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DIAGONAL FLIPS OF TRIANGULATIONS ON SURFACES, A SURVEY

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1. Introduction

A *triangulation* G on a surface is a simple graph embedded on the surface so that each face is triangular and any two faces share at most one edge. (The

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second condition is needed only to exclude K_3 on the sphere.) So each face can be identified with the triple $\{u, v, w\}$, simply denoted by uvw, of the vertices on its three corners. We shall denote the sets of vertices, edges and faces of G by $V(G)$, $E(G)$ and $F(G)$, respectively.

Two triangulations G_1 and G_2 on F^2 are said to be *equivalent* or *homeomorphic* to each other if there is a homeomorphism $h : F^2 \to F^2$ with $h(G_1) = G_2$, that is, if there is a graph-isomorphism $\varphi : V(G_1) \to V(G_2)$ which induces a bijection between $F(G_1)$ and $F(G_2)$. Furthermore, they are *ambient isotopic* or simply *isotopic* to each other if the homeomorphism $h : F^2 \to F^2$ is isotopic to the identity map $\mathrm{id}_{F^2} : F^2 \to F^2$, that is, if there is a continuous map $H : F^2 \times [0,1] \to F^2$ such that the restriction $H|_{F^2\times\{t\}}$ is a homeomorphism for each $t \in [0,1]$, $H(x,0) = x$ and $H(x,1) = h(x)$ for each $x \in F^2$. We shall say that two triangulations are *isomorphic* when they are isomorphic to each other as just abstract graphs.

A *diagonal flip* is a local deformation of a triangulation G which replaces a diagonal edge ac with the other bd in a quadrilateral region obtained from two triangular faces abc and acd, as shown in Figure 1. We however forbid flipping a diagonal ac if it breaks the simpleness of a graph, that is, if b and d are already adjacent in G. Two triangulations are said to be *equivalent under diagonal flips* if they can be transformed into each other by a finite sequence of diagonal flips.

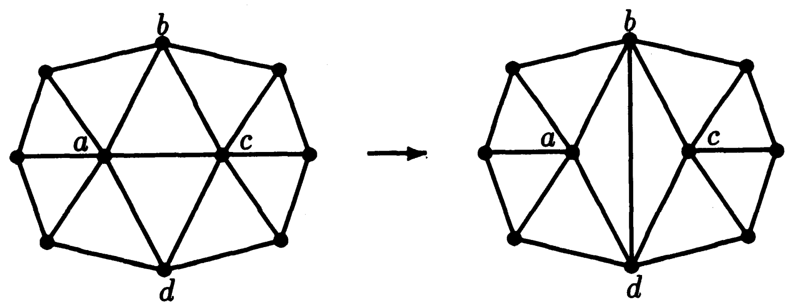

Figure 1. A diagonal flips in a triangulation

There have already been many studies on the diagonal flips in triangulatoins in the field of computational geometry; [9], [12], [20], [27], [34], [36], [79] and so on. In those papers, each triangulation subdivides the convex hull of a set of points with fixed position on the plane and each of its edges should be a straight segment joining a pair of points in the set. On the other hand, this survey will present many results in topological graph theory. So we may change the position of vertices on the surfaces freely and bend edges suitably by homeomorphisms.

Classically, Wagner [85] proved in 1936 that any two triangulations on the sphere with the same number of vertices are equivalent under diagonal flips.

Dewdney [25], Negami and Watanabe [64] also have shown the same results for the torus, the projective plane and the Klein bottle. Some of their arguments work in some general situation, but an essential part of their proofs strongly depend on those individual surfaces. So, it is hardly possible to generalize or extend their proofs for other general surfaces.

As we shall show later, the same fact does not hold as it is in general, that is, there exist a pair of triangulations which are not equivalent to each other under diagonal flips. However, Negami [66] has found some "breakthroughs" and showed that two triangulations on the same closed surface are equivalent under diagonal flips if they have the same and sufficienty large number of vertices, which is Theorem 15 in this survey. This result and his arguments to prove this have motivated some people to develop them and to find many phenomena related to triangulations of surfaces.

This survey includes those various topics, which are concerned with many concepts in topological graph theory, and hence it will work as a course to learn this field. Also, Negami and Nakamoto [65] have already written a survey on the same subject together with an analogue for quadrangulations [48], [49], [52], [53]. Our survey covers more recent studies and contains many questions for further research.

2. Wagner's theorem

The following theorem, which we shall refer as Wagner's theorem, is the starting point of our studies on diagonal flips of triangulations. Wagner's paper [85] including this theorem is written in German. So we shall show a full proof of his theorem for convenience of the reader.

Our proof is basicaly the same one as written in Ore's book [73] on "Four Color Problem", and gives us a simple algorithm to transform any triangulation on the sphere into the standard form as shown in Figure 2. Thus, two spherical triangulations can be transformed into each other via this standard form. We shall denote it by Δ_n with n the number of vertices inside the peripheral triangle. For example, Δ_0 is isomorphic to K_3, which is not however a triangulation by our definition, and Δ_1 is isomorphic to K_4, the tetrahedron.

THEOREM 1. (Wagner [85]) *Any two triangulations on the sphere with the same number of vertices are equivalent to each other under diagonal flips, up to homeomorphism.*

Proof. Let G be a triangulation on the sphere and identify it with a maximal

See the second edition of Ore's book. Someone told me that his proof in the first edition is incorrect.

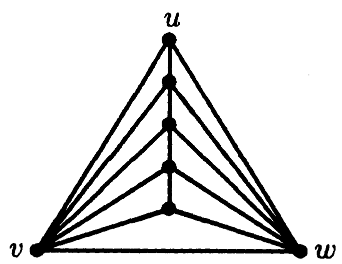

Figure 2. The standard form of spherical triagulations Δ_n

planar graph on the plane with uvw its outer region. First, suppose that $\deg u \geq 4$ and let $v, v_1, v_2, \ldots, v_m = w$ be the neighbors of u, lying around u in this cyclic order. If v is not adjacent to v_2 in G, then we can replace the diagonal uv_1 with vv_2 in the quadrilateral region uvv_1v_2. On the other hand, if v is adjacent to v_2, then v_1 cannot be adjacent to v_3, which is possibly w, and we can replace uv_2 with v_1v_3. In either case, we can decrease the degree of u by a diagonal flip.

Repeat this deformation as far as possible. Finally, we shall get a triangulation with $\deg u = 3$. Let u_1 be the unqiue neighbor of u except v and w, and consider the maximal planar graph G_1 bounded by u_1vw. Carrying out the same argument on G_1 as above, we shall obtain G_2 with $\deg u_1 = 3$ in G_1. Continue these. Then we shall get the path $uu_1u_2 \cdots u_n$ such that $\deg u = \deg u_n = 3$, $\deg u_i = 4$ $(i \neq n)$ and all u_i's are adjacent to both v and w. The final form is homeomorphic to the standard form Δ_n with $n+3$ vertices. ■

3. Basic tricks and pseudo-minimals

Our proof of Wagner's theorem strongly depends on the planarity of triangulations. The point is however how to make a vertex of degree 3. After making a vertex of degree 3, we would like to reduce our arguments to those on another triangulation with fewer vertices than the original. The following two tricks enable us to do it. Figures 3 and 4 are telling us all. See [64] or [66] for the detail.

TRICK 1. A vertex of degree 3 can be moved to any face by diagonal flips.

TRICK 2. Let G_1 and G_2 be two triangulations with vertices v_1 and v_2 of degree 3, respectively. A sequence of diagonal flips from $G_1 - v_1$ to $G_2 - v_2$ can be translated into that from G_1 to G_2.

By these tricks, if we can make vertices of degree 3 in two given triangulations by diagonal flips, then we can conclude that they are equivalent under diagonal flips, by the induction hypothesis and these tricks. So we have to consider those

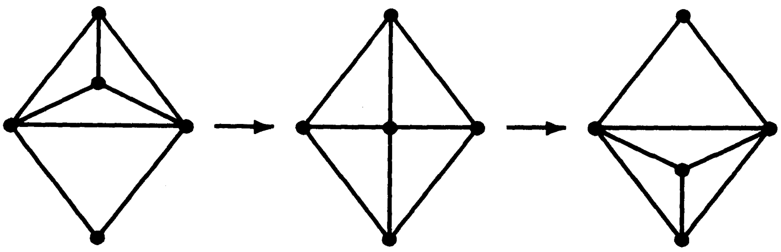

Figure 3. Moving a vertex of degree 3

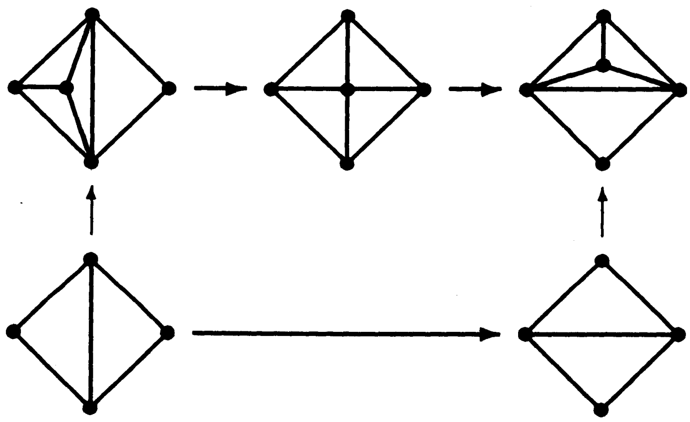

Figure 4. Lifting a daigonal flip

triangulations from which we cannot make a vertex of degree 3, to establish the first step of our induction. We shall call G a *pseudo-minimal* triangulation if it is not equivalent to any triangulation which has a vertex of degree 3 under diagonal flips.

Let G be a pseudo-minimal triangulation such that its minimum degree $\delta(G)$ is the smallest among those equivalent to G under diagonal flips. Choose a vertex u of G so that u attains $\delta(G)$ and consider the structure of its neighborhood in G. Let $v_0, v_1, \ldots, v_{n-1}$ be the neighbors of u, with indices taken modulo n, which form a cycle surrounding u in this order. Such a cycle is called the *link* of u in G, denoted by $\mathrm{lk}(u)$, and the *star neighborhood* $\mathrm{st}(u)$ of u is the wheel obtained as the union of u and $\mathrm{lk}(u)$ with edges between them. Since any diagonal flip cannot decrease $\deg u$ by our assumption, there have to be edges $u_{i-1}u_{i+1}$ outside the star neighborhood of u for all i, which looks like "a sunflower".

For example, if $\deg u \geq 7$, then such a flower-like neighborhood cannot be embedded in the torus. If $\deg u = 6$, then there is only one way to embed it in the torus, and it extends to the embedding of K_7 uniquely. Similarly, classify the embeddings of flower-like neighborhoods on a given susrface and extend them to be triangulations. If we could conclude that those extensions, which should be pseudo-minimal, are equivalent to one another, then we would obtain the theorem for the surface corresponding to Wagner's theorem.

Carrying out the above for the torus, the projective plane and the Klein bottle, Dewdney [25], Negami and Watanabe [64] have proved the following theorems. For example, the only pseudo-minimal triangulations of the torus is K_7 while that of the projective plane is K_6. On the other hand, the pseudo-minimal triangulations of the Klein bottle are not unique and they have 8 vertices. More detailed information on them will be shown in Section 5.

THEOREM 2. (Dewdney [25]) *Any two triangulations on the torus are equivalent under diagonal flips, up to homeomorphism, if they have the same number of vertices.*

THEOREM 3. (Negami and Watanabe [64]) *Any two triangulations on the projective plane are equivalent under diagonal flips, up to homeomorphism, if they have the same number of vertices.*

THEOREM 4. (Negami and Watanabe [64]) *Any two triangulations on the Klein bottle are equivalent under diagonal flips, up to homeomorphism, if they have the same number of vertices.*

As far as in cases of the torus, the projective plane and the Klein bottle, all the pseudo-minimal triangulations are the triangulations with fewest vertices on each of these sufaces, called *minimal triangulations*.

QUESTION 1. *Is a pseudo-minimal triangulation minimal?*

Negami [66] had conjectured that any pseudo-minimal triangulation is minimal, but the answer is negative in general, as shown in Section 18.

4. Contraction of edges

In their proofs of the previous theorems, the specified topology of each surface plays an essential role to classify the pseudo-minimal triangulations. So it is hardly possible to extend such a proof for general surface. However, Negami [66] has devised a nice trick to make it possible, as follows.

