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EXPONENTS OF SEMI-SELFSIMILAR PROCESSES

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Abstract. The exponent of a semi-selfsimilar process is shown to exist under the mere assumption of stochastic continuity at $t=0$, and related examples are given. A relationship between long range dependence of the increments and the exponent is also discussed.

1. Introduction

Let $\{X(t), t\geq 0\}$ be an $\mathbf{R}^d$-valued Lévy process in the sense that it is stochastically continuous for $t\geq 0$, $X(0)=0$ a.s. and it has independent stationary increments. If the law of $X(1)$ is strictly semi-stable in the sense that the characteristic function $\hat{\mu}$ of $X(1)$ satisfies $\hat{\mu}(z)^a=\hat{\mu}(bz)$ for some $a\in(0,1)\cup(1,\infty)$ and $b>0$, then the process $\{X(t)\}$ satisfies

$$\{X(at), t\geq 0\} \stackrel{d}{=} \{bX(t), t\geq 0\}, \tag{1.1}$$

where $\stackrel{d}{=}$ denotes the equality in all joint distributions.

Non-Gaussian stable distributions have heavy tails and are widely used in stochastic modeling. Non-Gaussian strictly semi-stable distributions have also heavy tails and since the class of those distributions is much wider than that of non-Gaussian strictly stable distributions, they offer more variety in stochastic modeling.

As strictly stable Lévy processes are extended to selfsimilar processes, strictly semi-stable Lévy processes can be extended to more general processes, namely, to semi-selfsimilar processes. In general, if an $\mathbf{R}^d$-valued process $\{X(t)\}$ satisfies a scaling property (1.1) for some $a\in(0,1)\cup(1,\infty)$ and $b>0$, then we want to call $\{X(t)\}$ semi-selfsimilar.

Selfsimilar processes have contributed a lot to stochastic modeling of phenomena with long range dependence, (see e.g. [2] and [3]). However, selfsimilar processes have to satisfy (1.1) for all $a>0$. Because of the weaker scaling prop-

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erty, semi-selfsimilar processes are expected to offer higher flexibility in stochastic modeling of phenomena with long range dependence.

Semi-selfsimilar (but not selfsimilar) processes in the sense of this paper have already appeared in the literature about diffusions on Sierpinski gaskets, (see [1], [5] and [7]), although such terminology is not used. Namely, if $\{X(t)\}$ is such a diffusion on $\mathbf{R}^d$, then $\{X((d+3)^n t)\} \stackrel{d}{=} \{2^n X(t)\}$ for any $n \in \mathbf{Z}$. So the semi-selfsimilarity might also be an important notion in mathematical physics.

In [9], several topics on semi-selfsimilar processes have been studied, and among them, there is the existence of exponents of semi-selfsimilarity. In this paper, we shall discuss again exponents of semi-selfsimilarity and show that stochastic continuity of semi-selfsimilar processes at the origin is delicately related to the existence of exponents of semi-selfsimilarity, which has a sharp contrast to selfsimilar processes. We also show when increments of semi-selfsimilar processes have long range dependence in terms of value of their exponents.

2. Examples

If b in (1.1) can be expressed as $b = a^H$ for a unique $H \geq 0$, then we call H the exponent of a selfsimilar or semi-selfsimilar process $\{X(t), t \geq 0\}$. Then we may say that $\{X(t)\}$ is H-selfsimilar or H-semi-selfsimilar. In this section, we give several examples of semi-selfsimilar processes to explain our motivation of this paper.

Throughout the examples below, let $\{Y(t), t \geq 0\}$ be an H-selfsimilar process with $H > 0$ suth that $Y(t)$ is stochastically continuous at any $t \geq 0$ and nonconstant for every $t > 0$. Note that $Y(0) = 0$ a.s., since $Y(0) \stackrel{d}{\sim} a^H Y(0)$ for any $a > 0$. Here $\stackrel{d}{\sim}$ denotes the equality in law.

