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REGULARITY OF SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS WITH H^2-INITIAL DATA

By

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Abstract. We consider the regularity estimate for the solution of nonlinear Schrödinger equation with power nonlinearity, where the initial data belongs to H^2. Since $e^{it\Delta}$ has the smoothing effect, the solution that we consider belongs to $H^{\frac{5}{2}}$ locally time in space.

1. Introduction

In this paper, we estimate regularity of local solutions to nonlinear Schrödinger equations with H^2-initial data.

We consider the following equation;

$$i\partial_t u = -\Delta u + F(u), \tag{1}$$

where u is a complex-valued function of $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$, Δ is the Laplacian on $\mathbb{R}^n$, and $F(u) = F \circ u$ is a local nonlinear operator given by a complex-valued function F on $\mathbb{C}$.

Here we consider the following assumptions on the nonlinear term F;

Assumption F1. $F \in C^1(\mathbb{C};\mathbb{C})$, with $F(0) = 0$.

Assumption F2. $|DF(\zeta)| \equiv \max\{|\partial_\zeta F|, |\partial_{\bar{\zeta}} F|\} \leq M|\zeta|^{p-1}$, for $|\zeta| \geq 1, 1 \leq p < \infty$, where $\partial_\zeta = \frac{1}{2}(\partial_\xi - i\partial_\eta), \partial_{\bar{\zeta}} = \frac{1}{2}(\partial_\xi + i\partial_\eta), (\zeta = \xi + i\eta)$.

The Cauchy problems of Eqn.(1) with above assumptions were studied by many authors. In Ginibre and Velo [2], Kato [4], [5], they discussed about local wellposedness in the case that the initial data belongs to H^1, and in Ginibre and Velo [3], Kato [4], [5], about the existence of global solution. Sjölin's result [12] that we noted below was based on the existence of local solutions discussed in

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Kato [4]. In Kato [5], he discussed about local wellposedness in the case that the initial data belongs to H^2, and in Kato [4], [5], Tsutsumi [14], about the existence of global solution. We try to estimate regularity of local solutions to Eqn.(1) with $u(0) = u_0 \in H^2$, which were obtained by Kato [5]. Here let the nonlinear term F be satisfied with the following (F1') instead of (F1), (F2) and the following (F3) for some p, which is two times continuously differentiable.

Assumption F1'. $F \in C^2(\mathbb{C};\mathbb{C})$, with $F(0) = 0$.

Assumption F3. $|D^2F(\zeta)| \leq M|\zeta|^{\max\{p-2,0\}}$, for $|\zeta| \geq 1, 1 \leq p < \infty$, where $|D^2F(\zeta)| \equiv \max\{|\partial_\zeta\partial_\zeta F|, |\partial_\zeta\partial_{\bar\zeta}F|, |\partial_{\bar\zeta}\partial_{\bar\zeta}F|\}$.

In Sjölin [11], he obtained the following inequality; For some $C > 0$, depending on $\phi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |\phi(t,x)(1-\Delta)^{\frac{1}{4}} e^{it\Delta} f|^2 dx dt \leq C\|f\|_{L^2}^2, \quad \forall f \in L^2, \tag{2}$$

This inequality manifests that the free Schrödinger propagator $e^{it\Delta}$ has the smoothing effect which can improve the differentiability property locally in time and space.

Later the similar property for e^{-itH}, where $H = -\Delta + V$ is a self-adjoint operator and $V = V(x)$ or $V(t,x)$ are various scalar potentials, was studied by many authors (Ben-Artzi [1], Ruiz and Vega [9], etc.). In particular, Kato and Yajima [7] obtained the inequality replacing ϕ in (2) by $(1+|x|^2)^{-\frac{1}{4}-\varepsilon}$, $\varepsilon > 0$, and Yajima [15] obtained the similar estimate for the propagators of Schrödinger equations with time dependent magnetic and scalar potentials which may increase at infinity $|x| \to \infty$.

In Sjögren and Sjölin [10], they obtained the extension of (2) in the following form; They defined

$$\mathcal{A} \equiv \{\varphi \in C^\infty(\mathbb{R}^n) \mid \text{There exists } \varepsilon > 0 \text{ such that } |\partial^\alpha \varphi(x)| \leq C_\alpha (1+|x|)^{-1/2-\varepsilon},\ \forall \alpha\},$$

where $\alpha = (\alpha_1, \alpha_2, .., \alpha_n)$ is multi-index, $\partial_j = \partial/\partial x_j$, and $\partial^\alpha = (\partial_1^{\alpha_1}, \partial_2^{\alpha_2}, .., \partial_n^{\alpha_n})$ and introduced mixed Sobolev spaces $H^{r,\rho} = H^{r,\rho}(\mathbb{R}\times\mathbb{R}^n) = (G_r \bigotimes G_\rho) * L^2(\mathbb{R}^{n+1})$, where G_r and G_ρ are Bessel kernels in $\mathbb{R}$ and $\mathbb{R}^n$, respectively. If $r \geq 0$, $\rho \geq 0$, then, for each $\varphi \in \mathcal{A}$, $\psi \in C_0^\infty(\mathbb{R})$,

$$\|\psi\varphi e^{-it\overline{H}} u\|_{H^{r,\rho}} \leq C_{\psi\varphi}\|u\|_{H^{mr+\rho-\frac{1}{2}(m-1)}}, \tag{3}$$

for some $C_{\varphi,\psi} > 0$. Here $\overline{H} = -P + V$, where P is a elliptic operators with constant coefficient which degree is $m \geq 2$ and $V = V(x)$ is a real-valued function in C^∞ with $D^\alpha V$ bounded for every α.