Contraction of an edge e in a triangulation G is to shrink e on the surface and to eliminate digonal regions, as shown in Figure 5. The resulting graph on

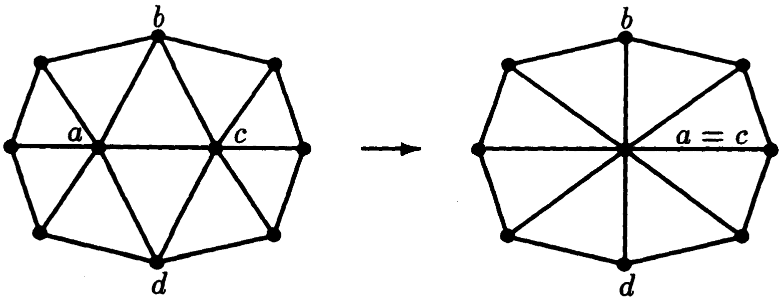

Figure 5. Contracting an edge in a triangulation

the surface is denoted by G/e. An edge e is said to be *contractible* if G/e also is a triangulations. When G is not isomorphic to K_4, an edge e is not contractible if and only if e lies on a cycle of length 3 which does not bound a face. We don't contract an edge e when it is not contractible. For, if we did, then G/e would be a nonsimple graph. A triangulation G is said to be *contractible* to another triangulation T if T can be obtained from G by contracting some edges in order.

Any sequence of diagonal flips does not change the number of vertices while a sequence of edge contraction decreases it. Although this impresses us with their difference, the following trick connects them and will enable us to establish a theory of diagonal flips in triangulations on general surfaces with the notion of irreducible triangulations defined in the next section.

TRICK 3. Contraction of an edge can be realized as a sequence of diagonal flips followed by removing a vertex of degree 3.

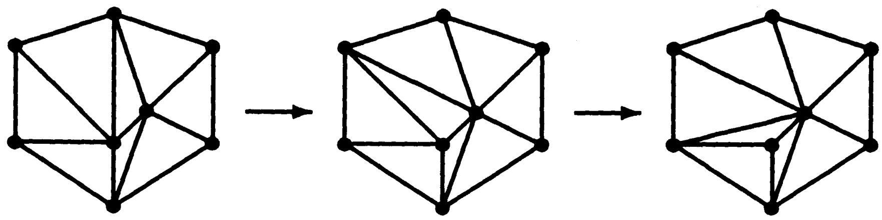

Figure 6. Contraction and diagonal flips

Let G be a triangulation and Δ one of its face. Attach the standard form Δ_n of spherical triangulations to the face Δ to obtain another triangulation on the same surface. The resulting triangulation is denoted by $G+\Delta_n$. This notation is however ambiguous, but all the triangulations with the same notation $G+\Delta_n$ are equivalent under diagonal flips, by Trick 1. The next lemma follows immediately from Trick 3.

LEMMA 5. *Let G and T be triangulations on a closed surface. If G is contractible to T, then G is equivalent to $T + \Delta_m$ with $m = |V(G)| - |V(T)|$.*

By this lemma, if two triangulations are contactible to a common triangulation T, then they are equivalent under diagonal flips via $T + \Delta_m$. For example, every triangulation on the sphere, except K_4, contains a contractible edge and hence it is contractible to K_4 [81]. Thus, it is equivalent to $K_4 + \Delta_m$, which is nothing but the standard form Δ_{m+1}. This is another proof of Wagner's theorem. There is not however such a common triangulation for two given triangulations in general.

5. Irreducible trianglations

A triangulation is said to be *irreducible* if it has no contractible edges. It is clear that any triangulation is contractible to one of irreducible triangulations. In other words, it can be obtained from some irreducible triangulation by a sequence of the inverse operations, called *splitting* of vertices or *vertex-splitting*. Note that the following implicaiton:

$$\text{minimal} \Longrightarrow \text{psendo-minimal} \Longrightarrow \text{irreducible}$$

The second implication follows from Trick 3.

There have been already classified the irreducible triangulations of the projective plane, the torus and the Klein bottle.

THEOREM 6. (Barnette [10]) *There are only two irreducible triangulations of the projective plane, up to homeomorphism, given in Figure 7.*

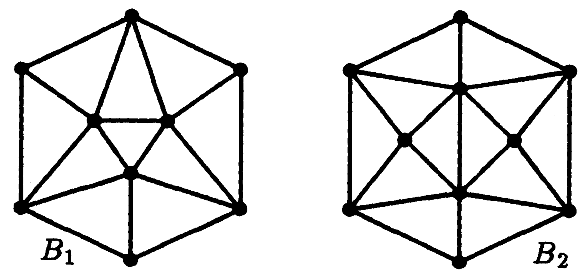

Figure 7. Irreducible triangulations of the projective plane

For our later arguments, we shall assign the notations B_1 and B_2 to the left and right ones in Figure 7. Then B_1 is isomorphic to K_6 while B_2 to $K_4 + \overline{K_3}$ as graphs. To obtain the actual triangulations on the projective plane, we should identify each pair of antipodal points on the boundary of each hexagon. Note that B_1 is psendo-minimal but B_2 is not.

THEOREM 7. (Lawrencenko [40]) *There are precisely* 21 *irreducible triangulations of the torus, up to homeomorphism.*

THEOREM 8. (Lawrencenko and Negami [41]) *There are precisely* 25 *irreducible triangulations of the Klein bottle, up to homeomorphism;* 21 *handle types* Kh1 *to* Kh21, *and* 4 *crosscap types* Kc1 *to* Kc4.

In general, there exist only a finite number of irreducible triangulations for each closed surface. This follows theoretically from Wagner's Conjecture, solved affirmatively by Robertson and Seymour [77]. A graph H is called a *minor* of a graph G, denoted by $H \leq {}_m G$, if H can be obtained from G by contracting and deleting edges. Note that the "contraction" used in graph minor contract only an edge and does not eliminate digonal regions so well as ours.

THEOREM 9. (Wagner's Conjecture; Robertson and Seymour [77]) *Any infinite series* $\{G_1, G_2, G_3, \ldots\}$ *of graphs contains a pair* G_i *and* G_j *one of which is a minor of the other.*

There has been some works [11], [28], [47], [50] to give an upper bound for the number of vertices of irreducible triangulations. Let $\chi(F^2)$ denote the Euler characteristic of a surface F^2. The following is the best result at the present and is best possible for the order with respect to the *Euler genus* $2 - \chi(F^2)$.

THEOREM 10. (Nakamoto and Ota [50]) *Any irreducible triangulations of a closed surface* F^2 *with* $\chi(F^2) \leq 0$ *has at most* $171(2 - \chi(F^2)) - 72$ *vertices.*

They have constructed a series of irreducible triangulations the number of whose vertices are linear with respect to their Euler genus, by adding many irreducible triangulations of the torus and the projective plane to faces of a suitable spherical triangulation. So each of their examples contains a cycle of length 3 which separates the surface into two pieces. Conversely, we ask:

QUESTION 2. *Does any irreducible triangulation of the maximum order on a closed surface with negative Euler characteristic contain a separating cycle of length* 3*?*

Now consider Figure 8 to visualize our theory. This expresses the set of all the triangulations on a closed surface. The vertical axis represents the number

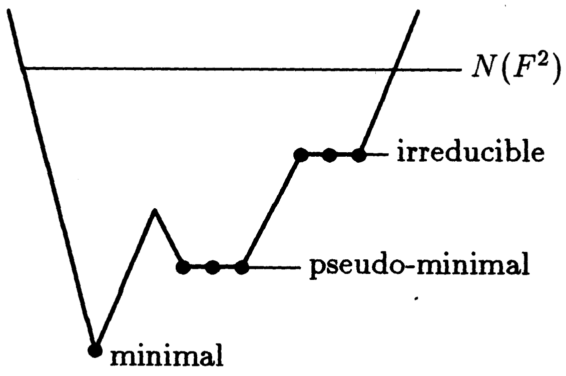

Figure 8. Minimal, pseudo-minimal and irreducible triangulations

of vertices and hence the triangulations with the same number of vertices are placed at the same horizontal level. Thus, the contraction of an edge moves a triangulation downward while a diagonal flip carries it horizontally. Any triangulation G goes down to either a valley or a plateau of this figure through contraction. The valley includes some pseudo-niminal triangulations, which can be transformed into one another by diagonal flips but connot go out of this valley. In particular, the bottom includes minimal triangulations. On the other hand, the plateau includes irreducible triangulations which are not pseudo-minimal. Thus, they goes horizontally and reach a triangulation which has a vertex of degree 3. For example, the set of triangulations of the torus has only one valley $\{T_1\}$ (7 vertices) and four plateaues:

$$\{T_2, T_3, T_4, T_5\} \text{ (8 vertices)}; \ \{T_7\}, \{T_6, T_8, \ldots, T_{20}\} \text{ (9 vertices)}$$
$$\{T_{21}\} \text{ (10 vertices)}$$

The irreducible triangulations have their own right and have been discussed in many other contexts. When we want to conclude that every triangulation has some property, it may suffices to show that every irreducible triangulation does after checking whether or not the property in question is preserved by vertex-splitting. For example, Brunet, Nakamoto and Negami [19] has shown the following theorem, based on the classification of irreducible triangulations of the Klein bottle [41].

THEOREM 11. (Brunet, Nakamoto and Negami [19]) *Every 5-connected triangulation of the Klein bottle includes a hamilton cycle which bounds a 2-cell region.*

Although the hamiltonicity is not preserved by vertex-splitting, the following fact is important for their proof; a triangulation of the Klein bottle includes two disjoint meridians if and only if it does not include an essential separating cycle of length 3. This can be proved by the way mentioned above basically.

The next theorem [42] is another more direct application of Theorems 7 and 8. The property in the theorem is actually preserved by vertex-splitting and it suffices to show that the intersection of the sets of irreducible triangulations of the torus and of the Klein bottle consists of only graphs which have the property. Such a graph in the intersection is unique and is T_3 for the torus and Kh1 for the Klein bottle.

THEOREM 12. (Lawrencenko and Negami [42]) *A graph G triangulates both the torus and the Klein bottle if and only if G has the structure given in Figure 9.*

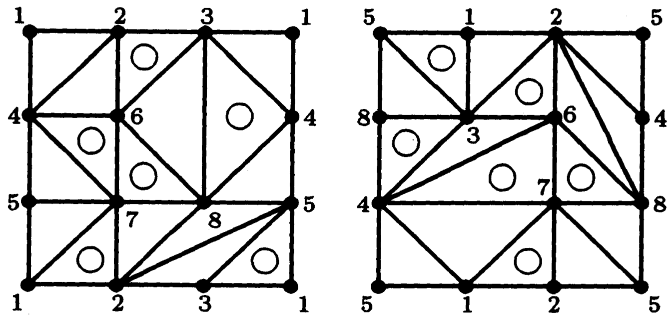

Figure 9. Graphs which triangulate both the torus and the Klein bottle

The labeling of vertices in Figure 9 indicates the identification to make T_3 in the torus and Kh1 in the Klein bottle, and also the isomorphism between T_3 and Kh1. The faces marked with ○ should be subdivided arbitrarily. Adding a suitable number of handles to those, we can construct easily an example of a graph which triangulates two different closed surfaces with the same even Euler characteristic, orientable and nonorientable, other than the torus and the Klein bottle. The complete graph K_n also is such a graph. Using Ringel's solution of "Map Color Theorem", we can show that K_n triangulates the different surfaces if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$. See section 8 for triangulations with complete graphs.

6. Common refinements and stable equivalence

First, consider the equivalence over triangulations of the projective plane. Let G_1 and G_2 be two triangulations on the projective plane with the same number of vertices. Each of them is contractible to either B_1 or B_2. As we have already seen, if they are contactible to the same irreducible triangulation, then they are equivalent to each other under diagonal flips. So we may assume that G_1 is contractible to B_1 and G_2 to B_2. In this case, G_i is equivalent to $B_i + \Delta_{m_i}$, with $m_1 = m_2 + 1$, by Lemma 5. On the other hand, $B_1 + \Delta_1$ can be transformed into B_2 by two diagonal flips. Then $B_1 + \Delta_1 + \Delta_{m_2}$ and $B_2 + \Delta_{m_2}$ are equivalent by Tricks 1 and 2. Since $B_1 + \Delta_1 + \Delta_{m_2}$ is equivalent to $B_1 + \Delta_{m_1}$ by Trick 1, $B_1 + \Delta_{m_1}$ and $B_2 + \Delta_{m_2}$ are equivalent, too. It follows that G_1 and G_2 are equivalent under diagonal flips. This is an easy proof of Theorem 3, as an applicaiton of our theory.

To generalize this argument, Negami [66] has considered the following trick. A *refinement* of a triangulation G is a triangulation which includes a subdivision of G as its subembedding. He has proved that any refinement of G is contractible to G and hence:

TRICK 4. Any common refinement of two triangulations is contractible to both of them.