If $\{X(t)\}$ is H-selfsimilar with $H > 0$, then $\{X(t)\}$ is always stochastically continuous at $t = 0$. For, $X(0) = 0$ a.s. and

$$P\{|X(t)| > \varepsilon\} = P\{t^H |X(1)| > \varepsilon\} \to 0$$

when $t \to 0$. This is not true for semi-selfsimilar processes. The first example below shows that there exists an H-semi-selfsimilar process, $H > 0$, which is not stochastically continuous at $t = 0$.

To construct such a process, let $g : \mathbf{R} \to \mathbf{R}$ be a function satisfying

$$g(u+v) = g(u) + g(v), \quad \forall u, v \in \mathbf{R}, \tag{2.1}$$

$$g(1) > 0, \tag{2.2}$$

$$\limsup_{u \to -\infty} g(u) = +\infty, \quad \liminf_{u \to -\infty} g(u) = -\infty. \tag{2.3}$$

The existence of such a function is shown in [6]. It follows easily from (2.1) that

$$g(ru) = rg(u)\,, \quad \forall r \in \mathbf{Q}\,,\ \forall u \in \mathbf{R}\,, \tag{2.4}$$

where $\mathbf{Q}$ is the set of rational numbers. Therefore

$$g(r+u) = rg(1) + g(u)\,, \quad \forall r \in \mathbf{Q}\,,\ \forall u \in \mathbf{R}\,. \tag{2.5}$$

Let $f(t) = e^{g(\log t)}$ for $t > 0$. We may and do suppose that $g(1) = H$, where H is the exponent of $\{Y(t)\}$. We see from (2.5) that

$$f(at) = a^H f(t) \quad \text{if } \log a \in \mathbf{Q}\,.$$

For any given process $\{X(t)\}$, let Γ be the set of all $a > 0$ such that there exists $b > 0$ satisfying (1.1), and $\log \Gamma$ be the set of $\log a$ with $a \in \Gamma$.

Example 1. Define $\{X(t), t \geq 0\}$ by

$$X(t) = \begin{cases} 0\,, & \text{if } t = 0\,, \\ f(t)\,, & \text{if } t > 0 \text{ and } \log t \notin \mathbf{Q}\,, \\ Y(t)\,, & \text{if } \log t \in \mathbf{Q}\,, \end{cases}$$

we have

$$\{X(at)\} \stackrel{d}{=} \{a^H X(t)\}\,, \quad \text{if } \log a \in \mathbf{Q}\,, \tag{2.6}$$

because $\log at \in \mathbf{Q}$ if and only if $\log t \in \mathbf{Q}$. Also, if $a > 0$ and $\log a \notin \mathbf{Q}$, then we cannot find $b > 0$ satisfying (1.1), as is seen by choosing $t = a^{-1}$. Thus we have $\log \Gamma = \mathbf{Q}$, and $\{X(t)\}$ is semi-selfsimilar with a unique exponent H. We have

$$\lim_{\substack{u \to -\infty \\ u \in \mathbf{Q}}} g(u) = -\infty$$

by (2.2) and (2.4). On the other hand,

$$\limsup_{\substack{u \to -\infty \\ u \notin \mathbf{Q}}} g(u) = +\infty$$

by (2.3). Hence,

$$\limsup_{\substack{t \downarrow 0 \\ \log t \notin \mathbf{Q}}} f(t) = +\infty\,.$$

Namely, $\{X(t)\}$ is not stochastically continuous at $t = 0$. (Actually, this $\{X(t)\}$ is not stochastically continuous at any $t \geq 0$.)

As we have seen above, some semi-selfsimilar processes with a unique exponent are not stochastically continuous at the origin. However, if we do not assume the stochastic continuity at the origin, we cannot assure the existence of a unique exponent for a semi-selfsimilar process, as seen in the next example.

