F)$, there is a natural continuous onto map $w : \Gamma(F,n) \to \Omega(F)$. Let vol (P) denote the oriented volume of the simplex spanned by $p_0, \ldots, p_n$. Let Γ_+, Γ_-, Γ_0 be the subsets of $\Gamma(F,n)$ consisting of the classes of configurations P with vol $(P) > 0$, vol $(P) < 0$, and vol $(P) = 0$, respectively. Then, $\Gamma(F,n) = \Gamma_+ \cup \Gamma_0 \cup \Gamma_-$. Further, $w : \Gamma_+ \cup \Gamma_0 \to \Omega(F)$, $w : \Gamma_- \cup \Gamma_0 \to \Omega(F)$, and $w : \Gamma_0 \to \dot{\Omega}(F)$ are all homeomorphisms. Hence, $\Gamma(F,n)$ is homeomorphic to the space obtained from two copies of $\Omega(F)$ by attaching along their boundaries by identity map. Since $\Omega(F)$ is a k-dimensional ball by Schoenberg's theorem, $\Gamma(F,n)$ is a k-dimensional sphere. ■

The configuration spaces of general pentagonal frameworks in $\boldsymbol{R}^2$ are classified by Theorem 6.2. Let F be a general pentagonal framework in $\boldsymbol{R}^4$ with joints $p_1, \ldots, p_5$. It is possible to deform F in $\boldsymbol{R}^4$ so that (1) the four joints p_1, p_2, p_3, p_4 span a 3-simplex, and (2) p_4, p_5, p_1 are not collinear. Then, we can rotate p_5 around the line p_1p_4 in the hyperplane perpendicular to p_1p_4 so that F does not lie on a hyperplane. Hence $\Gamma(F,4)$ is homeomorphic to a 5-dimensional sphere by Theorem 6.3.

PROBLEM 6.1. *Classify* $\Gamma(F,3)$ *for general pentagonal frameworks* F *in* $\boldsymbol{R}^3$.

It will not be difficult to see that for any general pentagonal framework F in $\boldsymbol{R}^3$, $\Gamma(F,3)$ is a 4-dimensional manifold. Kamiyama [17] proved that $\Gamma(F(1,1,1,1,1),3)$ is simply connected with Euler characteristic 7.

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