Two triangulations G_1 and G_2 are said to be *stably equivalent* under diagonal flips if $G_1 + \Delta_{m_1}$ and $G_2 + \Delta_{m_2}$ are equivalent with suitable numbers m_1 and m_2. Then, we can show the following lemma, using Trick 4 and Lemma 5. Put G_1 and G_2 together on the same surface which they triangulate and divide each region to be triangular. Then a common refinement of G_1 and G_2 will be obtained. (The way to construct such a common refinement will be important for our later argument.)

LEMMA 13. *Any two triangulations are stably equivalent to each other under diagonal flips.*

In particular, any pair of irreducible triangulations of the same surface are stably equivalent as well as B_1 and B_2 in case of the profective plane. Not only the notion of stable equivalence but also the finiteness of irredicible triangulations in number will play very important roles through our theory.

A natural question arises; for what m, are $G_1 + \Delta_m$ and $G_2 + \Delta_m$ equivalent under diagonal flips if G_1 and G_2 have the same number of vertices? Negami has already given an answer to this question with a new idea, which will appear in Section 19. Also see Section 12 for the phrase "up to isotopy".

THEOREM 14. (Negami [71]) *Let G_1 and G_2 be two triangulations on a closed surface F^2 with the same number of vertices, say n. Then $G_1 + \Delta_m$ and $G_2 + \Delta_m$ can be transformed into each other, up to isotopy, by a sequence of diagonal flips if $m \geq 18(n - \chi(F^2))$.*

7. Diagonal flips on general surfaces

Now we have prepared all we need to prove the following theorem. This is the first goal in our theory and has opened a new stage of the study on diagonal flips in triangulation on surfaces.

THEOREM 15. (Negami [66]) *Given a closed surface F^2, there exists a natural number $N = N(F^2)$ such that two triangulations G_1 and G_2 on F^2 are equivalent under diagonal flips, up to homeomorphism, if $|V(G_1)| = |V(G_2)| \geq N$.*

Proof. Let $T_1, T_2, \ldots$ be the irreducible triangulations of F^2. By Lemma 13, there are a pair of nonnegative integers m_{ij} and m_{ji}, for any pair T_i and T_j, such that $T_i + \Delta_{m_{ij}}$ and $T_j + \Delta_{m_{ji}}$ are equivalent under diagonal flips. Since the irreducible triangulations of F^2 are finite in number, there exists a constant N such that

$$|V(T_i + \Delta_{m_{ij}})| = |V(T_i)| + m_{ij} \leq N$$

for all pairs $\{i, j\}$.

Let G_1 and G_2 be any two triangulations of F^2 with $|V(G_1)| = |V(G_2)| \geq N$ and suppose that G_i is contractible to T_i. Then G_i is equivalent to $T_i + \Delta_{m_i}$ for each i by Lemma 5. Since $|V(T_i + \Delta_{m_{ij}})| \leq |V(T_i + \Delta_{m_i})|$, then $T_1 + \Delta_{m_1}$ and $T_2 + \Delta_{m_2}$ are equivalent to each other by Tricks 1 and 2. This implies that G_1 and G_2 are equivalent under diagonal flips via $T_1 + \Delta_{m_1}$ and $T_2 + \Delta_{m_2}$. ■

Let $N(F^2)$ denote the minimum value of N in the theorem. With this notation, Theorems 1 to 4 can be expressed by the following formulas:

$$N(S^2) = 4, \quad N(P^2) = 6, \quad N(T^2) = 7, \quad N(K^2) = 8$$

where S^2, P^2, T^2 and K^2 stand for the sphere, the projective plane, the torus and the Klein bottle in order. Each of these numbers is equal to the number of a minimal triangulation for each surface. However, $N(F^2)$ does not coincide with such a value in general. In fact, there exist a pair of minimal triangulations which are not equivalent under diagonal flips, as is shown in the next section. For example, if a complete graph triangulates a closed surface in two or more ways, those triangulations are not equivalent under diagonal flips. For, any diagonal flip is not applicable to a complete graph since it yields multiple edges.

Note that our arguments in the proof also work well, assuming that $T_1, T_2, \ldots$ are pseudo-minimal. A triangulation G is not contractible to a pseudo-minimal one in general, but it is equivalent to $T_i + \Delta_m$ under diagonal flips for some pseudo-minimal triangulation T_i and with $m = |V(G)| - |V(T_i)|$. This leads to the following exact formula for $N(F^2)$:

$$N(F^2) = \min\{N : \text{All } T_i + \Delta_{N-|V(T_i)|} \text{ are equivalent to one another under diagonal flips}\}$$

where $T_1, T_2, \ldots$ are the pseudo-minimal triangulations on F^2. However, this is just an abstract formula and we don't have a way to determine this value in a general form. We shall discuss on an upper bound for $N(F^2)$ in Section 15.

8. Inequivalent complete triangulations

Let $V_{\min}(F^2)$ denote the order of minimal triangulations on F^2, that is, the minimum number n such that there exist a triangulation on F^2 with n vertices. This value $V_{\min}(F^2)$ is a trivial lower bound for $N(F^2)$, and $V_{\min}(F^2) < N(F^2)$ if and only if there exist a pair of minimal triangulations of F^2 which are not equivalent to each other under diagonal flips.

The precise value of $V_{\min}(F^2)$ has been determined in [37] and [74], as an application of the solution of "Map Color Theorem".

$$V_{\min}(F^2) = \left\lceil \frac{7 + \sqrt{49 - 24\chi(F^2)}}{2} \right\rceil$$

where F^2 is neither the orientable closed surface of genus 2, the nonorientable ones of genus 3, nor the Klein bottle. The values of $V_{\min}(F^2)$ are 10, 9 and 8 for these exceptional surfaces, respectively.

It is well-known that if a closed surface F^2 admits a triangular embedding of a complete graph, called a *complete triangulation* here, then such a triangulation is minimal. Conversely, the complete graph K_n triangulates an orientable closed surface if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$, while K_n triangulates a nonorientable one if and only if $n \equiv 0, 1, 3$ or $4 \pmod 6$ and $n \neq 7$. Ringel's book [75] on "Map Color Theorem" includes a construction of possible triangulations with K_n for all n with the above conditions, but they are just "one of them". We are interested in the existence of another construction of complete triangulations, as shown in the previous section.

Archo, Bracho and Neumann-Lara [7], [8] have discussed on the complete triangulations in another context. A 3-uniform hypergraph or 3-graph H with vertex set $V(H)$ is said to be *tight*, in general, if for any partition $V(H) = X \cup Y \cup Z$ with $X \neq \emptyset$, $Y \neq \emptyset$ and $Z \neq \emptyset$, there is an edge $\{x, y, z\}$ of H such that $x \in X$, $y \in Y$ and $z \in Z$. A triangulation G of a closed surface also is said to be *tight* (or *untight*) if the 3-graph $(V(G), F(G))$ is tight (or not). They have shown a certain method to construct a series of tight triangulations with complete graphs and a series of untight ones. In particular, they have constructed tight and untight triangulations of the nonorientable closed surface of genus 117 with K_{30} and found three inequivalent triangular embeddings of K_{16} into the nonorientable closed surface of genus 26, discussing the types of partitions. However, their construction necessarily produces a triangulation on a nonorientable closed surface.

QUESTION 3. *Is there an untight triangulation on an orientable closed surface?*

Lawrencenko, Negami and White [43] also have found three inequivalent complete triangulations with K_{19} into the orientable closed surface of genus 20, which are all tight. Bracho and Strausz [17] has determined the smallest n for which K_n admits two or more inequivalent triangulations on the same closed surface; $n = 12$ for the orientable surfaces while $n = 9$ for the nonorientable ones. In particular, Strausz [82] has found a triangular embedding of K_9 on the nonorientable closed surface of genus 5, which is different from Ringel's construction. Very recently, Bonnington, Grannell, Griggs and J. Širáň have developed a method to construct exponentially many inequivalent complete triangulations, using Steiner triple systems, in [13], [31] and [32]

9. Loosely-tightness of triangulations

The tightness is a good notion to distinguish inequivalent complete triangulations, but does not work for non-complete triangulations at all; it is easy to see that a tight triangulation is necessarily complete.

To extend the notion of tightness for other triangulations, Negami and Midorikawa [68] have given the following definition. A triangulation G is said to be *k-loosely tight* if for any surjective assignment $f : V(G) \to \{1, 2, \ldots, 3 + k\}$, there is a face $xyz \in F(G)$ such that the three vertices x, y and z have three distinct "colors", that is, $|f(\{x, y, z\})| = 3$. Such a face is called a *heterochromatic face*. Clearly, any triangulation G of a closed surface is k-loosely tight for some $k \leq |V(G)| - 3$. The minimum value of k with G k-loosely tight is called the *looseness* of G and is denoted by $\xi(G)$. In particular, G is tight if and only if $\xi(G) = 0$.

They have discussed on various relationship between the looseness and other combinatorial structures or deformations of triangulations. For example, they have shown the following lemma.

LEMMA 16. *Let G be a triangulation on a closed surface F^2 and e a contractible* *edge in G. Then:*

$$\xi(G/e) \le \xi(G) \le \xi(G/e) + 1$$

Furthermore, they have shown an upper bound for the looseness of complete triangulations as follows, mimicking Arocha, Bracho and Neumann-Lara's arguments [8]. This inequality implies that K_n does not admit an untight triangulation if $n < 12$, but the smallest order of an untight complete triangulation is 16.

THEOREM 17. (Negami and Midorikawa [68]) *The looseness of any complete triangulation G with n vertices has the following upper bound:*

$$\xi(G) \le \sqrt{n + \frac{1}{4}} - \frac{5}{2}$$

Also, Negami and Midorikawa have proposed many problems related to the looseness of triangulations.

QUESTION 4. *Is there a constant c, independent of the surfaces, such that any complete triangulation has looseness at most c?*

Define the *looseness* $\xi(F^2)$ of a closed surface F^2 as the minimum value of $\xi(G)$ taken over all the triangulations G on F^2. By Lemma 16, some irreducible triangulation attains it. For example, K_4 in the sphere, K_6 in the projective plane and K_7 in the torus are tight and have looseness 0.

$$\xi(S^2) = 0, \quad \xi(P^2) = 0, \quad \xi(T^2) = 0, \quad \xi(K^2) = 1$$

QUESTION 5. *Find $\xi(F^2)$ for each closed surface F^2. Is there a minimum triangulation of F^2 which attains $\xi(F^2)$?*

We are expecting that the looseness of triangulations works as an invariant to distinguish the inequivalence of thier embeddings. However, it is also closely related to some combinatorial structures of triangulations, as follows. We denote the independence number and the diameter of a graph G by $\alpha(G)$ and $\mathrm{dia}(G)$, respectively.

THEOREM 18. (Tanuma [83]) *Let G be a triangulation on the sphere, the projective plane, the torus or the Klein bottle. Then G is 1-loosely tight if and only if $\alpha(G) \le 1$ and $\mathrm{dia}(G) \le 1$.*

THEOREM 19. (Tanuma [84]) *Two triangulations on the projective plane have the same looseness if they are isomorphic as graphs.*

To prove the last theorem, Tanuma has used a theory to investigate re-embeddings of triangulations on closed surfaces, developed in [69].

QUESTION 6. *Construct a pair of inequivalent triangulations on each closed surface with different looseness, which are isomorphic as graphs and are not complete.*

10. Diagonal flips in labeled triangulations

Through these four consecutive sections, we shall discuss on some variations of Theorem 15. First, we shall consider the *labeled triangulation*, that is, a triangulation with vertices $v_1, \ldots, v_n$ such that each v_i has a fixed label i. Two labeled triangulations G_1 and G_2 on a closed surface F^2 should be equivalent if there is a homeomorphism $h : F^2 \to F^2$ such that $h(G_1) = G_2$ and that $v \in V(G_1)$ and $h(v) \in V(G_2)$ have the same label.

TRICK 5. Let G be a labeled triangulation with a vertex v of degree 3. Then we can exchange the labels of v and of any other vertex by diagonal flips.

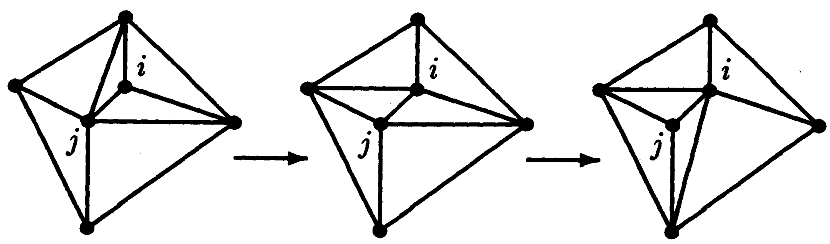

Figure 10. Exchanging labels

By the above trick, the vertex of degree 3 plays the role of a "label carrier". Since it can be moved freely to any place by Trick 1, it will carry the labels between two vertices whose labels we want to transpose. The following theorem is just a corollary of Theroem 15.