Example 2. Define $\{X(t), t \geq 0\}$ by

$$X(t) = \begin{cases} 0, & \text{if } t = 0 \text{ or } \log t \notin \mathbf{Q} + \sqrt{2}\mathbf{Q}, \\ Y(e^{r+s}), & \text{if } \log t = r + s\sqrt{2} \in \mathbf{Q} + \sqrt{2}\mathbf{Q}. \end{cases}$$

It is easily seen that $\{X(t)\}$ is not stochastically continuous at any $t \geq 0$. Let $\alpha \in \mathbf{Q}$. Then

$$X(e^{\alpha}t) = X(e^{\alpha\sqrt{2}}t) = \begin{cases} 0, & \text{if } t = 0 \text{ or } \log t \notin \mathbf{Q} + \sqrt{2}\mathbf{Q}, \\ Y(e^{\alpha+r+s}), & \text{if } \log t = r + s\sqrt{2} \in \mathbf{Q} + \sqrt{2}\mathbf{Q}. \end{cases}$$

It follows that $\{X(e^{\alpha}t)\} \stackrel{d}{=} \{X(e^{\alpha\sqrt{2}}t)\} \stackrel{d}{=} \{e^{\alpha H}X(t)\}$. Thus $\{X(t)\}$ is semi-selfsimilar but does not have an exponent, and $\log \Gamma = \mathbf{Q} + \sqrt{2}\mathbf{Q}$.

The third example in this section is an H-semi-selfsimilar process which is stochastically continuous at $t = 0$ but not at any $t > 0$.

Example 3. Define $\{X(t), t \geq 0\}$ by

$$X(t) = \begin{cases} 0, & \text{if } t = 0 \text{ or } \log t \notin \mathbf{Q}, \\ Y(t), & \text{if } \log t \in \mathbf{Q}. \end{cases}$$

Obviously $\{X(t)\}$ is stochastically continuous at $t = 0$ but not at any other $t > 0$. We have (2.6) by the same reasoning as for Example 1. If $a > 0$ and $\log a \notin \mathbf{Q}$, then there does not exist $b > 0$ satisfying (1.1), as is seen by choosing $t = a^{-1}$ again. Thus we have again $\log \Gamma = \mathbf{Q}$, and it follows from (2.6) that $\{X(t)\}$ is H-semi-selfsimilar.

A slight modification enables us to give the following example which is H-semi-selfsimilar, stochastically continuous at $t = 0$ but not at any $t > 0$, and has the additional property that $\{X(at)\} \stackrel{d}{=} \{a^H X(t)\}$ for some $a = a_1, a_2$ such that $\log a_1 / \log a_2$ is irrational.

Example 4. Define $\{X(t), t \geq 0\}$ by

$$X(t) = \begin{cases} 0, & \text{if } t = 0 \text{ or } \log t \notin \mathbf{Q} + \sqrt{2}\mathbf{Q}, \\ Y(t), & \text{if } \log t \in \mathbf{Q} + \sqrt{2}\mathbf{Q}. \end{cases}$$

Then

$$\{X(at)\} \stackrel{d}{=} \{a^H X(t)\}, \quad \text{if } \log a \in \mathbf{Q} + \sqrt{2}\mathbf{Q},$$

and $\log \Gamma = \mathbf{Q} + \sqrt{2}\mathbf{Q}$.

3. The existence of a unique exponent

In [9], to show the existence of a unique exponent, it was assumed that the semi-selfsimilar process $\{X(t), t \geq 0\}$ is stochastically continuous at any $t \geq 0$. However, the examples in the previous section suggest that the stochastic continuity only at $t = 0$ would be needed for the existence of a unique exponent. Actually this is true as the following theorem shows. We call $\{X(t)\}$ a zero process if, for each t, $P\{X(t) = 0\} = 1$. Otherwise it is called nonzero.

