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FOUR-DIMENSIONAL ALMOST KÄHLER EINSTEIN AND WEAKLY $*$-EINSTEIN MANIFOLDS

By

TAKASHI OGURO, KOUEI SEKIGAWA AND AKIRA YAMADA

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Abstract. Concerning the intergarability of almost Kähler manifolds, it is known the conjecture by S.I. Goldberg that a compact almost Kähler manifold is Kähler. In this paper we give a positive partial answer to the conjecture and further introduce the related example constructed by P. Nurowski and M. Przanowski.

1. Introduction

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the Kähler form is closed (or equivalently, $\mathfrak{S}_{X,Y,Z}\, g((\nabla_X J)Y, Z) = 0$ for $X, Y, Z \in \mathfrak{X}(M)$, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y, Z). By the definition, a Kähler manifold ($\nabla J = 0$) is an almost Kähler manifold. A non-Kähler, almost Kähler manifold is called a strictly almost Kähler manifold. The first example of compact strictly almost Kähler manifold was found by W.P. Thurston ([19]). It is well-known that an almost Kähler manifold with integrable almost Kähler manifold, the following conjecture by S.I. Goldberg is known ([5]).

Conjecture. *The almost complex structure of a compact almost Kähler Einstein manifold is integrable.*

The above conjecture is true in the case where the scalar curvature is non-negative ([17]). However, the conjecture is still open in the case where the scalar curvature is negative. For the other progresses, we refer to ([1]), ([3]), ([4]), ([10]), ([13]), ([14]), ([15]) and so on. The first and the second authors have proved the following ([15]).

Theorem A. *A four-dimensilnal almost Kähler Einstein and $*$-Einstein manifold is a Kähler manifold.*

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Theorem B. *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler Einstein and weakly $*$-Einstein manifold with negarive scalar curvature. Then, $\tau \leq \tau^* \leq \tau/3$ on M and $\tau = \tau^*$ holds at some point of M, where τ and τ^* are the scalar curvature and the $*$-scalar curvature, respectively.*

The main purpose of the present paper is to prove the following.

Main Theorem. *Let $M = (M, J, g)$ be a four-dimensional strictly almost Kähler Einstein and weakly $*$-Einstein manifold. Then, M is a Ricci-flat space of pointwise constant holomorphic curvature $\tau^*/8$ and hence M is self-dual, where τ^* is the $*$-scalar curvature of M.*

Quite recently, the authors knew through private communications with Dr. V. Apostolov that J. Armstrong has proved the above result by making use of spinorial method in his doctor thesis (Oxford University). Our proof of the above result is rather straight-forward and seems more understandable that the one by J. Armstrong.

Combining the Main Theorem and the result of [17], we have immediately the following improvement of Theorem B.

Corollary. *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler Einstein and weakly $*$-Einstein manifold. Then, M is a Kähler manifold.*

In the above corollary, we can not omit the assumption of compactness. In fact, recently, P. Nurowski and M. Przanowski ([12]) constructed an example of strictly almost Kähler Ricci-flat space of dimension four. We may easily check that their example is also a weakly $*$-Einstein space which is not $*$-Einstein. Therefore, we may also remark that the assumption of $*$-Einsteinness in Theorem A can not be replaced by weaker assumption of weakly $*$-Einsteinness and Main Theorem also supports their example. In the last section, we shall introduce their example and discuss it.

Through this paper, we assume that all manifolds are connected and smooth and further that all quantities on manifolds are smooth, unless otherwise specified.

2. Preliminaries

In this section, we prepare several formulas which we need in the proof of the Main Theorem.

Let $M = (M, J, g)$ be a four-dimensional almost Hermitian manifold with almost Hermitian structure (J, g) and Ω the Kähler form of M defined by

$\Omega(X,Y) = g(X,JY)$ for $X,Y \in \mathfrak{X}(M)$ ($\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M). We assume that M is oriented by the volume form $dM = \Omega^2/2$. We denote by ∇, R, ρ, and τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. The curvature tensor R is defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \tag{2.1}$$

for $X,Y,Z \in \mathfrak{X}(M)$. We denote by ρ^* the Ricci $*$-tensor of M defined by

$$\rho^*(x,y) = \frac{1}{2}\text{ trace of } (z \mapsto R(x,Jy)Jz) \tag{2.2}$$

for $x,y,z \in T_pM$, $p \in M$. Further, we denote by τ^* the $*$-scalar curvature of M which is the trace of the linear endomorphism Q^* defined by $g(Q^*, z, y) = \rho^*(x,y)$ for $x,y \in T_pM$, $p \in M$. By (2.2), we get immediately

$$\rho^*(x,y) = \rho^*(Jy,Jx) \tag{2.3}$$

for $x,y \in T_pM$, $p \in M$. From (2.3), it follows immediately that ρ^* is symmetric if and only if ρ^* is J-invariant. We may also note that if M is Kähler, $\rho^* = \rho$ holds on M. Now, if $\rho^* = \lambda^* g (\lambda^* = \tau^*/4)$ holds on M, then M is called a weakly $*$-Einstein manifold. Further, a weakly $*$-Einstein manifold with constant $*$-scalar curvature is called a $*$-Einstein manifold. It is easily observed that the following identity holds for any four-dimensional almost Hermitian manifold $M = (M,J,g)$ ([7]):

$$\frac{1}{2}\{\rho(x,y) + \rho(Jx,Jy)\} - \frac{1}{2}\{\rho^*(x,y) + \rho^*(y,x)\} = \frac{\tau - \tau^*}{r} g(x,y) \tag{2.4}$$

for $x,y \in T_pM$, $p \in M$. We denote by N the Nijenhuis tensor of M defined by

$$N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] \tag{2.5}$$

for $X,Y \in \mathfrak{X}(M)$. The celebrated theorem by A. Newlander and L. Nirenberg ([11]) says that the almost complex structure J is integrable if and only if the Nijenhuis tensor N vanishes identically on M. The curvature operator $\mathcal{R}$ is the symmetric endomorphism of the vector bundle $\wedge^2 M$ of real 2-forms over M defined by

$$g(\mathcal{R}(\iota(x) \wedge \iota(y)), \iota(z) \wedge \iota(w)) = -g(R(x,y)z,w) = -R(x,y,z,w) \tag{2.6}$$

for $x,y,z,w \in T_pM$, $p \in M$, where ι denotes the duality: $TM \to T^*M$ defined by means of the metric g. The following decomposition for the vector bundle $\wedge^2 M$ is useful in our arguments:

$$\wedge^2 M = \mathbb{R}\Omega \oplus \wedge_0^{1,1} M \oplus LM \tag{2.7}$$

where $\wedge_0^{1,1}M$ denotes the vector bundle of real primitive J-invariant 2-forms, LM the vector bundle of real primitive J-skew-invariant 2-forms over M, respectively. The bundle LM is endowed with the natural complex structure (also denoted by J) which is defined by $J\Phi(X,Y) = -\Phi(JX,Y)$ for any local section Φ of LM and $X, Y \in \mathfrak{X}(M)$. The bundle $\wedge_0^{1,1}M$ identifies itself with the bundle $\wedge_-^2 M$ of anti-self-dual 2-forms, while the sum $\mathbb{R} \oplus LM$ is the bundle $\wedge_+^2 M$ and $\wedge_-^2 M$ are preserved by the curvature operator $\mathcal{R}$ ([8]). In this paper, for any orthonormal basis (resp. any local orthonormal frame field) $\{e_i\}$ of any point $p \in M$ (resp. on a neighborhood of p), we shall adopt the following notational convention:

$$\begin{aligned} R_{ijkl} &= g(R(e_i,e_j)e_k,e_l) = R(e_i,e_j,e_k,e_l), \\ R_{\bar{i}jkl} &= g(R(Je_i,e_j)e_k,e_l) = R(Je_i,e_j,e_k,e_l), \\ &\cdots, \\ R_{\bar{i}\bar{j}\bar{k}\bar{l}} &= g(R(Je_i,Je_j)Je_k,Je_l) = R(Je_i,Je_j,Je_k,Je_l), \end{aligned} \tag{2.8}$$

and so on, where the latin indices run over the range 1, 2, 3, 4. Then, we have

$$J_{ij} = -J_{ji}, \quad \nabla_i J_{jk} = -\nabla_j J_{kj}, \quad \nabla_i J_{\bar{j}\bar{k}} = -\nabla_i J_{jk}. \tag{2.9}$$

In the sequel, we assume that $M = (M, J, g)$ is a four-dimensional almost Kähler manifold. Then, it is known that M is a quasi-Kähler manifold, i.e.,

$$\nabla_i J_{jk} = -\nabla_{\bar{i}} J_{\bar{j}k} \tag{2.10}$$

for $1 \leq i,j,k,l \leq 4$ ([20]). From (2.10), it follows immediately that M is a semi-Kähler manifold, i.e.,

$$\sum_a \nabla_a J_{ai} = 0 \tag{2.11}$$

holds on M ([20]). On one hand, it is also known that the Nijenhuis tensor N of M is expressed by

$$g(N(X,Y),JZ) = 2g((\nabla_Z J)X,Y) \tag{2.12}$$

for $X, Y, Z \in \mathfrak{X}(M)$. From (2.12), taking account of the Ricci identity, we have

$$\begin{aligned} &2\{R(w,x,Jy,z) + R(w,x,y,Jz)\} \\ &= \frac{1}{2}\{g(JN(N(y,z),x),w) - g(JN(N(y,z),w),x)\} \\ &\quad - g(J(\nabla_w N)(y,z),x) + g(J(\nabla_x N)(y,z),w) \end{aligned} \tag{2.13}$$

for $w, x, y, z \in T_pM$, $p \in M$ ([6]). From (2.12) and (2.13), by direct calculation, we have

$$\rho^*_{ij} + \rho^*_{ji} - \rho_{ij} - \rho_{\bar{i}\bar{j}} = \sum_{a,b} (\nabla_a J_{ib}) \nabla_a J_{jb} \tag{2.14}$$

and further

$$\|\nabla J\|^2 = \frac{1}{4}\|N\|^2 = 2(\tau^* - \tau)\,. \tag{2.15}$$

From (2.9), (2.10), (2.12), for any unitary basis $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_4\}$ of T_pM at any point $p \in M$, we have

$$(\nabla_z J)e_1, (\nabla_z J)e_2 \in \mathrm{span}\,\{e_3, e_4\}\,, \quad (\nabla_z J)e_3, (\nabla_z J)e_4 \in \mathrm{span}\,\{e_1, e_2\}\,, \tag{2.16}$$

for any $z \in T_pM$, and

$$N(e_1, e_2) = 0\,, \quad N(e_3, e_4) = 0\,.$$

Now, from (2.7), (2.9) and (2.10), we have

$$\nabla\,\Omega = \alpha \otimes \Phi - J\alpha \otimes J\Phi\,, \tag{2.17}$$

where $\{\Phi, J\Phi\}$ is a local orthonormal frame field of LM, α is a local 1-form and $J\alpha$ is the 1-form defined by $(J\alpha)(X) = -\alpha(JX)$, for $X \in \mathfrak{X}(M)$. From (2.15) and (2.17), we have easily

$$\|\alpha\|^2 = \frac{1}{2}\|\nabla\,\Omega\|^2 = \frac{1}{4}\|\nabla J\|^2 = \frac{\tau^* - \tau}{2}\,. \tag{2.18}$$

Let $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ be a local unitary frame field and put $e^i = \iota(e_i)$ $(1 \le i \le 4)$. Then, $\{e^i\} = \{e^1, e^2 = Je^1, e^3, e^4 = Je^3\}$ is the local unitary frame field dual to $\{e_i\}$ and the Kähler form Ω is expressed by

$$\Omega = -e^1 \wedge e^2 - e^3 \wedge e^4\,.$$

Further, we may observe that

$$\{\Phi, J\Phi\} = \left\{\frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4), \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3)\right\}$$

is a local orthonormal frame field of LM. We have note that the expression (2.17) for $\nabla\,\Omega$ is not unique (cf. [15], p. 108). We set

$$\nabla_{e_i} e_j = \sum_k \Gamma_{ijk} e_k \qquad (1 \le i, j \le 4)\,. \tag{2.19}$$

Then, we may check that $\Gamma_{ijk} = -\Gamma_{ijk} e_k$ $(1 \le i, j, k \le 4)$. By (2.19), we have also

$$\nabla_{e^i} e^j = \sum_k \Gamma_{ijk} e^k \qquad (1 \le i, j \le 4)\,. \tag{2.20}$$

By (2.19) and (2.20), we have

$$\begin{aligned} \Gamma_{j24} - \Gamma_{j13} &= -\frac{1}{\sqrt{2}}\alpha_{\bar{j}}\,, \\ \Gamma_{j23} + \Gamma_{j14} &= -\frac{1}{\sqrt{2}}\alpha_j \end{aligned} \tag{2.21}$$

for $1 \le j \le 4$. Taking account of (2.19)∼(2.21), we have

$$\begin{aligned} \nabla_j \Phi &= (\Gamma_{j12} + \Gamma_{j34}) J\Phi - \frac{1}{2}\alpha_j\,\Omega\,, \\ \nabla_j (J\Phi) &= -(\Gamma_{j12} + \Gamma_{j34})\Phi - \frac{1}{2}\alpha_{\bar{j}}\,\Omega \end{aligned} \tag{2.22}$$