THEOREM 20. *Given a closed surface F^2, there exists a natural number $L = L(F^2)$ such that two labeled triangulations G_1 and G_2 are equivalent under diagonal flips, up to homeomorphism, if $|V(G_1)| = |V(G_2)| \geq L$. Futhermore, we have:*

$$N(F^2) \leq L(F^2) \leq N(F^2) + 1$$

Proof. Let G_i be any labeled triangulations with at least $N(F^2)+1$ vertices. Choose and fix another triangulation G_0 with $|V(G_i)|-1$ vertices. Then G_i is equivalent to $G_0+\Delta_1$ under diagonal flips as unlabeled triangulations, by Theorem 15. Since $G_0+\Delta_1$ contains a vertex of degree 3, we can change the labeling of $G_0+\Delta_1$ by Trick 5 and conclude that G_i and $G_0+\Delta_1$ are equivalent as labeled triangulations, too. Thus, any two triangulation G_1 and G_2 can be transformed into each other via $G_0+\Delta_1$ by diagonal flips in the labeled sense. ∎

By the same logic, we can show that if $V_{\min}(F^2) < N(F^2)$, then $N(F^2) = L(F^2)$. If a closed surface F^2, except the sphere, has a complete triangulation, which is necesarily minimal, then $V_{\min}(F^2) < L(F^2) \leq N(F^2)+1$, since such a complete triangulation is not faithful as an embedding of a graph. An embedding of a graph G on a surface F^2 is said to be *faithful*, in general, if any automorohism of G extends to an auto-homeomorphism over F^2. This concept has been defined in [55] and discussed with the uniqueness of embedding through a series of Negami's papers [55] to [63]. Whitney's theorem on the unique dual [87] implies that K_4 on the sphere is faithful, that is, any two K_4's on the sphere labeled arbitrarily are equivalent up to homeomorphism. These arguments implies the following equalities. Decide the value of $L(K^2)$ for the Klein bottle; 8 or 9.

$$L(S^2)=4, \quad L(P^2)=7, \quad L(T^2)=8, \quad L(K^2)=8 \text{ or } 9$$

11. Triangulations with boundaries

Now consider a triangulations G of a surface F^2 with boundary ∂F^2, that is, a graph embedded on F^2 so that each face is triangular and that each component of ∂F^2 is a cycle in G. Our theory works also for those triangulations in the same manner as for these without boundary and will conclude the following theorem easily:

THEOREM 21. *Given a surface F^2 which has k boundary components assigned with natural numbers $c_1,\ldots,c_k \geq 3$, there exists a natural number $D = D(F^2; c_1,\ldots,c_k)$ such that two triangulations G_1 and G_2 on F^2 are equivalent under diagonal flips, up to homeomorphism, if $|V(G_1)| = |V(G_2)| \geq D$ and if the corresponding boundary cycles of both G_1 and G_2 have the prescribed length c_i.*

It should be noticed that both G_1 and G_2 in the theorem have to have many vertices not on ∂F^2. We need other arguments to establish a similar theorem if we forbid any vertex in the interior of F^2.

A triangulation G on a surface F^2 is called a *Catalan triangulation* on F^2 if all the vertiecs of G lie on ∂F^2. For example, a Cataran triangulation on a disk is nothing but a polygon subdivided into triangles with only its diagonals. The number of ways to subdivide an n-gonal disk is well-known as the Catalan number:

$$\frac{2 \cdot 6 \cdot 10 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdots (n-1)}$$

This is the reason why such a triangulation is called a Catalan triangulation.

The dual of a Cataran triangulation of a disk is a binary tree and a diagonal flip in the Cataran triangulation corresponds to what is called a rotation in a binary tree, which is an important notion in computer science. Sleator, Tarjan and Thurston [80] have shown a nice result on Catalan triangulations of the disk, as follows.

THEOREM 22. (Sleator, Tarjan and Thurston [80]) *Any two Cataran triangulations of an n-gonal disk with $n \geq 13$ can be transformed into each other by diagonal flips in at most $2n - 10$ steps.*

It is important that this theorem specifies an upper bound for the number of diagonal flips needed to transform a given two Cataran triangulations. Their proof of this theorem itself is not difficult, but they use a big theory of 3-dimensional hyperbolic geometry and computer experiments to show the existence of a pair of Catalan triangulations of an n-gonal disk for $n \geq 13$ which attain the bound in the theorem.

The fist result for Cataran triangulations of a nonplanar surface has been established by Edelman and Reiner [26]. They have enumerated Cataran triangulations of the Möbius band, showing the following theorem and some observations.

THEOREM 23. (Edelman and Reiner [26]) *Any two Cataran triangulations of the Möbius band with the same number of vertices are equivalent under diagonal flips, up to homeomorphism.*

A *punctured surface* is a surface with precisely one boundary component. For example, the punctured torus can be obtained from the torus by removing an open disk to make a hole. The following two theorems might be said to be applicatios of our theory in a sense, but other special arguments are needed to prove them in fact.

THEOREM 24. (Cortés and Nakamoto [22]) *Any two Cataran triangulations of the punctured torus with the same number of vertices are equivalent under diagonal flips, up to homeomorphism.*

THEOREM 25. (Cortés and Nakamoto [23]) *Any two Cataran triangulations of the punctured Klein bottle with the same number of vertices are equivalent under diagonal flips, up to homeomorphism.*

To prove the following theorem, we need another breakthrough, which will be shown in Section 19. This is a very rencent work.

THEOREM 26. (Cortés, Glima, Márquez and Nakamoto [24]) *Given a puctured surface F^2, there exists a natural number $N(F^2)$ such that two Cataran triangulations G_1 and G_2 of F^2 are equivalent under diagonal flips, up to homeomorphism, if $|V(G_1)| = |V(G_2)| \geq N(F^2)$.*

12. Isotopy and Dehn twists

Many theorems in this survey contain the phrase "up to homeomorphism". This means that any homeomorphic image of a triangulation should be regarded as the same one as the original. For example, the mirror image $\overline{G}$ of a triangulation G on a closed surface F^2 is homeomorphic or is equivalent to G *up to homeomorphism*, but they cannot be transformed into each other continuously on F^2. In this case, we say that $\overline{G}$ and G are not equivalent *up to isotopy*.

Consider the unique embedding of K_7 into the torus, which is a triangulation of the torus. Cutting it along a non-trivial cycle of length 3, we obtain a triangulated annulus. Now twist one of the two boundary cycles several times and identify them again. This deformation is called a *Dehn twist* and induces an auto-homeomorphism over the torus in this case. The resulting triangulation is also K_7 on the torus. The latter looks more complicated than the former and they are not equivalent up to isotopy. However, if we didn't regard them as the same one, then Theorem 2 would not hold since there is no way to transform them into each other by diagonal flips. So we need such identification between those triangulations, *up to homeomorphism*.

What we can say about the equivalence under diagonal flips, up to isotopy? The following trick will enable us to answer this question.

TRICK 6. A Dehn twist along a cycle C in a triangulation can be realized by a sequence of diagonal flips if the cycle C is a part of a zigzag ladder as shown in Figure 11.

Using this trick, Nakamoto and Ota [53] have improved Theorem 15 so that it holds up to isotopy, as follows. However, the lower bound for the number of vertices, corresponding to $N(F^2)$, will be so larger than $N(F^2)$ in Theorem 15.

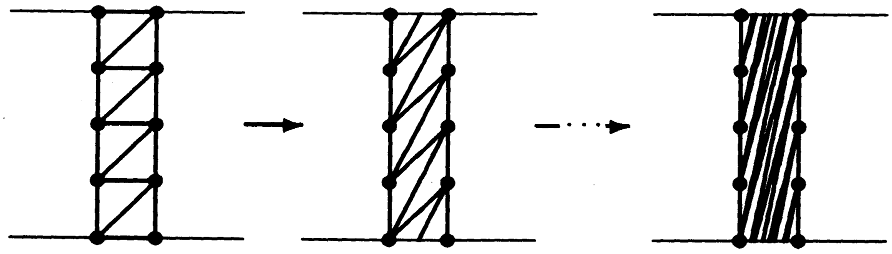

Figure 11. Dehn twist and diagonal flips

THEOREM 27. (Nakamoto and Ota [53]) *Given a closed surface F^2, there exists a natural number $\tilde{N} = \tilde{N}(F^2)$ such that two triangulations G_1 and G_2 are equivalent under diagonal flips, up to isotopy, if $|V(G_1)| = |V(G_2)| \geq \tilde{N}$.*

Let $\Lambda(F^2)$ denote the *mapping class group* or the *homeotopy group* of F^2, which is the group consisting of isotopy classes of auto-homeomorphisms over F^2. For example, $\Lambda(S^2) \cong \boldsymbol{Z}_2$ and any orientation-reversing auto-homeomorphism represents its nontrivial element while $\Lambda(P^2)$ is a trivial group. By Lickorish [44], [45] and [46], it has been shown that all the Dehn twists generate a subgroup $\Lambda_0(F^2)$ in $\Lambda(F^2)$ of index 2 and that $\Lambda_0(F^2)$ is finitely generated. Humphries [33] has determined a finite set of generators of $\Lambda_0(F^2)$ for each orientable closed surface, while Chillingworth [21] has done the same for each nonorientable one. To generate the whole of $\Lambda(F^2)$, we need an arbitrarily chosen orientation-reversing auto-homeomorphism for an orientable one and another kind of an auto-homeomorphism, called a *Y-homeomorphism*, for a nonorientable one.

Nakamoto and Ota [53] have shown that if a triangulation G can be transformed into one which contains a subgraph consisting of "zigzag ladders" placed along Humphries' or Chillingworth's generators, then G can be transformed into $h(G)$ by a sequence of diagonal flips for any auto-homeomorphism $h : F^2 \to F^2$ which belongs to $\Lambda_0(F^2)$. Not only the finiteness of irreducible triangulations but also that of the index of $\Lambda_0(F^2)$ in $\Lambda(F^2)$ play essential roles in their proof of Theorem 27.

13. Splitting-closed classes

Now consider a coditional version of Theorem 15. That is, we would like to restrict the class of triangulations on a closed surface to the class consisting of only triangulations with some property. Burunet, Nakamoto and Negami have already formulated this problem abstractly, as follows.

A class $\mathcal{P}$ of triangulations of F^2 is said to be *splitting-closed* if it is closed under vertex splittings. Any member of $\mathcal{P}$ is called a *$\mathcal{P}$-triangulation* (or a triangulation with property $\mathcal{P}$). An edge e of a $\mathcal{P}$-triangulation G is said to be *$\mathcal{P}$-contractible* if e is contractible and if $G/e \in \mathcal{P}$. A $\mathcal{P}$-triangulation G is *$\mathcal{P}$-irreducible* if no edge of G is $\mathcal{P}$-contractible. A *$\mathcal{P}$-diagonal flip* in a $\mathcal{P}$-triangulation G is a diagonal flip such that the resulting graph is also a $\mathcal{P}$-triangulation. Two triangulations G_1 and G_2 are said to be *$\mathcal{P}$-equivalent* under diagonal flips if they can be transformed into each other by a finite sequence of $\mathcal{P}$-diagonal flips.

THEOREM 28. (Brunet, Nakamoto and Negami [18]) *For any closed surface F^2 and for any splitting-closed class $\mathcal{P}$ of triangulations of F^2, there exists a natural number $N_{\mathcal{P}}(F^2)$ such that if G_1 and G_2 are two $\mathcal{P}$-triangulations with $|V(G_1)| = |V(G_2)| \geq N_{\mathcal{P}}(F^2)$, then G_1 and G_2 are $\mathcal{P}$-equivalent, up to homeomorphism.*

A class $\mathcal{P}$ of triangulations of a closed surface F^2 is said to be *closed under homeomorphism* if $h(G) \in \mathcal{P}$ for any member $G \in \mathcal{P}$ and for any homeomorphism $h : F^2 \to F^2$.

THEOREM 29. (Brunet, Nakamoto and Negami [18]) *For any closed surface F^2 and for any splitting-closed class $\mathcal{P}$ of triangulations of F^2 which is closed under homeomorphism, there exists a natural number $M_{\mathcal{P}}(F^2)$ such that if G_1 and G_2 are two $\mathcal{P}$-triangulations with $|V(G_1)| = |V(G_2)| \geq M_{\mathcal{P}}(F^2)$, then G_1 and G_2 are $\mathcal{P}$-equivalent, up to isotopy.*

Although these theorems are quite abstract, we can create many theorems, assigning concrete classes to $\mathcal{P}$. For example, the classes of triangulations with each of the following properties are splitting-closed and closed under homeomorphism, and hence they can be used as $\mathcal{P}$ in Theorem 29.