Theorem 1. *Let $\{X(t), t \geq 0\}$ be an $\mathbf{R}^d$-valued, nonzero semi-selfsimilar process. Suppose that it is stochastically continuous at $t = 0$. Then the following statements are true.*
(i) *There exists a unique $H \geq 0$ such that, if $a \in (0,1) \cup (1,\infty)$ and $b > 0$ satisfy (1.1), then $b = a^H$.*
(ii) *$X(0) = 0$ a.s. if and only if $H > 0$. $X(t) = X(0)$ a.s. for every t if and only if $H = 0$.*

Recall Example 2, which shows that a semi-selfsimilar process does not necessarily have a unique exponent, unless it is stochastically continuous at $t = 0$. So, we cannot entirely remove the assumption of the stochastic continuity at $t = 0$ to prove the existence of a unique exponent.

Proof of Theorem 1. The nonzero property of $\{X(t)\}$ implies that b is uniquely determined by a. Thus we write $b = b(a)$ for $a \in \Gamma$. It is easy to prove the following.

(1) $1 \in \Gamma$ and $b(1) = 1$.
(2) If $a \in \Gamma$, then $a^{-1} \in \Gamma$ and $b(a^{-1}) = b(a)^{-1}$.
(3) If a and a' are in Γ, then $aa' \in \Gamma$ and $b(aa') = b(a)b(a')$.

Furthermore, we observe

(4) If $X(0) = 0$ a.s., then $b(a) > 1$ for any $a \in \Gamma \cap (1,\infty)$.
(5) If $b(a) \neq 1$ for some $a \in \Gamma \cap (1,\infty)$, then $X(0) = 0$ a.s.
(6) If $b(a) = 1$ for some $a \in \Gamma \cap (1,\infty)$, then $X(t) = X(0)$ a.s. for every t.

To see (4), suppose that $X(0) = 0$ a.s. and that $b(a) \leq 1$ for some $a \in \Gamma \cap (1,\infty)$. Denote the characteristic function of $X(t)$ by $\hat{\mu}_t(z)$. Then by semi-selfsimilarity, we have $\hat{\mu}_{a^n t}(b(a)^{-n} z) = \hat{\mu}_t(z)$ for all $n \in \mathbf{Z}$ and for all $z \in \mathbf{R}^d$.

Since $X(0) = 0$ a.s. and $\{X(t)\}$ is stochastically continuous at $t = 0$, we have $\hat{\mu}_{a^n t}(w) \to 1$ uniformly in w in any compact set as $n \to -\infty$. Thus $\hat{\mu}_t(z) = 1$ and this contradicts the nonzero property of the process. This proves (4). (5) is obvious from the relationship $\hat{\mu}_0(z) = \hat{\mu}_0(b(a)^n z)$ for all $n \in \mathbf{Z}$. If $b(a) = 1$ for some $a > 1$, then $\{X(t)\} \stackrel{d}{=} \{X(a^n t)\}$ for all $n \in \mathbf{Z}$ and the stochastic continuity at $t = 0$ yields

$$P\{|X(t) - X(0)| > \varepsilon\} = P\{|X(a^n t) - X(0)| > \varepsilon\} \to 0$$

as $n \to -\infty$, which proves (6).

Hence, by (4) and (5), $b(a) \geq 1$ for any $a \in \Gamma \cap (1, \infty)$. By (1), (2), and (3), $\log \Gamma$ is an additive subgroup of $\mathbf{R}$ and $(\log \Gamma) \cap (0, \infty) \neq \emptyset$. Let r_0 be the infimum of $(\log \Gamma) \cap (0, \infty)$.