Further, we have

$$\nabla_j (J\alpha)_i = -\nabla_j\,\alpha_{\bar{i}} + \sum_t (\nabla_j J_{ti})\alpha_t\,, \tag{2.23}$$

for $1 \le i,\, j \le 4$. From (2.17)∼(2.23), by direct calculation, we have

$$\begin{aligned} \nabla^2_{ji}\,\Omega = -\frac{1}{2}(\alpha_i\alpha_j + \alpha_{\bar{i}}\alpha_{\bar{j}})\,\Omega + \left\{\nabla_j\alpha_i - \alpha_{\bar{i}}\,(\Gamma_{j12} + \Gamma_{j34})\right\}\Phi \\ + \left\{\nabla_j\alpha_{\bar{i}} + \alpha_i\,(\Gamma_{j12} + \Gamma_{j34}) - \sum_t (\nabla_j J_{ti})\,\alpha_t\right\} J\Phi\,. \end{aligned} \tag{2.24}$$

Since $\nabla^2_{ji}\,\Omega_{kl} = -\nabla^2_{ji}\,J_{kl}$, from (2.17), (2.18) and (2.24), we have

$$\begin{aligned} & R_{jik\bar{l}} + R_{ji\bar{k}l} \\ &= \left\{\nabla_i\alpha_j - \nabla_j\alpha_i + \alpha_{\bar{i}}\,(\Gamma_{j12} + \Gamma_{j34}) - \alpha_{\bar{j}}\,(\Gamma_{i12} + \Gamma_{i34})\right\}\Phi_{kl} \\ &\quad + \left\{\nabla_i\alpha_{\bar{j}} - \nabla_j\alpha_{\bar{i}} - \alpha_i\,(\Gamma_{j12} + \Gamma_{j34}) + \alpha_j\,(\Gamma_{i12} + \Gamma_{i34}) + \frac{\tau^* - \tau}{2}\Phi_{ij}\right\}(J\Phi)_{kl}\,. \end{aligned} \tag{2.25}$$

By (2.6), we see that the left hand side of (2.25) is rewritten as

$$R_{jik\bar{l}} + R_{ji\bar{k}l} = -R_{ijk\bar{l}} - R_{ij\bar{k}l} = g\left(\mathcal{R}(e^k \wedge Je^l + Je^k \wedge e^l), e^i \wedge e^j\right). \tag{2.26}$$

Now, we set

$$A_{ij} = g\left(e_i, (\nabla_{e_j} N)(e_1, e_3)\right), \qquad (1 \le i, j \le 4)\,. \tag{2.27}$$

In the remainder of this section, we assume that $M = (M, J, g)$ is a four-dimensional almost Kähler Einstein manifold. First, we recall several formulas established in [15]. Since $\mathcal{R}(\wedge^2_\pm M) \subset \wedge^2_\pm M$, we have

$$R_{1212} = R_{3434}\,, \qquad R_{1313} = R_{2424}\,, \qquad R_{1414} = R_{2323}\,. \tag{2.28}$$

Further, we have the following:

$$A_{24} = -A_{13}\,, \quad A_{42} = -A_{31}\,, \quad A_{32} = A_{41}\,, \quad A_{14} = A_{23}\,, \tag{2.29}$$

$$R_{1324} - R_{1313} = \frac{1}{2}(A_{13} - A_{31})\,, \tag{2.30}$$

$$R_{1423} + R_{1414} = \frac{1}{2}(A_{13} - A_{31}) + \frac{\tau^* - \tau}{4}\,, \tag{2.31}$$

$$\begin{aligned} -R_{1314} + R_{1424} &= -R_{1314} - R_{1323} = R_{2324} + R_{1424} \\ &= R_{2324} - R_{1323} = \frac{1}{2}(A_{14} - A_{41})\,, \end{aligned} \tag{2.32}$$

(cf. [15], pp. 99–101). The equality (2.4) reduces to

$$\rho^*_{ij} + \rho^*_{ji} = \frac{\tau^*}{2}\delta_{ij}\,. \tag{2.33}$$

Since $\mathcal{R}(\wedge^2_+ M) \subset \wedge^2_+ M$, we may set

$$\mathcal{R}(\Phi) = u\Phi + wJ\Phi + A\,\Omega\,, \qquad \mathcal{R}(J\Phi) = w\Phi + vJ\Phi + B\,\Omega\,. \tag{2.34}$$

Then, by (2.2), (2.6), (2.28) and (2.34), we have

$$\begin{aligned} &A = \frac{1}{2\sqrt{2}}(\rho^*_{14} - \rho^*_{41})\,, \quad B = -\frac{1}{2\sqrt{2}}(\rho^*_{13} - \rho^*_{31})\,, \\ &u = -(R_{1313} - R_{1324})\,, \quad v = -(R_{1414} + R_{1423})\,, \quad w = -(R_{1314} + R_{1323})\,. \end{aligned} \tag{2.35}$$

From (2.25), (2.26) and (2.34), we have

$$\begin{aligned} w &= \frac{1}{\sqrt{2}}\{\nabla_1\alpha_3 - \nabla_3\alpha_1 + \alpha_2\,(\Gamma_{312} + \Gamma_{334}) - \alpha_4\,(\Gamma_{112} + \Gamma_{134})\}\,, \\ w &= -\frac{1}{\sqrt{2}}\{\nabla_2\alpha_4 - \nabla_4\alpha_2 - \alpha_1\,(\Gamma_{412} + \Gamma_{434}) + \alpha_3\,(\Gamma_{212} + \Gamma_{234})\}\,, \\ w &= -\frac{1}{\sqrt{2}}\{-\nabla_1\alpha_3 - \nabla_4\alpha_2 - \alpha_1\,(\Gamma_{412} + \Gamma_{434} + \alpha_4\,(\Gamma_{112} + \Gamma_{134})\}\,, \\ w &= -\frac{1}{\sqrt{2}}\{\nabla_2\alpha_4 + \nabla_3\alpha_1 - \alpha_2\,(\Gamma_{312} + \Gamma_{334}) + \alpha_3\,(\Gamma_{212} + \Gamma_{234})\}\,, \end{aligned} \tag{2.36}$$

$$\begin{aligned} u &= -\frac{1}{\sqrt{2}}\Big\{\nabla_1\alpha_4 - \nabla_3\alpha_2 + \frac{\tau^* - \tau}{2\sqrt{2}} - \alpha_1\,(\Gamma_{312} + \Gamma_{334}) + \alpha_3\,(\Gamma_{112} + \Gamma_{134})\Big\}\,, \\ u &= \frac{1}{\sqrt{2}}\Big\{-\nabla_2\alpha_3 + \nabla_4\alpha_1 - \frac{\tau^* - \tau}{2\sqrt{2}} - \alpha_2\,(\Gamma_{412} + \Gamma_{434}) + \alpha_4\,(\Gamma_{112} + \Gamma_{234})\Big\}\,, \end{aligned} \tag{2.37}$$