- Being k-representative.
- Intersecting any non-separating simple closed curve in at least k points.
- Containing at least k disjoint homotopic cycles.
- Containing at least k disjoint cycles.
- Containing k distinct spanning trees.
- Having looseness at least k. (It follows from Lemma 16.)

In general, a graph G embedded in a closed surface F^2 is said to be *k-representative* if G intersects any essential simple closed curve in at least k points. The minimum value k such that G is k-representative is called the *representativity* of G and is usually denoted by $\rho(G)$. (These definitions can be found in [78].) The second property in the above is a restrictive form of the representativity.

Consider the property that G contains k disjoint cycles homotopic to a fixed essential simple closed curve γ on a closed surface F^2. It is clear that this property is not closed under homeomorphism. However, we don't know whether or not the theorem with "up to isotopy" holds actually for this property.

QUESTION 7. *Is there a splitting-closed class of triangulations of a closed surface that includes a pair of triangulations with the same but arbitrarily large number of vertices which are not equivalent under diagonal flips, up to isotopy?*

Let $\mathcal{P}_d(F^2)$ be tha class of triangulations on a closed surface F^2 with minimum degree at least d. The class $\mathcal{P}_3(F^2)$ coincides with the class consisting of all the triangulations on F^2 while $\mathcal{P}_d(F^2)$ is not however splitting-closed for any $d \geq 4$. Although we cannot apply the result given in this section to these classes, we shall show a good theorem for them in Section 17.

14. Crossing numbers of embedding pairs

Recall that we embed two irreducible triangulations T_i and T_j together on a closed surface F^2 to make a common refinement of them on F^2 in the proof of Theorem 15. If they have few crossing points on their edges, then the order of their refinement will be small. This leads us to an idea of a new invariant which should be called the *crossing number* of two graphs.

We define the crossing number of two graphs in three phases as follows. Let G_1 and G_2 be two graphs embedded separately on closed surfaces F_1^2 and F_2^2, both of which is homeomorphic to a common closed surface F^2. Consider homeomorphisms $h_1 : F_1^2 \to F^2$ and $h_2 : F_2^2 \to F^2$ and count the crossing points of $h_1(G_1)$ and $h_2(G_2)$ embedded on F^2.

- The *crossing number* $\mathrm{cr}(G_1, G_2)$: This is defined as the minimum number of crossing points when h_1 and h_2 range over those homeomorphisms such that $h_1(G_1)$ and $h_2(G_2)$ intersect each other only in their edges transversely.
- The *oriented crossing number* $\mathrm{cr}_+(G_1, G_2)$: This is defined for oriented surfaces. Fix the orientations over F^2, F_1^2 and F_2^2 and consider only orientation-preserving homeomorphisms as h_1 and h_2 with the same condition as above.
- The *diagonal crossing number* $\mathrm{cr}_\Delta(G_1, G_2)$: This is the minimum number of crossing points evaluated under the following conditions.
  - (i) Any vertex does not lie on the interior of edges.
  - (ii) A pair of edges coincide fully or cross each other in a finite number of points transversely with or without common ends if they intersect.

By the definition, we have the following inequality:

$$\mathrm{cr}_\Delta(G_1, G_2) \ \leq \ \mathrm{cr}(G_1, G_2) \ \leq \ \mathrm{cr}_+(G_1, G_2)$$

In particular, if G_1 and G_2 are embedded on the sphere, then $\mathrm{cr}(G_1, G_2) = 0$ necessarily, since they can be embedded together disjointly on the sphere. It is clear that $\mathrm{cr}(G, G^*) \leq |E(G)|$ for any 2-cell embedding G and its dual G^*, but the equality will not hold in most cases. What is $\mathrm{cr}(G, G)$? Don't be confused with the crossing number of a single graph, which is the minimum number of crossing points in its drawings on the plane.

These notions have been defined first by Negami in [67]. He has given a general upper bound for the crossing number of embedding pairs, as follows. We denote the *Betti number* or the *cycle rank* of a graph G by $\beta(G)$, and $\beta(G) = |E(G)| - |V(G)| + 1$ if G is connected.

THEOREM 30. (Negami [67]) *Let G_1 and G_2 be two graphs embedded on a closed surface of genus g, orientable or nonorientable. Then we have:*

$$\mathrm{cr}(G_1, G_2) < 4g \cdot \beta(G_1) \cdot \beta(G_2)$$

Following his proof in [67], we can construct an embedding pair $\{G_1, G_2\}$ so that each pair of edges contains at most $4g$ crossings if $g \geq 2$. In case of the torus, we can carry out more accurate arguments and conclude that:

$$\mathrm{cr}(G_1, G_2) \leq \frac{2}{3}|\beta(G_1)| \cdot |\beta(G_2)|$$

The upper bound given in Theorem 30 contains a factor depending on the surface F^2. This is enough natural, but Negami has established an upper bound for $\mathrm{cr}(G_1, G_2)$ independent of F^2, as follows.

THEOREM 31. (Negami [67]) *Given a closed surface F^2, there is a natural number $R = R(F^2)$ such that if G_1 and G_2 are R-representative graphs on F^2, then $\mathrm{cr}(G_1, G_2) < |E(G_1)| \cdot |E(G_2)|$.*

This theorem may be said to be an application of the well-known powerful theorem called "Graph Minor Theorem", which state that:

THEOREM 32. (Graph Minor Theorem [76]) *Given an embedding H, there exists a natural number R_H such that $H \leq_m G$ for any embedding G with $\rho(G) \geq R_H$.*

Choose a 3-regular graph H 2-cell embedded on a closed surface F^2 with only one face. If both G_1 and G_2 have sufficiently large representativity, then each of G_1 and the dual G_2^* of G_2 includes a subdivision of H as its subembedding. Embed G_1 and G_2 together on F^2 so that the subdivisions of H in G_1 and in G_2^* coincide with each other. Then each pair of their edges will contain at most one crossing as well as any pair of line segments on the plane does. This is a sketch of Negami's proof of Theorem 31.

Define the *edge-representativity* $\rho_e(G)$ of a graph G embedded on a closed surface F^2 as the minimum value of $|G \cap \gamma|$ taken over essential simple closed curves γ on F^2 which do not meet any vertex of G. Then we have $\rho(G) \leq \rho_e(G)$ in general. Archdeacon and Bonnington [3] have established the exact formula for $\mathrm{cr}(G_1, G_2)$ for any pair of graphs G_1 and G_2 embedded on the projective plane, using this invariant. Their proof is very nice and they also have discussed about $\mathrm{cr}(G_1, G_2)$ and $\mathrm{cr}_+(G_1, G_2)$ for pairs of graphs on the torus in [3].

THEOREM 33. (Archdeacon and Bonnington [3]) *For any two graphs G_1 and G_2 embedded on the projective plane, we have:*

$$cr(G_1, G_2) = \rho_e(G_1) \cdot \rho_e(G_2)$$

Theorem 31 might give a supporting evidence for the positive answer to the following question, but most of people will conjecture the negative answer to it.

QUESTION 8. *Is there a costant c with $\mathrm{cr}(G_1, G_2) \leq c \cdot |E(G_1)| \cdot |E(G_2)|$, independent of the surface?*

The diagonal crossing number $\mathrm{cr}_\Delta(G_1, G_2)$ might not be so interesting for general graphs, but it is important for our arguments on diagonal flips in triangulations. For example, conider the irreducible triangulations of the torus. Each of those has a rectangle presentation, as given in [40]. We can pile each pair of such rectangles so that they have many edges in common and very few crossings. Such a picture suggests how to transform one to the other by diagonal flips. However, it seems difficult to establish a general theory to analyze the diagonal crossing number.

QUESTION 9. *Find a reasonable upper bound for $\mathrm{cr}_\Delta(G_1, G_2)$, applicable to any pair of triangulations G_1 and G_2 on a closed surface.*

15. Bounding $N(F^2)$

Combining the arguments on the crossing number in the previous section with those in our proof of Theorem 15, we can give a theoretical upper bound for $N(F^2)$.

THEOREM 34. (Negami [70]) *Let F^2 be a closed surface of geuns g, orientable or nonorientable. Then:*

$$N(F^2) = O(g^3)$$

Proof. Let T_i and T_j be any two irreducible triangulations of F^2. We can embed T_i and T_j together on F^2 with $\mathrm{cr}(T_i, T_j)$ crossing points on edges and construct their common refinement T_{ij}, adding new edges to $T_i \cup T_j$. By Theorem 30, we have:

$$\begin{aligned} |V(T_{ij})| &\leq |V(T_i)| + |V(T_j)| + \mathrm{cr}(T_i, T_j) \\ &\leq |V(T_i)| + |V(T_j)| + 4g \cdot |E(T_i)| \cdot |E(T_j)| \end{aligned}$$

Then, T_{ij} is equivalent to both $T_i + \Delta_{m_{ij}}$ and $T_j + \Delta_{m_{ji}}$ for suitable numbers m_{ij} and m_{ji} by Lemma 5 and Trick 4.

Following our proof of Theorem 15, we can show that $N(F^2)$ does not exceed $\max\{|V(T_{ij})|\}$ and this maximum is bounded by a cubic function of g since both $|V(T_i)|$ and $|E(T_i)|$ are bounded by a linear function of g for any irreducible triangulation T_i of F^2 by Theorem 10. Thus, we have $N(F^2) = O(g^3)$. ■

Evaluating the cubic function which bounds $N(F^2)$ with the above argument, we will obtain the following inequality for the orientable closed surface F^2 of genus g:

$$N(F^2) \leq 4,260,096g^3 - 1,832,832g^2 + 197,820g - 144$$

This is however so far from the truth.

Negami [71] has already improved the order of an upper bound for $N(F^2)$ to be linear, using Theorem 14. Recall that $\tilde{N}(F^2)$ is the quantity which has appeared in Theorem 27, which is the isotopy version of Theorem 15, and we have $N(F^2) \leq \tilde{N}(F^2)$ in general.

THEOREM 35. (Negami [71]) *Let $V_{\mathrm{pse}}(F^2)$ denote the maximum number of vertices taken over all the pseudo-minimal triangulations of a closed surface F^2 with Euler characteristic $\chi(F^2)$. Then we have:*

$$\tilde{N}(F^2) \leq 19\, V_{\mathrm{pse}}(F^2) - 18\, \chi(F^2)$$

16. Distance between triangulations

How many diagonal flips do we need to transform two given triangulations into each other? This question is closely related to an algorithm which generates a sequence of diagonal flips from one to the other. For example, our proof of Wagner's theorem suggests one of such algorithms which transforms a given triangulation on the sphere into the standard form. The algorithm decreases the degree of u by $\deg u - 3$ diagonal flips and hence it will generate at most $\sum_{i=4}^{n}(i-3) = \frac{1}{2}(n-3)(n-2)$ diagonal flips for a given triangulation on the sphere with n vertices. For example, put the standard form Δ_{n-3} after rotating it so that $v \to u$, $u \to w$ and $w \to v$ and carry out the algorithm for it. This will attain the upper bound.

Komuro [38] has already improve this algorithm, introducing the following invariant for triangulations on the sphere. Let G be a spherical triangulation embedded on the plane so that a triangle uvw bounds it, as like the standard form Δ_{n-3} in Figure 2. Consider $3\deg v + \deg w$. This value is equal to $4n-4$ for Δ_{n-3} and is the largest among the triangulations on the sphere with n vertices. Roughly speaking, his algorithm generates a sequence of diagonal flips so that one diagonal flip increases $3\deg v + \deg w$ by one.

LEMMA 36. (Komuro [38]) *Any triangulation on the sphere with n vertices bounded by a triangle uvw can be transformed into the standard form* Δ_{n-3} *by* $4n - 4 - (3\deg v + \deg w)$ *diagonal flips, up to isotopy.*

Finding a pair of vertices v and w so as to maximize $3\deg v + \deg w$, he has established the following theorem.

THEOREM 37. (Komuro [38]) *Any two triangulations with n vertices on the sphere can be transformed into each other, up to isotopy, by at most* $8n - 54$ *diagonal flips if* $n \geq 13$ *and by at most* $8n - 48$ *diagonal flips if* $n \geq 7$.

Gao, Urrutia and Wang [30] also have proved the similar theorem for labeled triangulations on the sphere, considering an algorithm to sort the labels of vertices with a binary-tree method. However, their theorem specifies only the order of the numebr of diagonal flips with respect to the number of vertices.