Suppose that $r_0 > 0$. Then $r_0 \in \log \Gamma$. In fact, if $r_0 \notin \log \Gamma$, then there are $s_n, n = 1, 2, \ldots$, in $\log \Gamma$ strictly decreasing to r_0 and we have $r_0 > s_n - s_{n+1} \in (\log \Gamma) \cap (0, \infty)$ for sufficiently large n, contrary to the definition of r_0. This implies that $\{nr_0 : n \in \mathbf{Z}\} \subset \log \Gamma$. If $nr_0 < r < (n+1)r_0$ for some $r \in \log \Gamma$, then $r - nr_0 \in \log \Gamma$ and $0 < r - nr_0 < r_0$, which is a contradiction. Hence $\log \Gamma = \{nr_0 : n \in \mathbf{Z}\}$ and there is a unique exponent $H \geq 0$.

In the rest of the proof assume that $r_0 = 0$. Maejima and Sato [9] proved that $\log \Gamma = \mathbf{R}$ in this case. But it does not work any more, since the argument in [9] to show the closedness of $\log \Gamma$ uses the stochastic continuity of $\{X(t)\}$ for $t \geq 0$. (Actually $\log \Gamma$ is not always closed under the condition that $\{X(t)\}$ is stochastically continuous only at $t = 0$, as have been seen in Example 3 in the previous section.) So we have to use another idea to show the existence of an exponent. Suppose that $H \geq 0$ with the desired property does not exist. Then there exist $0 \leq H_1 < H_2$ such that, for $i = 1, 2$, the set Γ_i defined by $\Gamma_i = \{a \in \Gamma : b(a) = a^{H_i}\}$ contains some $a_i \neq 1$. If $a \in \Gamma_i$, then $a^{-1} \in \Gamma_i$, since $b(a^{-1}) = b(a)^{-1} = a^{-H_i}$. Hence, for each i, there is $a_i \in \Gamma_i \cap (1, \infty)$. If $H_1 = 0$ (and $H_2 > 0$), then, by (5) and (6), $X(t) = 0$ a.s. for each $t \geq 0$, which contradicts the nonzero assumption of $\{X(t)\}$. Hence $H_1 > 0$. For any positive integer m, there exists a positive integer n such that

$$\left| n - \frac{H_2 \log a_2}{H_1 \log a_1} m \right| \leq 1 .$$

Therefore we can find two sequences $\{m_k\}$, $\{n_k\}$ such that $m_k, n_k \to \infty$ and

$$-n_k H_1 \log a_1 + m_k H_2 \log a_2 \to b \quad \text{as } k \to \infty$$

for some $b \in (-\infty, \infty)$. Let $s_k = a_1^{-n_k} a_2^{m_k}$. Since

$$\frac{n_k}{m_k} \to \frac{H_2 \log a_2}{H_1 \log a_1} \quad \text{as } k \to \infty ,$$

we have, as $k \to \infty$,

$$\log s_k = m_k \left(-\frac{n_k}{m_k} \log a_1 + \log a_2 \right) \to -\infty ,$$

namely $s_k \to 0$. Noting that $X(0) = 0$ a.s. by (5), we can choose $t_0 > 0$ so that $X(t_0)$ is nonzero. Notice that

$$X(s_k t_0) = X(a_1^{-n_k} a_2^{m_k} t_0) \overset{d}{\sim} a_1^{-H_1 n_k} a_2^{H_2 m_k} X(t_0) .$$

Let k tend to ∞ here. Use the stochastic continuity of $\{X(t)\}$ at $t = 0$ and the fact that $X(0) = 0$ a.s. Then we have $e^b X(t_0) = 0$ a.s., which contradicts that $X(t_0)$ is nonzero. Therefore the exponent $H \geq 0$ uniquely exists. Hence (i) is true. The assertion (ii) follows from (4), (5) and (6). This completes the proof of Theorem 1. □

4. Long range dependence of increments of semi-selfsimilar processes

Suppose that $\{X(t), t \geq 0\}$ has stationary increments and finite second moments with mean zero, and let $\xi_n = X(n+1) - X(n)$, $n = 0, 1, 2, \dots$ and $r(n) = E[\xi_n \xi_0]$. If $\sum_{n=0}^{\infty} r(n)$ is divergent, then it is said that stationary increments $\{\xi_n\}$ have long range dependence, because the slow decrease of the correlations is considered as an expression of long range dependence. (See e.g. [2], [3], and [10].)