$$
\begin{aligned}
v &= \frac{1}{\sqrt{2}}\{\nabla_1\alpha_4 - \nabla_4\alpha_1 + \alpha_2\,(\Gamma_{412}+\Gamma_{434}) + \alpha_3\,(\Gamma_{112}+\Gamma_{134})\}\,,\\
v &= \frac{1}{\sqrt{2}}\{\nabla_2\alpha_3 - \nabla_3\alpha_2 - \alpha_1\,(\Gamma_{312}+\Gamma_{334}) - \alpha_4\,(\Gamma_{212}+\Gamma_{234})\}\,,
\end{aligned}
\tag{2.38}
$$

$$
\begin{aligned}
\frac{1}{2}(\rho^*_{13}-\rho^*_{31}) &= \frac{1}{\sqrt{2}}\{\nabla_1\alpha_2 - \nabla_2\alpha_1 + \alpha_1\,(\Gamma_{112}+\Gamma_{134}) + \alpha_2\,(\Gamma_{212}+\Gamma_{234})\}\,,\\
\frac{1}{2}(\rho^*_{13}-\rho^*_{31}) &= \frac{1}{\sqrt{2}}\{\nabla_3\alpha_4 - \nabla_4\alpha_3 - \alpha_3\,(\Gamma_{312}+\Gamma_{334}) + \alpha_4\,(\Gamma_{412}+\Gamma_{434})\}\,,
\end{aligned}
\tag{2.39}
$$

$$
\begin{aligned}
\frac{1}{2}(\rho^*_{14}-\rho^*_{41}) &= -\frac{1}{\sqrt{2}}\{\nabla_1\alpha_1 + \nabla_2\alpha_2 + \alpha_1\,(\Gamma_{212}+\Gamma_{234}) - \alpha_2\,(\Gamma_{112}+\Gamma_{134})\}\,,\\
\frac{1}{2}(\rho^*_{14}-\rho^*_{41}) &= -\frac{1}{\sqrt{2}}\{\nabla_3\alpha_3 + \nabla_4\alpha_4 + \alpha_3\,(\Gamma_{412}+\Gamma_{434}) - \alpha_4\,(\Gamma_{312}+\Gamma_{334})\}\,.
\end{aligned}
\tag{2.40}
$$

Now, we denote by $\xi = (\xi_i)$ the smooth vector field on M defined by

$$
\xi_i = \sum_t \nabla_t\,\rho^*_{ti}\,. \tag{2.41}
$$

Then, taking account of (2.2) and the equality $\mathfrak{S}_{i,j,k}\nabla_i J_{jk} = 0$, we have

$$
\xi_i = \frac{1}{2}\sum_{a,b}(\nabla_i J_{ab})\,\rho^*_{\bar{b}\bar{a}} - \frac{1}{2}\sum_{a,b,t} R_{t\bar{i}ab}\nabla_t J_{ab}\,. \tag{2.42}
$$

From (2.42), taking account of (2.3), (2.17), (2.30)~(2.32), we have

$$
\begin{aligned}
\xi_1 &= \frac{3}{\sqrt{2}}(\alpha_1\rho^*_{14} - \alpha_2\rho^*_{13})\,,\\
\xi_2 &= \frac{3}{\sqrt{2}}(\alpha_1\rho^*_{13} + \alpha_2\rho^*_{14})\,,\\
\xi_3 &= \frac{1}{\sqrt{2}}\left\{-\alpha_1(A_{14}-A_{41}) + \alpha_2\left(A_{13}-A_{31}+\frac{\tau^*-\tau}{4}\right)\right\},\\
\xi_4 &= \frac{1}{\sqrt{2}}\left\{\alpha_1\left(A_{13}-A_{31}+\frac{\tau^*-\tau}{4}\right) + \alpha_2\,(A_{14}-A_{41})\right\}.
\end{aligned}
\tag{2.43}
$$

3. Proof of the Main Theorem

Let $M = (M, J, g)$ be a four-dimensional strictly almost Kähler Einstein and weakly $*$-Einstein manifold. Then, $M_0 = \{p \in M \mid \tau^* - \tau > 0 \text{ at } p\}$ is a non-empty open submanifold of M. We denote by $\mathcal{D}$ the 2-dimensional J-invariant smooth distribution on M_0 spanned by $\{\alpha^*, J\alpha^*\}$ $(\iota(\alpha^*) = \alpha)$ and by $\mathcal{D}^\perp$ the orthogonal complement of $\mathcal{D}$ in TM. Let $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ be any local smooth unitary frame field on a neighborhood of any point of M_0 such that $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are local frame field of $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. Then, since $\alpha_3 = \alpha_4 = 0$, taking account of (2.21) and (2.33), we see that the equalities (2.36)~(2.40) reduce respectively to

$$\begin{aligned}
w &= \frac{1}{\sqrt{2}}\{-\Gamma_{131}\alpha_1 - \Gamma_{132}\alpha_2 - \nabla_3\alpha_1 + \alpha_2(\Gamma_{312} + \Gamma_{334})\}, \\
w &= \frac{1}{\sqrt{2}}\{\Gamma_{241}\alpha_1 + \Gamma_{242}\alpha_2 + \nabla_4\alpha_2 + \alpha_1(\Gamma_{412} + \Gamma_{434})\}, \\
w &= -\frac{1}{\sqrt{2}}\{\Gamma_{131}\alpha_1 + \Gamma_{132}\alpha_2 - \nabla_4\alpha_2 - \alpha_1(\Gamma_{412} + \Gamma_{434})\}, \\
w &= -\frac{1}{\sqrt{2}}\{-\Gamma_{241}\alpha_1 - \Gamma_{242}\alpha_2 + \nabla_3\alpha_1 - \alpha_2(\Gamma_{312} + \Gamma_{334})\},
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
u &= \frac{1}{\sqrt{2}}\left\{\Gamma_{141}\alpha_1 + \Gamma_{142}\alpha_2 + \nabla_3\alpha_2 - \frac{\tau^* - \tau}{2\sqrt{2}} + \alpha_1(\Gamma_{312} + \Gamma_{334})\right\}, \\
u &= \frac{1}{\sqrt{2}}\left\{\Gamma_{231}\alpha_1 + \Gamma_{232}\alpha_2 + \nabla_4\alpha_1 - \frac{\tau^* - \tau}{2\sqrt{2}} - \alpha_2(\Gamma_{412} + \Gamma_{434})\right\},
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
v &= \frac{1}{\sqrt{2}}\{-\Gamma_{141}\alpha_1 - \Gamma_{142}\alpha_2 - \nabla_4\alpha_1 + \alpha_2(\Gamma_{412} + \Gamma_{434})\}, \\
v &= \frac{1}{\sqrt{2}}\{-\Gamma_{231}\alpha_1 - \Gamma_{232}\alpha_2 - \nabla_3\alpha_2 - \alpha_1(\Gamma_{312} + \Gamma_{334})\},
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
0 = \rho^*_{13} &= \frac{1}{\sqrt{2}}\{(\Gamma_{431} - \Gamma_{341})\alpha_1 + (\Gamma_{432} - \Gamma_{342})\alpha_2\}, \\
0 = \rho^*_{14} &= \frac{1}{\sqrt{2}}\{(\Gamma_{342} - \Gamma_{432})\alpha_1 - (\Gamma_{341} - \Gamma_{431})\alpha_2\}.
\end{aligned} \tag{3.4}$$