THEOREM 38. (Gao, Urrutia and Wang [30]) *Any two labeled triangulations with n vertices on the sphere can be transformed into each other, up to isotopy, by* $O(n \log n)$ *diagonal flips.*

It is not so difficult to construct a pair of triangulations on the sphere with n vertices such that we need at least $O(n)$ diagonal flips to transform them into each other. For example, Komuro [38] has given a series of triangulations G_n such that at least $n-7$ diagonal flips are needed to transform G_n into Δ_{n-3}, considering the difference of their degree sequences and the effect of diagonal flips to them. Gao, Urrutia and Wang [30] have conjectured that there exist pairs of triangulations which guarantee the best possibility of their theorem, but never constructed those yet.

For general closed surfaces, Negami has evaluated the length of a sequence of diagonal flips which Theorem 15 guarantees. Let G_1 and G_2 be two triangulations on a closed surface F^2. Define the *distance* $d(G_1, G_2)$ as the minimum number of diagonal flips which we need to transform G_1 into G_2, up to homeomorphism.

THEOREM 39. (Negmai [70]) *Given a closed surafce F^2, there are two constants α_1 and α_0, depending only on F^2, such that*

$$d(G_1, G_2) \leq 2n^2 + \alpha_1 n + \alpha_0$$

for any pair of triangulations G_1 and G_2 on F^2 with precisely n vertices.

Note that if we defined $d(G_1, G_2)$ up to isotopy, then there would not exist an upper bound for $d(G_1, G_2)$. Consider any triangulation G_1 on F^2 and another one G_2 which is obtained from G by applying Dehn twists. They are equivalent to each other, up to homeomorphism. If they have enough many vertices, then they are equivalent under diagonal flips, up to isotopy, by Theorem 27, but we can make $d(G_1, G_2)$ arbitrarily large, applying Dehn twists many times.

17. Minimum degree conditions

Recall the first paragraph in Section 3. The vertices of degree 3 has been playing a very important role in our theory through the previous sections. They can be neglected to obtain a triangulation with fewer vertices, can be moved to elsewhere when it disturbs a diagonal flip we want to do, and carry labels between two vertices in a labeled triangulation to exchange their labels. What happens if we forbid the existence of those?

As is mentioned in Section 13, the class $\mathcal{P}_4(F^2)$, consisting of triangulations with minimum degree at least 4, is not splitting-closed and hence we cannot use the theory given in that section. However, Komuro, Nakamoto and Negami have proved the following theorem.

THEOREM 40. (Komuro, Nakamoto and Negami [39]) *For any closed surface F^2 except the sphere, there exists a natural number $N_4(F^2)$ such that two triangulations G_1 and G_2 on F^2 with minimum degree at least 4 can be transformed into each other by a finite sequence of diagonal flips, up to homeomorphism, through those triangulations if $|V(G_1)| = |V(G_2)| \geq N_4(F^2)$.*

The point of their proof is to distinguish two kinds of vertices of degree 4. Let G be a triangulation on a closed surface F^2 with minimum degree 4 and v a vertex of degree 4 in G with $lk(v) = abcd$. The vertex v is said to be *inserted* if either $(G - v) + ac$ or $(G - v) + bd$ is a triangulation on F^2. Any inserted vertex of degree 4 can be moved freely over a triangulation while a vertex not inserted cannot be. Thus, the inserted vertices of degree 4 play the same role as the vertices of degree 3 do in the previous. According to their theory, we can show the following equalities:

$$N_4(P^2) = 6, \quad N_4(T^2) = 7 \quad N_4(K^2) = 8$$

It should be noticed that the sphere is excepted from Theorem 40. Actually, the theorem does not hold for the sphere. For example, consider $C_n + \overline{K}_2$ embedded on the sphere; there is a cycle C_n of length n along its equator and two vertices corresponding to $\overline{K}_2$ are placed at the north pole and the south pole so that they are adjacent to all the vertices along the equator. We call this triangulation a *double wheel* with *rim* C_n. The octahedron can be regarded as a double wheel with rim C_4. Since each edge of a double wheel is incident to a vertex of degree 4, any diagonal flip cannot be applied to it without producing vertices of degree 3. Thus, a double wheel cannot be transformed into any other triangulation on the sphere with the degree condition. However, they have shown that the double wheels are isolated exceptions for the theorem on the sphere, as follows.

THEOREM 41. (Komuro, Nakamoto and Negami [39]) *Two triangulations on the sphere, except the double wheels, with minimum degree at least 4 can be transformed into each other, up to isotopy, by a finite sequence of diagonal flips through those triangulations if they have the same number of vertices.*

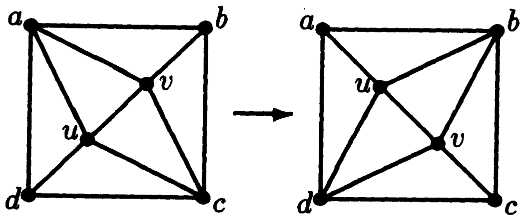

Figure 12. Rhombus twist

We need another kind of deformations, other than diagonal flips, to transform the double wheels into other triangulations on the sphere. Let G be a triangulation on a closed surface and $abcd$ a cycle of length 4 in G bounding a quadrilateral region R which is not a face of G. Suppose that R contains only a pair of adjacent vertices u and v and that u is adjacent to a, c, d and v to a, b, c. The *rhombus twist* is to deform the inside of R so that u is adjacent to a, b, d and v to b, c, d afterward, as shown in Figure 12. We do not perform this deformation to keep the minimum degree at least 4 unless $\deg a \geq 5$ and $\deg c \geq 5$ in G.

THEOREM 42. (Komuro, Nakamoto and Negami [39]) *Two triangulations on the sphere with minimum degree at least* 4 *can be transformed into each other, up to isotopy, by a finite sequence of diagonal flips and rhombus twists through those triangulations if they have the same number of vertices.*

We can define $N_d(F^2)$ just formally as the minimum number N such that two triangulations G_1 and G_2 on F^2 with minimum degree at least d can be transformed into each other, up to homeomorsphim, by a finite sequence of diagonal flips through those triangulations if $|V(G_1)| = |V(G_2)| \geq N$. However, $N_d(F^2)$ for $d \geq 7$ is not so meaningful since there are only finitely many graphs G embedded on F^2 with $\delta(G) \geq 7$. (It follows from Euler's formula that $|V(G)| \leq 6|\chi(F^2)|$ for such a graph G.) On the other hand, $N_d(F^2)$ connot be a finite constant for $d = 5$ and 6, as shown below.

A triangulation G is said to be *d-covered* if each edge is incident to a vertex of degree d. If G is d-covered with $\delta(G) = d$, then flipping any edge decreases the degrees of its both ends and results in a triangulation with minimum degree $d - 1$. Thus, we cannot transform G into any other triangulation, keeping their minimum degree at least d. It is not difficult to construct an infinite series of 5-covered triangulations on each closed surface F^2 and that of 6-covered ones on F^2 with $\chi(F^2) \leq 0$. This denies the existence of $N_d(F^2)$ for $d = 5$ and 6.

Furthermore, Nakamoto and Negami [54] have shown a constructive characterization of 5-covered and of 6-covered triangulations, and discussed on an upper bound for such an integer d that a closed surface F^2 admits a d-covered triangulation. For example, if $\chi(F^2) \leq 0$, then we have

$$d \leq 2\left\lfloor \frac{5 + \sqrt{49 - 24\chi(F^2)}}{2} \right\rfloor$$

and there are only finitely many d-covered triangulations, up to homeomorphism, with any positive integer $d \geq 13$. See [54] for details.

18. Frozen triangulations

Any d-covered triangulation, discussed in the previous section, does not admit any diagonal with the minimum degree condition. Here, we shall consider a similar property for usual triangulations.

A triangulation is said to be *frozen* if any diagonal flip is not applicable to it, that is, if any diagonal flip yields a pair of multiple edges. Thus, a triangulation G is frozen if and only if the four vertices lying on the quadrilateral with e as its diagonal induces K_4 for any edge $e \in E(G)$.

For example, any complete triangulation is frozen. Is there a fozen triangulation which is not complete? The answer to this question is "Yes". We can construct such a frozen triangulation, as follows.

Let $K_{n(m)}$ denote the complete n-partitile graph with partite sets of the same size m, that is, $K_{m,\dots,m}$ with n m's. The following theorem is one of corollaries of Theorem 7.5 in [2], and suggests a covering construction of a frozen triangulation.

THEOREM 43. (Archdeacon [2]) *If the complete graph K_n triangulates a closed surface F^2 and if each prime factor of m is at least $n-1$ except the case of $n = 4$, $m = 3$, then $K_{n(m)}$ triangulates another surface.*

Following Archdeacon's method in [2], we can construct the triangulation with $K_{n(m)}$ on a closed surface $\tilde{F}^2$ as a *wrapped covering* of K_n embedded on F^2. That is, there is an m^2-fold branched covering $p : \tilde{F}^2 \to F^2$, only branched over $V(K_n)$, such that the neighborhood of each vertex $v \in V(K_{n(m)})$ wraps that of $p(v) \in V(K_n)$ cyclically. (See [2] and [35] for the precise definition of a wrapped covering.) If F^2 is non-orientable, then $\tilde{F}^2$ is non-orientable, too, since any feasible m in the theorem is odd for $n \geq 4$.

This structure guarantees that if $abcd$ is a quadrilateral with ac its diagonal in $K_{n(m)}$, then b and d belong to two different partite sets of $K_{n(m)}$ for $n \geq 4$. Thus, b and d are joined with an edge and hence ac cannot be flipped. This implies that $K_{n(m)}$ is a frozen triangulation of $\tilde{F}^2$. Of cource, $K_{n(m)}$ is not complete unless $m = 1$.

For example, since K_4 triangulates the sphere, $K_{4(m)}$ triangulates an orientable closed surface with any odd integer $m \neq 3$ and each of the triangulations with $K_{4(m)}$ is frozen. To construct a non-complete frozen triangulations on non-orientable closed surfaces, we can use $K_{6(m)}$ with m not divisible by 2 and 3 since K_6 triangulates the projective plane. (We can find other constructions of triangulations with $K_{n(m)}$ in [14], [15] and [16].)

This discovery of frozen triangulations gives the negative answer to Question 1 in Section 3. Any frozen triangulation is pseudo-minimal, by their definition. It is not difficult to show that the frozen triangulations $K_{4(m)}$ and $K_{6(m)}$ are not minimal, evaluating the number of their vertices. It is clear that if F^2 admits a frozen triangulation G, then $|V(G)| < N(F^2)$. These imply the following theorems.

THEOREM 44. (Negami [72]) *Let F^2 be the orientable closed surface of genus g. If g is an even square number more than 4, then we have:*

$$V_{\min}(F^2) < 4 + 4\sqrt{g} < N(F^2)$$

THEOREM 45. (Negami [72]) *Let F^2 be the non-orientable closed surface of genus q. If $5q-1$ is an even square number and if $q \not\equiv 2 \bmod 3$, then we have:*

$$V_{\min}(F^2) < 18 + 6\sqrt{5q-1} < N(F^2)$$

See [72] for details. In that paper, Negami has discussed on the relationship among many properties of triangulations; namely complete, minimal, pseudo-minimal, irreducible and frozen.

19. Pseudo-triangulations

A triangulation on a closed surface is a simple graph, that is, a graph without loops and multiple edges. This restriction on the simpleness of triangulations yields some difficulity in our theory on diagonal flips. What happens if we allowed loops or multiple edges in triangulations. One might expect that everything becomes easier. In fact, we can establish a good theory on such unusual triangulations, as follows.

We call a graph G on a closed surface F^2 a *pseudo-triangulation* on F^2 if each face is just three-edged. It may have loops or multiple edges. To avoid the reader's confusion, we shall call a usual triangulation a *simple triangulation* in this section.

The theorem corresponding to Theorem 15 holds for pseudo-triangulations, too, but with no restriction on the number of vertices. That is, $N(F^2) = 1$ for all of closed surfaces F^2. Moreover, we can evaluate the distance between two given pseudo-triangulations, using a kind of crossing numbers different from those given in Section 14.

Let G_1 and G_2 be two labeled pseudo-triagulations on a closed surface F^2. The *diagonal crossing number* $\mathrm{cr}_\nabla(G_1, G_2)$ of G_1 and G_2 *under vertex coincidence* is defined as the minimum number of crossing points on edges counted in $h(G_1) \cup G_2$ for all the homeomorphisms $h : F^2 \to F^2$ such that:

(i) The homeomorphism h induces the label-preserving bijection between $V(G_1)$ and $V(G_2)$.