Note that if $\{X(t)\}$ is H-selfsimilar, $H > 0$, with stationary increments and has finite first moment, then H of this process should satisfy $H \leq 1$, and in case $H < 1$ we have $E[X(t)] = 0$, while in case $H = 1$ we have $X(t) = tX$ a.s. for some random variable X. (See e.g. [8].) Similarly, if $\{X(t)\}$ is H-semi-selfsimilar, $H > 0$, with stationary increments and stochastically continuous at $t = 0$, and if it has finite first moment and

$$\sup_{0 \leq t < \varepsilon} E\left[|X(t)|\right] < \infty , \quad \text{for some } \varepsilon > 0 , \tag{4.1}$$

then $H \leq 1$. This can be shown as follows.

There is $a > 1$ such that $\{X(at)\} \overset{d}{=} \{a^H X(t)\}$. Thus, for any $n \in \mathbf{Z}$, $X(a^n) \overset{d}{\sim} a^{nH} X(1)$. We have

$$E\left[|X(a^n)|\right] \leq \sum_{k=0}^{[a^n]-1} E\left[|X(a^n - k) - X(a^n - k - 1)|\right] + E\left[|X(a^n - [a^n])|\right] ,$$

where $[a^n]$ is the largest integer less than or equal to a^n. Thus, using $X(0) = 0$ a.s., we have

$$a^{nH} E\left[|X(1)|\right] \le [a^n] E\left[|X(1)|\right] + \sup_{0\le t<1} E\left[|X(t)|\right],$$

and for any $n \in \mathbf{Z}$,

$$(a^{nH} - [a^n]) E\left[|X(1)|\right] \le \sup_{0\le t<1} E\left[|X(t)|\right]. \tag{4.2}$$

If $H > 1$, then letting $n \to \infty$ gives $\sup_{0\le t<1} E\left[|X(t)|\right] = \infty$, which contradicts (4.1) together with stationary increments. Hence $H \le 1$.

Also observe that if furthermore $H < 1$, then $E\left[X(1)\right] = 0$ and thus $E\left[X(a^n)\right] = 0$ for all $n \in \mathbf{Z}$. For, similarly to (4.2) we have

$$\left|(a^{nH} - [a^n]) E\left[X(1)\right]\right| \le \sup_{0\le t<1} E\left[|X(t)|\right]$$

and, if $H < 1$ and $E\left[X(1)\right] \neq 0$, then letting $n \to \infty$ yields again a contradiction.

We now consider only the case $0 < H < 1$. Then, if $\{X(t)\}$ is H-selfsimilar with stationary increments having finite second moment, we have

$$r(n) = \frac{1}{2}\left\{(n+1)^{2H} + |n-1|^{2H} - 2n^{2H}\right\} E\left[X(1)^2\right]$$

and we see that $\sum_{n=0}^{\infty} r(n) = \infty$ if and only if $\frac{1}{2} < H < 1$.

This argument of using $r(n)$ does not work for semi-selfsimilar processes, because the semi-selfsimilarity is not enough to give an explicit expression for $r(n)$. However, there are several other ways to understand long range dependence. Among those, one of the characterizations of long range dependence of stationary increments is that the variance of the process $\{X(t)\}$ diverges to infinity faster than the order t. (See [2]. Also see the non-central limit theorems for strongly dependent stationary random variables in [4] and [10].)