From (3.4), we see that the distribution $\mathcal{D}^\perp$ is integrable. By $(3.1)_{2,3}$, we get

$$(\Gamma_{131} + \Gamma_{241})\alpha_1 + (\Gamma_{132} + \Gamma_{242})\alpha_2 = 0. \tag{3.5}$$

By $(3.2)_2$ and $(3.3)_1$, we get

$$u + v = \frac{1}{\sqrt{2}}\left\{-(\Gamma_{141} - \Gamma_{231})\alpha_1 - (\Gamma_{142} - \Gamma_{232})\alpha_2 - \frac{\tau^* - \tau}{2\sqrt{2}}\right\}. \tag{3.6}$$

Similarly, by $(3.2)_1$ and $(3.3)_2$, we get also

$$u + v = \frac{1}{\sqrt{2}}\left\{(\Gamma_{141} - \Gamma_{231})\alpha_1 + (\Gamma_{142} - \Gamma_{232})\alpha_2 - \frac{\tau^* - \tau}{2\sqrt{2}}\right\}. \tag{3.7}$$

Thus, by (3.6) and (3.7), we have

$$(\Gamma_{141} - \Gamma_{231})\,\alpha_1 + (\Gamma_{142} - \Gamma_{232})\,\alpha_2 = 0\,. \tag{3.8}$$

On one hand, from (2.21), we have

$$\begin{aligned} \Gamma_{142} - \Gamma_{131} &= \frac{\alpha_2}{\sqrt{2}}, \quad \Gamma_{242} - \Gamma_{231} = -\frac{\alpha_1}{\sqrt{2}} \\ \Gamma_{132} + \Gamma_{141} &= \frac{\alpha_1}{\sqrt{2}}, \quad \Gamma_{232} + \Gamma_{241} = \frac{\alpha_2}{\sqrt{2}}\,. \end{aligned} \tag{3.9}$$

Thus, from (3.5) and (3.8), taking account of (2.18) and (3.9), we have

$$\begin{aligned} &(\Gamma_{131} - \Gamma_{232})\,\alpha_1 + (\Gamma_{132} + \Gamma_{231})\,\alpha_2 = 0 \\ &-(\Gamma_{132} + \Gamma_{231})\,\alpha_1 + (\Gamma_{131} - \Gamma_{232})\,\alpha_2 = -\frac{\tau^* - \tau}{2\sqrt{2}}\,. \end{aligned} \tag{3.10}$$

From (2.18) and (3.10), since $(\alpha_1, \alpha_2) \neq (0,0)$, we have

$$\Gamma_{131} - \Gamma_{232} = -\frac{\alpha_2}{\sqrt{2}}, \quad \Gamma_{132} + \Gamma_{231} = \frac{\alpha_1}{\sqrt{2}}\,. \tag{3.11}$$

Since M is a weakly $*$-Einstein manifold, we may easily observe that the first Chern form γ of M takes of the following form:

$$8\pi\gamma = \tau e^1 \wedge e^2 + \tau^* e^3 \wedge e^4 \tag{3.12}$$

(cf. [18], p. 151). Now, we may choose a local smooth unitary frame field $\{e_i\}$ on a neighborhood of any point of M_0 with the property that $\mathcal{D} = \mathrm{span}\,\{e_1, e_2 = Je_1\}$, $\mathcal{D}^\perp = \mathrm{span}\,\{e_3, e_4 = Je_3\}$ and further

$$\alpha = \|\alpha\| e^1\,, \qquad J\alpha = \|\alpha\| e^2\,,$$

$$\nabla\Omega = \|\alpha\| \left\{ e^1 \otimes \frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4) - e^2 \otimes \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3) \right\}$$

hold (cf. [15], p. 108). Then, from (3.9) and (3.11), with respect to this unitary frame field $\{e_i\}$, we have

$$\begin{aligned} &\Gamma_{142} = \Gamma_{131} = \Gamma_{232} = -\Gamma_{241} \\ &\Gamma_{132} + \Gamma_{231} = \frac{\|\alpha\|}{\sqrt{2}}, \qquad \Gamma_{231} = \Gamma_{141}\,, \\ &\Gamma_{141} + \Gamma_{242} = \Gamma_{231} - \Gamma_{132}\,. \end{aligned} \tag{3.13}$$

Since the form γ is closed, taking account of (2.20) and (3.13), by direct calculation, we have

$$\Gamma_{131} + \Gamma_{232} = 0\,, \qquad \Gamma_{132} - \Gamma_{231} = 0\,, \qquad e_1\tau^* = e_2\tau^* = 0\,. \tag{3.14}$$

Thus, from (3.13) and (3.14), we have

$$\begin{aligned}
&\Gamma_{131} = \Gamma_{232} = \Gamma_{142} = \Gamma_{241} = 0\,, \\
&\Gamma_{132} = \Gamma_{231} = \Gamma_{141} = -\Gamma_{242} = \frac{\|\alpha\|}{2\sqrt{2}}\,.
\end{aligned} \tag{3.15}$$

Taking account of (3.15), we may note that the distribution $\mathcal{D}$ is also integrable. Since $\rho^*_{ij} = 0$ $(i \neq j)$ and $e_1\tau^* = e_2\tau^* = 0$, by (2.39) and (2.40), we have

$$2\Gamma_{112} + \Gamma_{134} = 0\,, \qquad 2\Gamma_{212} + \Gamma_{234} = 0\,. \tag{3.16}$$

From (3.2), (3.3) and (3.15), we have

$$e_4\|\alpha\| = \sqrt{2}\,u + \frac{\tau^* - \tau}{4\sqrt{2}} = -\sqrt{2}\,v - \frac{\tau^* - \tau}{4\sqrt{2}} = \|\alpha\|\,(2\Gamma_{312} + \Gamma_{334})\,. \tag{3.17}$$

Similarly, from $(3.1)_{1,2}$ and (3.15), we have

$$e_3\|\alpha\| = -\sqrt{2}\,w = -\|\alpha\|\,(2\Gamma_{412} + \Gamma_{434})\,. \tag{3.18}$$

From (3.17) and (3.18), taking account of (2,18) and $(2.43)_{1,2}$, we have

$$\begin{aligned}
&e_3\tau^* = -2\sqrt{2}\|\alpha\|\,(A_{14} - A_{41}) = -2(\tau^* - \tau)(2\Gamma_{412} + \Gamma_{434})\,, \\
&e_4\tau^* = 2\sqrt{2}\|\alpha\|\left(A_{13} - A_{31} + \frac{\tau^* - \tau}{4}\right) = 2(\tau^* - \tau)(2\Gamma_{312} + \Gamma_{334})\,.
\end{aligned} \tag{3.19}$$