(ii) Each pair of edges $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$ either coincide with each other ($h(e_1) = e_2$) or cross each other transversely in a finte number of points, via h.

We also define $\widetilde{\mathrm{cr}}_\nabla(G_1, G_2)$, requiring the homeomorphisms h to be isotopic to the identity map over F^2 in addition. When G_1 and G_2 are regarded as unlabeled graphs, the homeomorphisms h should be ones such that $h(V(G_1)) = V(G_2)$, instead of the first condition. We denote the diagonal crossing number in this unlabeled sense by $\overline{\mathrm{cr}}_\nabla(G_1, G_2)$, and we have:

$$\overline{\mathrm{cr}}_\nabla(G_1, G_2) \leq \mathrm{cr}_\nabla(G_1, G_2) \leq \widetilde{\mathrm{cr}}_\nabla(G_1, G_2)$$

THEOREM 46. (Negami [71]) *Let G_1 and G_2 be two labeled pseudo-triangulations on a closed surface F^2 with the same number of vertices. Then they can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips of length at most* $\mathrm{cr}_\nabla(G_1, G_2)$.

The unlabeled verion and the isotopy verion of this theorem also hold if $\mathrm{cr}_\nabla(G_1, G_2)$ is replaced with $\overline{\mathrm{cr}}_\nabla(G_1, G_2)$ and $\widetilde{\mathrm{cr}}_\nabla(G_1, G_2)$, respectively. The proof given in [71] presents a "greedy" algorithm to transform G_1 into G_2 by diagonal flips in a sense, decreasing $\mathrm{cr}_\nabla(G_1, G_2)$.

After establishing this theorem, Negami [71] has proved Theorems 14 and 35, as follows. Let G_1 and G_2 be two simple triangulations on a closed surface F^2. Since they are also pseudo-triangulations, they can be transformed into each other by a sequence of diagonal flips, by Theorem 46, although such a sequence may include non-simple triangulations. Make a refinement of each pseudo-triangulation in the sequence in a well-mannered way so that it becomes a simple triangulation, and join each consecutive pair of those refinements by a sequence of diagonal flips. In his method, such a refinement can be obtained from its original G by adding $18(|V(G)| - \chi(F^2))$ extra vertices. This idea has motivated Nakamoto to prove Theorem 26.

Now we shall forbit only loops. In fact, we can establish a beautiful theory on pseudo-triangulations without loops. Basically, the same arguments as for simple triangulations proceed and the vertices of degree 2 play the same role in turn as those of degree 3 in the previous. Thus, a pseduo-triangulation without loops is said to be *pseudo-minimal* if it cannot be transformed into one which includes a vertex of degree 2 by diagonal flips. To prove the following theorem, it suffices, as well as for simple triangulations, to show that any two pseudo-minimal pseudo-triangulations are stable equivalent.

THEOREM 47. (Negami and Watanabe [86]) *Given a closed surface F^2, there exists a natural number $n(F^2)$ such that two pseudo-triangulations G_1 and G_2 on F^2 without loops can be transformed into each other, up to homeomorphism, by a sequence of diagonal flips through those pseudo-triangulations if $|V(G_1)| = |V(G_2)| \geq n(F^2)$.*

The key fact in the theory for pseudo-triangulations without loops is the following lemma, which tells us what is pseudo-minimal. Frozen, minimal and irreducible pseudo-triangulations are defined in the same way as those for simple triangulations. It is not difficult to find irreducible pseudo-triangulations which are not pseudo-minimal.

LEMMA 48. (Negami and Watababe [86]) *For a pseudo-triangulation G on a closed surface without loops, the following four are equivalent to one another:*

(i) *G is frozen.*

(ii) *G is pseudo-minimal.*

(iii) *G is minimal.*

(iv) *G has precisely three vertices.*

Let $n(F^2)$ denote its minimum value, as well as $N(F^2)$. Since we can construct or classify pseudo-minimal ones concretely, we can give a good upper bound for $n(F^2)$, which does not contain any unknown quantity.

THEOREM 49. (Negami and Watanabe [86]) *If a closed surface F^2 is one of the sphere, the projective plane, the torus and the Klein bottle, then $n(F^2) = 3$. Otherwise, we have:*

$$4 \leq n(F^2) \leq 18 - 5\chi(F^2)$$

For example, we can show that there are only two inequivalent pseudo-minimal pseudo-triangulations on the orientable closed surface of genus 2 and that they become equivalent under diagonal flips after adding one vertex of degree 2 to them. Thus, we have $n(F^2) = 4$ for the orientable closed surface F^2 of genus 2.

QUESTION 10. *Determine the precise value of $n(F^2)$ for each closed surfaces F^2 with $\chi(F^2) < 0$.* (Conjecture: $n(F^2) = 4$ for all.)

20. Quadrangulations

A *quadrangulation* on a closed surface F^2 is a simple graph embedded on F^2 so that each face is bounded by a cycle of length 4. In particular, a bipartite quadrangulation appears as what is called the *radial graph* $R(G)$, associated with a general graph G embedded on a closed surface F^2. (Put a vertex at the center of each face of G and join it to all the vertices lying on the boundary of the face. Delete the edges of G. the resulting graph is $R(G)$ and is a bipartite graph with partite sets $V(G)$ and $V(G^*)$, where G^* is the dual of G. It is clear that $R(G) = R(G^*)$ and hence $R(G)$ is useful to investigate both the primal G and its dual G^* together. See [2], [4] and [5] for the usage of radial graphs. The *medial graph* $M(G)$ is the dual of $R(G)$.) Note that any quadrangulation on the sphere is bipartite and there are non-bipartite quadrangulations on other closed surfaces. Let $V_B(G)$ and $V_W(G)$ denote the two partite sets of a bipartite quadrangulation G, black and white, with $V(G) = V_B(G) \cup V_W(G)$.

There have been many studies on quadrangulations; [1], [48], [49], [52], [53] and so on. In particular, Nakamoto's thesis [51] includes most of those. Both theories for triangulations and for quadrangulations have been devepoled together and some of their arguments proceed actually in parallel. There are however some special phenomena for only quadrangulations, which we shall describe here.

We need two operations for quadrangulations, called a *diagonal slide* and a *diagonal rotation*, corresponding to a diagonal flip for triangulations. The diagonal slide just slides a diagonal in a hexgonal region while the diagonal rotation rotates two edges around a vertex of degree 2, as shown in the left and right of Figure 13, respectively. The following theorem gives us the starting point for the studies on quadrangulations, and is one corresponding to Theorem 15 for triangulations.

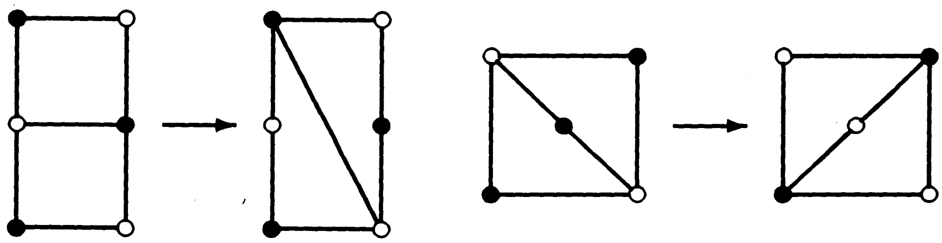

Figure 13. Diagonal slide and diagonal rotation

THEOREM 50. (Nakamoto [48]) *For any closed surface F^2, there is a natural number $Q(F^2)$ such that two bipartite quadrangulations G_1 and G_2 on F^2 can be transformed into each other, up to homeomorphism, by a sequence of diagonal slides and diagonal rotations if $|V(G_1)| = |V(G_2)| \geq Q(F^2)$.*

It should be noticed that both a diagonal slide and a diagonal rotation preserve the bipartiteness of quadrangulations, as Figure 13 suggests. Thus, a bipartite quadrangulation and a non-bipartite one cannot be transformed into each other by these operations. Also, two bipartite quadrangulations G_1 and G_2 cannot be transformed into each other by only diagonal slides if $|V_B(G)| \neq |V_B(G)|$ or if $|V_W(G)| \neq |V_W(G)|$. The diagonal rotation changes the size of $V_B(G)$ and of $V_W(G)$ and is needed actually to establish the theorem for bipartite quadrangulations.

To establish another theorem which covers all quadrangulations, we shall prepare a topological invariant, called a *cycle parity* of G. Let G be a quadrangulation on a closed surface F^2. It is easy to see that the lengths of two cycles in G have the same parity, even or odd, if they are homotopic on F^2. Thus, we can define a homomorphism $\sigma : \pi(F^2) \to \boldsymbol{Z}_2$ so that $\sigma([C]) = 0$ (or 1) if the length of a cycle C in G is even (or odd), where $[C]$ stands for the homotopy class including C. This homomorphism σ can be represented uniquely as the composition of the canonical projection $\pi(F^2) \to H_2(F^2, \boldsymbol{Z}_2)$ and a homomorphism $\sigma_* : H_2(F^2, \boldsymbol{Z}_2) \to \boldsymbol{Z}_2$ and σ_* can be regarded as an element in the $\boldsymbol{Z}_2$-cohomology group $H^2(F^2, \boldsymbol{Z}_2)$. The cycle parity of G on F^2 is defined as this $\sigma_* = \sigma_*(G) \in H^2(F^2, \boldsymbol{Z}_2)$. Two cycle parities σ_* and $\sigma'_* \in H^2(F^2, \boldsymbol{Z}_2)$ are said to be *congruent* if there is a homeomorphism $h : F^2 \to F^2$ such that $h^*(\sigma_*) = \sigma'_*$, where h^* stands for the homomorphism $H^2(F^2, \boldsymbol{Z}_2) \to H^2(F^2, \boldsymbol{Z}_2)$ induced by h.

It is easy to see that a diagonal slide and a diagonal rotation preserve the cycle parity. Thus, if $\sigma_*(G_1) \neq \sigma_*(G_2)$, then those quadrangulations G_1 and G_2 cannot be transformed into each other by diagonal flips and diagonal rotations. For example, a quadrangulation G is bipartite if and only if $\sigma_*(G) = 0$. Thus, the following theorem contains the previous.

THEOREM 51. (Nakamoto [49]) *For any closed surface F^2, there is a natural number $Q'(F^2)$ such that two quadrangulations G_1 and G_2 on F^2 with $|V(G_1)| = |V(G_2)| \geq Q'(F^2)$ can be transformed into each other, up to heomeomorphism, by a sequence of diagonal slides and diagonal rotations if their cycle parities are congruent.*

In fact, there are not so many congruence classes of cycle parities. For any orientable closed surface F^2 except the sphere, there are only two congruence classes, trivial or non-trivial. Thus, any non-bipartite quadrangulations G_1 and G_2 on F^2 can be transformed into each other by diagonal slides if $|V(G_1)| = |V(G_2)| \geq Q'(F^2)$. In this case, a diagonal rotation is not needed. Furthermore, Nakamoto and Ota [52] have already established the isotopy verion of these theorems for orietable closed surfaces, discussing on transition of cycle parities by Dehn twists. How about such a theorem for nonorientable closed surface?

21. Conclusion

The theory of diagonal flips in triangulations on surfaces has been developed so much, related to many other notions in topological graph theorey, some of which are new. However, we knows about only abstract phenomena for hyperbolic closed surfaces, that is, ones with negative Euler characteristic and do not know much about the equivalence over triangulations on such surfaces with the number of vertices smaller than $N(F^2)$. We would like to expect further studies to make them clear in near future.

References

[1] K. Ando, H. Komuro and A. Nakamoto, Plane quadrangulations with minimum degree 3, to appear in *Discrete Math.*

[2] D. Archdeacon, The medial graph and voltage-current duality, *Discrete Math.*, **104** (1992) 111–141.

[3] D. Archdeacon and C.P. Bonnington, Two maps on one surface, preprint.

[4] D. Archdeacon and R.B. Richter, The construction and classification of self-dual spherical polyhedra, *J. Combin. Theory, Ser. B*, **54** (1992), 37–63.

[5] D. Archdeacon and S. Negami, The construction of self-dual projective polyhedra, *J. Combin. Theory, Ser. B*, **59** (1993), 122–131.

[6] D. Archdeacon, Self-dual embeddings of complete multipartite graphs, *J. Graph Theory*, **18** (1994), 735-749.

[7] J.L. Arocha, J. Bracho and V. Neumann-Lara, On the minimum size of tight hypergraphs, *J. Graph Theory*, **16** (1992), 319-326.

[8] J.L. Arocha, J. Bracho and V. Neumann-Lara, Tight and untight triangulated surfaces, *J. Combin. Theory, Ser. B*, **63** (1995), 185–199.