Now suppose that $\{X(t)\}$ is H-semi-selfsimilar with $\frac{1}{2} < H < 1$, has finite second moment, and satisfies (4.1). Then, recalling that $E\left[X(a^n)\right] = 0$ for all $n \in \mathbf{Z}$, we have

$$\begin{aligned}\limsup_{t\to\infty} t^{-1}\mathrm{Var}X(t) &\ge \lim_{n\to\infty} a^{-n}\mathrm{Var}X(a^n)\\ &= \lim_{n\to\infty} a^{-n} E\left[X(a^n)^2\right]\\ &= \lim_{n\to\infty} a^{n(2H-1)} E\left[X(1)^2\right] = \infty.\end{aligned} \tag{4.3}$$

If $\{X(t)\}$ has stationary increments, then (4.3) means that the variance of sums of stationary increments (the variance of the original process itself) cannot have the same order as the number of the summands. So, it is reasonable to understand that stationary increments of H-semi-selfsimilar processes have long range dependence if $H > \frac{1}{2}$.

5. Concluding remarks

Let $\{X(t), t \geq 0\}$ be stochastically continuous for $t \geq 0$. In [9], it is shown that if $\{X(t)\}$ is semi-selfsimilar, then

(i) $\Gamma = \{a_0{}^n, n \in \mathbf{Z}\}$ for some $a_0 > 1$ and $\{X(t)\}$ is not selfsimilar but semi-selfsimilar, or

(ii) $\Gamma = (0, \infty)$ and $\{X(t)\}$ is selfsimilar

Suppose one wants to check the selfsimilarity of a process. If we follow the definition of selfsimilarity, one has to check (1.1) for all $a > 0$. However, suppose one could show the relationship (1.1) only for two a's, (a_1, a_2, say), such that $\log a_1 / \log a_2$ is irrational. Then by the observation above, the fact that $a_1, a_2 \in \Gamma$ implies $\Gamma = (0, \infty)$, and one can conclude that $\{X(t)\}$ is selfsimilar. This gives us an easy way to check selfsimilarity of a given process. Namely, we have

Proposition 1. *Suppose $\{X(t), t \geq 0\}$ is stochastically continuous at any $t \geq 0$. If $\{X(t)\}$ satisfies (1.1) for some a_1 and a_2 such that $\log a_1 / \log a_2 \notin \mathbf{Q}$, then it is selfsimilar with some unique exponent $H \geq 0$.*

In this proposition, we cannot relax the assumption of stochastic continuity at any $t \geq 0$ to that at $t = 0$, because of Example 4 in Section 2.

Another remark is concerning extension of Theorem 1 to wide-sense semi-selfsimilar processes. If [9] an $\mathbf{R}^d$-valued stochastic process $\{X(t), t \geq 0\}$ is called wide-sense semi-selfsimilar if there are $a \in (1, 0) \cup (1, \infty)$, $b > 0$, and an $\mathbf{R}^d$-valued nonrandom function $c(t)$ such that

$$\{X(at), t \geq 0\} \stackrel{d}{=} \{bX(t) + c(t), t \geq 0\}. \tag{5.1}$$

Semi-stable Lévy processes are special cases of such processes with linear functions $c(t)$. Let us call a process $\{X(t)\}$ trivial if, for each t, $P\{X(t) = \text{const.}\} = 1$. Otherwise it is called nontrivial. The existence of a unique exponent for any nontrivial wide-sense semi-selfsimilar process stochastically continuous for $t \geq 0$ is proved in [9]. Its proof combined with the proof of our Theorem 1 readily gives the following result.

Proposition 2. *Let $\{X(t), t \geq 0\}$ be an $\mathbf{R}^d$-valued, nontrivial wide-sense semi-selfsimilar process, stochastically continuous at $t = 0$. Then the following statements are true.*

(i) *There exists a unique $H \geq 0$ such that, if $a \in (0, 1) \cup (1, \infty)$, $b > 0$, and $c(t)$ satisfy (5.1), then $b = a^H$.*

(ii) *$X(0) = \text{const.}$ a.s. if and only if $H > 0$. There is an $\mathbf{R}^d$-valued function $h(t)$ satisfying $X(t) = X(0) + h(t)$ a.s. for every t if and only if $H = 0$.*

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