Now, let $K = \{A_{13} - A_{31} + (\tau^* - \tau)/4\}^2 + (A_{14} - A_{41})^2 = (\tau^* - \tau)\{(2\Gamma_{412} + \Gamma_{434})^2 + (2\Gamma_{312} + \Gamma_{334})^2\}$ be the smooth function on M which is introduced in the previous paper ([15], p. 105). From (2.18) and (3.15), we have

$$\begin{aligned}
R_{1212} &= e_1\Gamma_{212} - e_2\Gamma_{112} + \Gamma_{112}^2 + \Gamma_{212}^2 + \Gamma_{213}\Gamma_{132} - \Gamma_{114}\Gamma_{242} \\
&= e_1\Gamma_{212} - e_2\Gamma_{112} + \Gamma_{112}^2 + \Gamma_{212}^2 - \frac{\tau^* - \tau}{8}\,.
\end{aligned} \tag{3.20}$$

Thus, taking account of (3.16) and (3.20), we have

$$\begin{aligned}
R_{1234} &= e_1\Gamma_{234} - e_2\Gamma_{134} + \Gamma_{231}\Gamma_{114} - \Gamma_{132}\Gamma_{224} - \Gamma_{121}\Gamma_{134} + \Gamma_{212}\Gamma_{234} \\
&= -\frac{\tau^* - \tau}{8} - 2\,(e_1\Gamma_{212} - e_2\Gamma_{112} + \Gamma_{112}^2 + \Gamma_{212}^2) \\
&= -\frac{\tau^* - \tau}{8} - 2\left(R_{1212} + \frac{\tau^* - \tau}{8}\right) \\
&= -2R_{1212} - \frac{3}{8}(\tau^* - \tau)\,.
\end{aligned} \tag{3.21}$$

Thus, by (3.20) and (3.21), we have

$$-\frac{\tau^*}{4} = R_{1234} + R_{1212} = -R_{1212} - \frac{3}{8}(\tau^* - \tau)\,,$$

and hence

$$R_{1212} = \frac{\tau^*}{4} - \frac{3}{8}(\tau^* - \tau) = -\frac{\tau^* - 3\tau}{8}. \tag{3.22}$$

By (3.22), we have also

$$R_{1234} = -\frac{\tau^*}{4} + \frac{\tau^* - 3\tau}{8} = -\frac{\tau^* + 3\tau}{8}. \tag{3.23}$$

From (2.21) and (3.15), we have further

$$\begin{aligned} R_{3412} &= e_3\Gamma_{412} - e_4\Gamma_{312} + \Gamma_{431}\Gamma_{323} + \Gamma_{414}\Gamma_{342} \\ &\quad - \Gamma_{313}\Gamma_{432} - \Gamma_{341}\Gamma_{424} - \Gamma_{342}\Gamma_{312} + \Gamma_{434}\Gamma_{412} \\ &= e_3\Gamma_{412} - e_4\Gamma_{312} + 2H + \Gamma_{334}\Gamma_{312} - \Gamma_{434}\Gamma_{412}, \end{aligned} \tag{3.24}$$

where $H = \Gamma_{341}^2 + \Gamma_{342}^2$. Similarly, we have

$$\begin{aligned} R_{3434} &= e_3\Gamma_{434} - e_4\Gamma_{334} + \Gamma_{431}\Gamma_{314} + \Gamma_{432}\Gamma_{324} \\ &\quad - \Gamma_{331}\Gamma_{414} - \Gamma_{332}\Gamma_{424} + \Gamma_{334}^2 + \Gamma_{434}^2 \\ &= e_3\Gamma_{434} - e_4\Gamma_{334} - 2H + \Gamma_{334}^2 + \Gamma_{434}^2. \end{aligned} \tag{3.25}$$

We here recall the following formula

$$\Delta\tau^* = 4K + \frac{1}{4}(\tau^* - \tau)(3\tau^* - \tau) \tag{3.26}$$

(cf. [15], p. 106). Now, we shall calculate $\Delta\tau^*$ bye making use of the formulas (3.14), (3.15), (3.16) and (3.19), we get

$$\begin{aligned} \Delta\tau^* &= e_3(e_3\tau^*) + e_4(e_4\tau^*) - \sum_i \Gamma_{ii3}e_3\tau^* - \sum_i \Gamma_{ii4}e_4\tau^* \\ &= -2(e_3(\tau^* - \tau))(2\Gamma_{412} + \Gamma_{434}) - 2(\tau^* - \tau)(2e_3\Gamma_{412} + e_3\Gamma_{434}) \\ &\quad + 2(e_4(\tau^* - \tau))(2\Gamma_{312} + \Gamma_{334}) + 2(\tau^* - \tau)(2e_4\Gamma_{312} + e_4\Gamma_{334}) \\ &\quad - 2\Gamma_{434}(\tau^* - \tau)(2\Gamma_{412} + \Gamma_{434}) - 2\Gamma_{334}(\tau^* - \tau)(2\Gamma_{312} + \Gamma_{334}) \\ &= 4(\tau^* - \tau)(2\Gamma_{412} + \Gamma_{434})^2 + 4(\tau^* - \tau)(2\Gamma_{312} + \Gamma_{334})^2 \\ &\quad - 2(\tau^* - \tau)\{2e_3\Gamma_{412} - 2e_4\Gamma_{312} + e_3\Gamma_{434} - e_4\Gamma_{334} \\ &\quad + \Gamma_{434}(2\Gamma_{412} + \Gamma_{434}) + \Gamma_{334}(2\Gamma_{312} + \Gamma_{334})\} \end{aligned} \tag{3.27}$$

From (3.27), taking account of (3.22)~(3.25), we have

$$\begin{aligned} \Delta\tau^* &= 4K - 2(\tau^* - \tau)\left(-\frac{\tau^*}{4} + R_{1234} - 2H\right) \\ &= 4K + \frac{3}{4}(\tau^* - \tau)(\tau^* + \tau) + 4(\tau^* - \tau)H. \end{aligned} \tag{3.28}$$

Thus, from (3.26) and (3.28), we have

$$4(\tau^* - \tau)H = \frac{1}{4}(\tau^* - \tau)(3\tau^* - \tau) = -\tau(\tau^* - \tau),$$

and hence

$$H = -\frac{\tau}{4}. \tag{3.29}$$

Since $H \geq 0$, we see that $\tau \leq 0$. To prove Main Theorem, we assume that $\tau < 0$. We also recall the following formula

$$\|\mathrm{grad}\,(\tau^* - \tau)\|^2 = 4(\tau^* - \tau)K \tag{3.30}$$

(cf. [15], p. 106). From (2.21), (3.15), (3.16) and (3.18), we have

$$\begin{aligned} R_{2313} &= e_2\Gamma_{313} + e_3\Gamma_{231} + \Gamma_{314}\Gamma_{243} - \Gamma_{212}\Gamma_{323} \\ &\quad - \Gamma_{234}\Gamma_{413} + \Gamma_{323}\Gamma_{313} + \Gamma_{324}\Gamma_{413} \\ &= e_2\Gamma_{313} + 5\Gamma_{212}\Gamma_{314} - \frac{w}{2}, \end{aligned} \tag{3.31}$$

$$\begin{aligned} R_{1314} &= e_1\Gamma_{314} - e_3\Gamma_{114} + \Gamma_{313}\Gamma_{134} - \Gamma_{112}\Gamma_{324} \\ &\quad - \Gamma_{134}\Gamma_{414} + \Gamma_{313}\Gamma_{314} + \Gamma_{314}\Gamma_{414} \\ &= e_1\Gamma_{314} - 5\Gamma_{112}\Gamma_{313} - \frac{w}{2}. \end{aligned} \tag{3.32}$$