[9] B. Aronov, R. Sedel and D. Souvaine, On compatible triangulations of simple polygons, *Computational Geometry: Theory and Applications*, **3** (1993), 27–35.

[10] D. Barnette, Generating the triangulations of the projective plane, *J. Combin. Theory, Ser. B*, **33** (1982), 222-230.

[11] D.W. Barnette and A.L. Edelson, All 2-manifolds have finitely many minimal triangulations, *Isr. J. Math.*, **67** (1989), 123–128.

[12] M. Bern and D. Eppstein, Mesh generation and optimal triangulation, in "Computing in Educlidean Geometry", D.Z. Du and F.K. Hwang, eds., World Scientific, 1992, 23–90.

[13] C.P. Bonnington, M.J. Grannell, T.S. Griggs and J. Širáň, Exponential families of non-isomorphic triangulations of complete graphs, preprint.

[14] A. Bouchet, Constructing a covering triangulation by means of a nowhere-zero flow, *J. Combin. Theory, Ser. B*, **32** (1982), 316–325.

[15] A. Bouchet, Constructions of covering triangulations with folds, *J. Graph Theory*, **6** (1982), 57–74.

[16] A. Bouchet, Triangular imbeddings into surfaces of a join of equicardinal independent sets following an Eulerian graph, *Ann. Discrete Math.*, **17** (1983), 119–130.

[17] J. Bracho and R. Strausz, Nonisomophic complete triangulations, preprint.

[18] R. Brunet, A. Nakamoto and S. Negami, Diagonal flips of triangulations on closed surfaces preserving specified properties, *J. Combin. Theory, Ser. B*, **68** (1996), 295–309.

[19] R. Brunet, A. Nakamoto and S. Negami, Every 5-connected triangulation of the Klein bottle is hamiltonian, *Yokohama Math. J.*, **47**, special issue (1999), 239–244.

[20] B. Chazelle, Triangulating a simple polygon in linear time, *Discrete and Computational Geometry*, **6** (1991), 485–524.

[21] D.R.J. Chillingworth, A finite set of generators for the homeotopy group of a non-orientable surface, *Proc. Camb. Phil. Soc.*, **65** (1969), 409–430.

[22] C. Cortés and A. Nakamoto, Diagonal flips in outer-torus triangulations, to appear in *Discrete Math.*

[23] C. Cortés and A. Nakamoto, Diagonal flips in outer-Klein-bottle triangulations, preprint.

[24] C. Cortés, C. Glima, A. Márquez and A. Nakamoto, Diagonal flips in outer-triangulations of surfaces, preprint.

[25] A.K. Dewdney, Wagner's theorem for the torus graphs, *Discrete Math.*, **4** (1973), 139–149.

[26] P.H. Edelman and V. Reiner, Catalan triangulations of Möbius band, *Graphs and Combin.*, **13** (1997), 231–243.

[27] S. Fortune, Voronoi diagrams and Delaunay triangulations, in "Computing in Educlidean Geometry", D.Z. Du and F.K. Hwang, eds., World Scientific, 1992, 193–234.

[28] Z. Gao, L.B. Richmond and C. Thomassen, Irreducible triangulations and triangular embeddings on a surface, *CORR*, 91-07, University of Waterloo.

[29] Z. Gao, R.B. Richter and P. Seymour, Irreducible triangulations of surfaces, *J. Combin. Theory Ser. B*, **68** (1996), 206–217.

[30] Z. Gao, J. Urrutia and J. Wang, Diagonal flips in labelled planar triangulations, to appear in *Graph and Combin.*

[31] M.J. Grannell, T.S. Griggs and J. Širáň, Face 2-colourable triangular embeddings of complete graphs, *J. Combin. Theory, Ser. B*, **74** (1998), 8–19.

[32] M.J. Grannell, T.S. Griggs and J. Širáň, Surface embeddings of Steiner triple systems, *J. Combin. Designs*, **6** (1998), 325–336.

[33] S.P. Humphries, Generators for the mapping class group, in "Topology of Low-dimentional Manifolds", *Lecture Notes in Mathematics* 722 (1979), Springer, pp. 44–47.

[34] F. Hurt ado, M. Noy and J. Urrutia, Flipping edges on triangulation, in "Proc. 12nd ACM Symposium on Computational Geometry, May 24–26", 1996, 214–223.

[35] B. Jackson, T.D. Parsons and T. Pisanski, A duality theorem for graph embeddings, *J. Graph Theory*, **5** (1981) 55–77.

[36] E. Kranakis and J. Urrutia, Isomorphic triangulations with a small number of Steiner points, to appear in *Int. J. on Computational Geometry.*

[37] M. Jungerman and G. Ringel, Minimal triangulations on orientable surfaces, *Acta Math.*, **145** (1980), 121–154.

[38] H. Komuro, The diagonal flips of triangulations on the sphere, *Yokohama Math. J.*, **44** (1997), 115–122.

[39] H. Komuro, A. Nakamoto and S. Negami, Diagonal flips in triangulations on closed surfaces with minimum degree at least 4, *J. Combin. Theory, Ser. B*, **76** (1999), 68–92.

[40] S. Lawrencenko, The irreducible triangulations of the torus, *Ukrain. Geom. Sb.*, **30** (1987), 52–62. [In Russian; MR 89c:57002; English translation: J. Soviet Math. **51**, No.5 (1990), 2537–2543.]

[41] S. Lawrencenko and S. Negami, Irreducible triangulations of the Klein bottle, *J. Combin. Theory, Ser. B*, **70** (1997), 265–291.

[42] S. Lawrencenko and S. Negami, Constructing the graphs that triangulate both the torus and the Klein bottle, *J. Combin. Theory, Ser. B*, **77** (1999), 211–218.

[43] S. Lawrencenko, S. Negami and A.T. White, Three nonisomorphic triangulations of an orientable surface with the same complete graph, *Discrete Math.*, **135** (1994), 367–369.

[44] W.B.R. Lickorish, Homeomorphism of non-orientable 2-manifolds, *Proc. Camb. Phil. Soc.*, **59** (1963), 307–317.

[45] W.B.R. Lickorish, A finite set of generators for the homeotopy group of a 2-manifold, *Proc. Camb. Phil. Soc.*, **60** (1964), 769–778.

[46] W.B.R. Lickorish, A finite set of generators for the homeotopy group of a 2-manifold (corrigendum), *Proc. Camb. Phil. Soc.*, **62** (1966), 679–681.

[47] A. Malnic and R. Nedela, k-minimal triangulations of surfaces, *Acta Math. Univ. Comenian* (N.S.), **64** (1995), 57–76.
[48] A. Nakamoto, Diagonal transformations in quadrangulations of surfaces, *J. Graph Theory*, **21** (1996), 289–299.
[49] A. Nakamoto, Diagonal transformations and cycle parities of quadrangulations on surfaces, *J. Combin. Theory, Ser. B*, **67** (1996), 202–211.
[50] A. Nakamoto and K. Ota, Note on Irreducible triangulations of surfaces, *J. Graph Theory*, **20** (1995), 227–233.
[51] A. Nakamoto, "Triangulations and quadrangulations of surfaces", *Doctoral thesis*, Keio University, 1996.
[52] A. Nakamoto and K. Ota, Diagonal transformations in quadrangulations and Dehn twists preserving cycle parities, *J. Combin. Theory Ser. B*, **69** (1997), 125–141.
[53] A. Nakamoto and K. Ota, Diagonal transformations of graphs and Dehn twists of surfaces, *J. Combin. Theory, Ser. B*, **70** (1997), 292–300.
[54] A. Nakamoto and S. Negami, Triangulations on closed surfaces covered by vertices of given degree, preprint.
[55] S. Negami, Uniqueness and faithfulness of embedding of toroidal graphs, *Discrete Math.*, **44** (1983), 161–180.
[56] S. Negami, Uniquely and faithfully embeddable projective-planar triangulations, *J. Combin. Theory, Ser. B*, **36** (1984), 189–193.
[57] S. Negami, Classification of 6-regular Klein-bottlal graphs, *Res. Rep. Inf. Sci. T.I.T.*, **A-96** (1984).
[58] S. Negami, Unique and faithful embeddings of projective-planar graphs, *J. Graph Theory*, **9** (1985), 235–243.
[59] S. Negami, Construction of graphs which are not uniquely and not faithfully embeddable in surfaces, *Yokohama Math. J.*, **33** (1985), 67–91.
[60] S. Negami, Uniqueness and faithfulness of embedding of graphs into surfaces, *Doctor thesis,*, Tokyo Institute of Technology, 1985.
[61] S. Negami, Heredity of uniqueness and faithfulness of embedding of graphs into surfaces, *Res. Rep. Inf. Sci. T.I.T.*, **A-103** (1986).
[62] S. Negami, Enumeration of projective-planar embeddings of graphs, *Discrete Math.*, **62** (1986), 299–306.
[63] S. Negami, Re-embedding of projective-planar graphs, *J. Combin. Theory, Ser. B*, **44** (1988), 276–299.
[64] S. Negami and S. Watanabe, Diagonal transformations of triangulations on surfaces, *Tsukuba J. Math.*, **14** (1990), 155–166.
[65] S. Negami and A. Nakamoto, Diagonal transformations of graphs on closed surfaces, *Sci. Rep. Yokohama Nat. Univ., Sec. I*, **40** (1993), 71–97.
[66] S. Negami, Diagonal flips in triangulations of surfaces, *Discrete Math.*, **135** (1994), 225–232.
[67] S. Negami, Crossing numbers of graph embedding pairs on closed surfaces, preprint.
[68] S. Negami and T. Midorikawa, Loosely-tightness of triangulations of closed surfaces, *Sci. Rep. Yokohama Nat. Univ., Sec. I*, **43** (1996), 25–41.
[69] S. Negami, A. Nakamoto and T. Tanuma, Re-embedding structures of triangulations on closed surfaces, *Sci. Rep. Yokohama Nat. Univ., Sec. I*, **44** (1997), 41–55.
[70] S. Negami, Diagonal flips in triangulations on closed surfaces, estimating upper bounds, *Yokohama Math. J.*, **45** (1998), 113–124.
[71] S. Negami, Diagonal flips in pseudo-triangulations on closed surfaces, preprint.
[72] S. Negami, Note on frozen triangulations on closed surfaces, *Yokohama Math. J.*, **47**, special issue (1999), 191–202.

[73] O. Ore, "The four-color problem", Academic Press, New York, 1967.

[74] G. Ringel, Wie man die geschlossenen nichtorientierbaren Flächen in möglichst wenig Dreiecke zerlengen kann, *Math. Annalen*, **130** (1955), 317–326.

[75] G. Ringel, "Map Color Theorem", Springer Verlag, 1974.

[76] N. Robertson and P. Seymour, Graph Minor VII, "Disjoint paths on a surface", *J. Combin. Theory, Ser. B*, **39** (1988), 212–254.

[77] N. Robertson and P. Seymour, Graphs Minors XVI, "Wagner's conjecture", preprint.

[78] N. Robertson and R. Vitray, Representativity of surface embedding, *Paths, Flows, and VLSI-Layout*, ("Algorithm and Combinatorics" Vol. 9, B. Korte, L. Lovász, H.J. Prömel and A. Schrijver, eds), Springer-Verlag, Berlin, (1990), 293–328.

[79] A. Saalfeld, Joint triangulations and triangulated maps, in "Proc. 3rd Annual ACM Symposium in Computational Geometry", (1987), 195–204.

[80] D. Sleator, R. Tarjan and W. Thurston, Rotation distance, triangulations, and hyperbolic geometry, *J. Amer. Math. Soc.*, **1** (1988), 647–681.

[81] E. Steinitz and H. Rademaher, "Vorlesungen über die Theorie der Polyeder", Springer, Berlin, 1934.

[82] R. Strausz, "Triangulaciones de superficies de orden pequeño", B. Sc. thesis, UNAM in process.

[83] T. Tanuma, One-loosely tight triangulations of closed surfaces, *Yokohama Math. J.*, **47**, special issue (1999), 203–211.

[84] T. Tanuma, Looseness of triangulations of the projective plane, preprint.

[85] K. Wagner, Bemekungen zum Vierfarbenproblem, *J. der Deut. Math.*, Ver. **46**, Abt. 1, (1936), 26–32.

[86] T. Watanabe and S. Negami Diagonal flips in pseudo-triangulations on closed surfaces without loops, *Yokohama Math. J.*, **47**, special issue (1999), 213–223.

[87] H. Whitney, Congruent graphs and the connectivity of graphs, *Amer. J. Math.*, **54** (1932), 150–168.

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