Thus, by (3.31), (3.32) and (2.35), we have

$$e_1\Gamma_{314} + e_2\Gamma_{313} - 5\Gamma_{112}\Gamma_{313} + 5\Gamma_{212}\Gamma_{314} = 0. \tag{3.33}$$

Similarly, we have

$$R_{1313} = e_1\Gamma_{313} + \frac{\tau^* - \tau}{16} + H - \frac{\|\alpha\|}{2\sqrt{2}}(2\Gamma_{312} + \Gamma_{334}) + 5\Gamma_{112}\Gamma_{314}, \tag{3.34}$$

$$R_{2424} = e_2\Gamma_{314} + \frac{\tau^* - \tau}{16} + H - \frac{\|\alpha\|}{2\sqrt{2}}(2\Gamma_{312} + \Gamma_{334}) - 5\Gamma_{212}\Gamma_{313}. \tag{3.35}$$

Since $R_{1313} = R_{2424}$, from (3.34) and (3.35), we have

$$e_1\Gamma_{313} - e_2\Gamma_{314} + 5\Gamma_{112}\Gamma_{314} + 5\Gamma_{212}\Gamma_{313} = 0. \tag{3.36}$$

Taking accouont of (3.29), we may set

$$\Gamma_{313} = \lambda\cos\xi, \qquad \Gamma_{314} = \lambda\sin\xi, \tag{3.37}$$

for some local smooth function ξ. From (3.33) and (3.36), we have respectively

$$\lambda(e_1\xi)\cos\xi - \lambda(e_2\xi)\sin\xi = 5\lambda\Gamma_{112}\cos\xi - 5\lambda\Gamma_{212}\sin\xi, \tag{3.38}$$

$$-\lambda(e_1\xi)\sin\xi - \lambda(e_2\xi)\cos\xi = -5\lambda\Gamma_{112}\sin\xi - 5\lambda\Gamma_{212}\cos\xi. \tag{3.39}$$

Thus, by (3.38) and (3.39), we have

$$e_1 \xi = 5\Gamma_{112}\,, \qquad e_2 \xi = 5\Gamma_{212}\,. \tag{3.40}$$

Since $\mathcal{D}$ is integrable, taking account of (3.40), we have

$$[e_1, e_2]\,\xi = \Gamma_{121}\, e_1\xi - \Gamma_{212}\, e_2\xi = -5(\Gamma_{112}^2 + \Gamma_{212}^2)\,. \tag{3.41}$$

On one hand, by (3.40), we have also

$$[e_1, e_2]\,\xi = e_1(e_2\,\xi) - e_2(e_1\,\xi) = 5e_1\Gamma_{212} - 5e_2\Gamma_{212}\,. \tag{3.42}$$

Thus, by (3.20), (3.22), (3.41) and (3.42), we have

$$0 = 5\big(e_1\Gamma_{212} - e_2\Gamma_{112} + \Gamma_{112}^2 + \Gamma_{212}^2\big) = 5\left(R_{1212} + \frac{\tau^* - \tau}{8}\right) = \frac{5}{4}\tau\,.$$

This contradicts to the assumption $\tau < 0$. Therefore, we conclude $\tau = 0$ and hence M is Ricci-flat. Since $\tau = 0$, (3.22) and (3.23) reduce to

$$R_{1212} = R_{3434} = R_{1234} = -\frac{\tau^*}{8}\,. \tag{3.43}$$

Further, from (3.29) and (4.7)$\sim$(4.9) in [15], we have

$$\Gamma_{331} = \Gamma_{332} = \Gamma_{341} = \Gamma_{342} = \Gamma_{431} = \Gamma_{432} = \Gamma_{441} = \Gamma_{442} = 0\,. \tag{3.44}$$

From (3.15), (3.17), (3.34) and (3.44), we have

$$R_{1313} = \frac{\tau^*}{16} - \frac{\|\alpha\|}{2\sqrt{2}}(2\Gamma_{312} + \Gamma_{334}) = \frac{\tau^*}{16} - \left(\frac{u}{2} + \frac{\tau^*}{16}\right) = -\frac{u}{2}\,, \tag{3.45}$$

$$R_{1414} = e_4\Gamma_{141} + \Gamma_{141}^2 = \frac{1}{2\sqrt{2}}e_4\|\alpha\| + \frac{\tau^*}{16} = \frac{u}{2} + \frac{\tau^*}{8}\,. \tag{3.46}$$

Thus, from (2.35), (3.17), (3.45) and (3.46), we have

$$R_{1324} = u + R_{1313} = \frac{u}{2}\,, \tag{3.47}$$

$$R_{1423} = -v - R_{1414} = u + \frac{\tau^*}{4} - \left(\frac{u}{2} + \frac{\tau^*}{8}\right) = \frac{u}{2} + \frac{\tau^*}{8}\,. \tag{3.48}$$

Since $\|\alpha\|^2 = \tau^*/2$ and (3.14), we see $e_1\|\alpha\| = e_2\|\alpha\| = 0$. Thus, from (3.15) and (3.16), we have

$$\begin{aligned} R_{1224}(= -R_{1334}) &= e_1\Gamma_{224} + \Gamma_{123}\Gamma_{243} - \Gamma_{141}\Gamma_{221} + \Gamma_{212}\Gamma_{224} \\ &= \frac{1}{2\sqrt{2}}e_1\|\alpha\| + \frac{\|\alpha\|}{2\sqrt{2}}(2\Gamma_{212} + \Gamma_{234}) = 0 \end{aligned} \tag{3.49}$$

$$\begin{aligned} R_{1213}(= -R_{2434}) &= e_1\Gamma_{213} + \Gamma_{114}\Gamma_{234} - \Gamma_{132}\Gamma_{212} + \Gamma_{212}\Gamma_{213} \\ &= -\frac{1}{2\sqrt{2}}e_1\|\alpha\| - \frac{\|\alpha\|}{2\sqrt{2}}(2\Gamma_{212} + \Gamma_{234}) = 0 \end{aligned} \tag{3.50}$$

$$R_{1214}(=-R_{2334})=-e_2\Gamma_{114}+\Gamma_{112}\Gamma_{242}-\Gamma_{143}\Gamma_{213}-\Gamma_{121}\Gamma_{114} \tag{3.51}$$
$$=\frac{1}{2\sqrt{2}}e_2\|\alpha\|-\frac{\|\alpha\|}{2\sqrt{2}}(2\Gamma_{112}+\Gamma_{134})=0$$

$$R_{1223}(=-R_{1434})=-e_2\Gamma_{123}+\Gamma_{121}\Gamma_{231}-\Gamma_{134}\Gamma_{224}+\Gamma_{112}\Gamma_{123} \tag{3.52}$$
$$=\frac{1}{2\sqrt{2}}e_2\|\alpha\|-\frac{\|\alpha\|}{2\sqrt{2}}(2\Gamma_{112}+\Gamma_{134})=0$$

Further, from (3.31), (3.32) and (3.44), we have

$$R_{1323}(=-R_{1424})=-\frac{w}{2}\,, \tag{3.53}$$
$$R_{1314}(=-R_{2324})=-\frac{w}{2}\,. \tag{3.54}$$

On one hand, from (2.47)~(2.51), (2.53), (2.58) and (2.59) in [15], we have

$$A_{24}=-A_{13}\,,\quad A_{42}=-A_{31}\,,\quad A_{32}=A_{41}\,,\quad A_{23}=A_{14}\,, \tag{3.55}$$
$$A_{12}-A_{21}=A_{34}-A_{43}=A_{11}+A_{22}=A_{33}+A_{44}=0\,. \tag{3.56}$$

Further, from (2.61) in [15] and (2.35), we have

$$A_{13}-A_{31}=2u\,,\qquad A_{14}-A_{41}=2w\,. \tag{3.57}$$

Taking account of Proposition 2.3 in [2], we conclude that M is a space of pointwise constant holomorphic sectional curvature $\tau^*/8$. This completes the proof of Main Theorem.

4. Example of Ricci-flat strictry almost-Kähler manifold

In this section, we shall introduce the example of four-dimensional Ricci-flat strictly almost Kähler manifold constructed by P. Nurowski and M. Przanowski ([12]) and discuss it. First, we write down their example. Let M be a four-dimensional real half-space given by $M=\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \mid x_1>0,\ (x_2,x_3,x_4)\in\mathbb{R}^3\}$. We define a Riemannian metric g and almost complex structure J on M respectively by

$$g=(g_{ij})=\begin{pmatrix} x_1 & 0 & 0 & 0\\ 0 & x_1+\dfrac{x_3^2}{4x_1} & -\dfrac{x_2x_3}{4x_1} & \dfrac{x_3}{2x_1}\\ 0 & -\dfrac{x_2x_3}{4x_1} & x_1+\dfrac{x_2^2}{4x_1} & -\dfrac{x_2}{2x_1}\\ 0 & \dfrac{x_3}{2x_1} & -\dfrac{x_2}{2x_1} & \dfrac{1}{x_1}\end{pmatrix}, \tag{4.1}$$

$$J=(J^i_j)=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & \dfrac{x_3}{2x_1} & -\dfrac{x_2}{2x_1} & \dfrac{1}{x_1} \\ 1 & 0 & 0 & 0 \\ \dfrac{x_2}{2} & -x_1-\dfrac{x_3^2}{4x_1} & \dfrac{x_2x_3}{4x_1} & -\dfrac{x_3}{2x_1} \end{pmatrix}, \tag{4.2}$$

where $g_{ij}=g(\partial/\partial x_i,\partial/\partial x_j)$ and $J(\partial/\partial x_j)=\sum_i J^i_j(\partial/\partial x_i)$. Then, we see easily that (J,g) is an almost Hermitian structuer on M and the Kähler form Ω is given by

$$\Omega=-x_1dx_1\wedge dx_3-\frac{x_2}{2}dx_2\wedge dx_3+dx_2\wedge dx_4\,. \tag{4.3}$$

From (4.3), we see immediately that $d\Omega=0$, and hence (M,J,g) is an almost Kähler manifold. Now, we define vector fields e_1,e_2,e_3,e_4 on M respectively by

$$\begin{aligned} e_1&=\frac{1}{\sqrt{x_1}}\frac{\partial}{\partial x_1}, & e_2&=\frac{1}{\sqrt{x_1}}\frac{\partial}{\partial x_3}+\frac{x_2}{2\sqrt{x_1}}\frac{\partial}{\partial x_4},\\ e_3&=\sqrt{x}\frac{\partial}{\partial x_4}, & e_4&=\frac{1}{\sqrt{x_1}}\frac{\partial}{\partial x_2}-\frac{x_3}{2\sqrt{x_1}}\frac{\partial}{\partial x_4}. \end{aligned} \tag{4.4}$$

Then, we see easily that $\{e_i\}_{i=1,2,3,4}$ is a unitary frame field on M with $e_2=Je_1$, $e_4=Je_3$. By straightforward calculation, we have

$$\begin{aligned} R(e_1,e_2)e_1&=-\frac{1}{2x_1^3}\,e_2\,, & R(e_1,e_2)e_3&=-\frac{1}{2x_1^3}\,e_4\,,\\ R(e_3,e_4)e_1&=-\frac{1}{2x_1^3}\,e_2\,, & R(e_3,e_4)e_3&=-\frac{1}{2x_1^3}\,e_4\,,\\ R(e_1,e_3)e_1&=\frac{1}{x_1^3}\,e_3\,, & R(e_1,e_3)e_2&=-\frac{1}{x_1^3}\,e_4\,,\\ R(e_1,e_4)e_1&=-\frac{1}{2x_1^3}\,e_4\,, & R(e_1,e_4)e_2&=-\frac{1}{2x_1^3}\,e_3\,,\\ R(e_2,e_3)e_2&=-\frac{1}{2x_1^3}\,e_3\,, & R(e_2,e_3)e_1&=-\frac{1}{2x_1^3}\,e_4\,,\\ R(e_2,e_4)e_2&=\frac{1}{x_1^3}\,e_4\,, & R(e_2,e_4)e_1&=-\frac{1}{x_1^3}\,e_3\,. \end{aligned} \tag{4.5}$$

From (4.5) and (2.2), we see easily that

$$\rho=0 \qquad \text{and} \qquad \rho^*=\frac{1}{x_1^3}\,g\,, \tag{4.6}$$

and hence (M,J,g) is a strictly almost Kähler Ricc-flat weakly $*$-Einstein manifold with $*$-scalar curvature $\tau^*=4/x_1^3$.

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Department of Mathematical Sciences,
College of Science and Engineering,
Tokyo Denki University,
Saitama, 350-0394,
JAPAN

Department of Mathematics,
Faculty of Science,
Niigata University,
Niigata, 950-2181,
JAPAN

Division of General Education,
Nagaoka National College of Technology,
Nagaoka, Niigata 940-8532,
JAPAN