In Sjölin [12], [13] he adapted these estimates for Eqn.(1) with H^1-initial data. The conclusion is as follows;

Theorem A. (Sjölin [13]) *Assume* (F1) *and* (F2) *with* $1 < p < \infty$ *if* $n = 1, 2$ *with* $1 < p < (n+2)/(n-2)$ *if* $n \geq 3$. *And let* $u_0 \in H^1(\mathbb{R}^n)$, $\varphi \in \mathcal{A}$, *and* u *denote the local solution of the Eqn.*(1) *on* $I = [0, T]$ *with* $u(0) = u_0$. *If* $1 \leq n \leq 6$, *then* $\varphi u \in L^2(I; H^{\frac{3}{2}})$. *If* $n \geq 7$ *with* $p < (n-2)/(n-4)$, *then* $\varphi u \in L^2(I; H^{\frac{3}{2}})$.

Remark 1.1. This result was improved in Nakamura [8], that is, for any space-dimension n it holds that $\varphi u \in L^2(I; H^{\frac{3}{2}})$.

In this paper we shall adapt these estimates for Eqn.(1) with H^2-initial values. Then we need (F3) to estimate the second derivative of F. Sjölin [12] obtained the following theorem;

Theorem B. (Sjölin [12]) *Let* $1 \leq n \leq 7$. *Assume* (F1') *and* (F2-3) *with* $1 < p < \infty$ *if* $n = 1, 2$, *with* $1 < p < (n+2)/(n-2)$ *if* $n \geq 3$. *And let* $u_0 \in H^2(\mathbb{R}^n)$, $\varphi \in \mathcal{A}$. *Then the solution* $u \in C(I; H^2)$ *of the Eqn.(1) on* $I = [0, T]$ *with* $u(0) = u_0$ *satisfies* $\varphi u \in L^2(I; H^{\frac{5}{2}})$ *if* $T > 0$ *is sufficiently small.*

In Kato [5], he proved that assume (F1) and if $n \geq 4$, (F2) with $1 \leq p < n/(n-4)$, then there exists a unique solution $u \in C(I; H^2)$ of Eqn.(1) with $u(0) = u_0 \in H^2$. Moreover $\partial_t u \in \mathcal{X} \equiv (\bigcap L^{q,s}) \cap C(I; L^2)$, where $\bigcap$ is the intersection in (q, s) satisfying $1/q + 2/ns = 1/2$ and $1/2 - 1/n < 1/q \leq 1/2$. Here we will apply Sjölin's estimates to the above solution, that is, we will extend Theorem B for the wider range of p.

It is sufficient to consider the only case that $p > 2$, which implies $|D^2 F(\zeta)| \leq M|\zeta|^{p-2}$, for $|\zeta| \geq 1$. Actually, if $1 < p \leq 2$, then the nonlinear term F satisfy s (F2-3) for any $p_1 > 2$.

We obtain the following Theorem. We denote $\partial = (\partial_1, \partial_2, .., \partial_n)$, $\partial^2 = (\partial_j \partial_k)_{j,k=1}^n$. We often use ∂, ∂^2 instead of ∂_j, $\partial_j \partial_k$ for any $j, k = 1, 2, .., n$, respectively.

Theorem. *Let* $1 \leq n \leq 7$. *Assume* (F1') *and if* $n \geq 4$, (F2-3) *with* $1 < p < n/(n-4)$. *And let* $u_0 \in H^2$. *Then the unique solution* $u \in C(I; H^2)$ *of the Eqn.*(1) *with* $u(0) = u_0$ *satisfies the following properties;*

(i) $u, \partial u, \partial^2 u \in \mathcal{X}$.

(ii) $\varphi u \in L^2(I; H^{\frac{5}{2}})$ *for each* $\varphi \in \mathcal{A}$.

Remark 1.2. We consider in $1 \leq n \leq 7$ since the maximum of space-dimension n satisfied with $2 < n/(n-4)$ is $n = 7$.

Remark 1.3. Applying this theorem to Theorem 1 in Yajima [15], we can obtain the regularity estimate of solutions to nonlinear Shrödinger equations with magnetic fields in the case that the initial data belongs to H^2.

Set $I = [0, T]$, for $0 < T < \infty$. For $f \in C^2(\mathbb{C};\mathbb{C})$, that is, f so that Df and $D^2 f$, as defined in (F2-3), exist, we introduce the real linear map f' as follows;

$$f'(\zeta)(\omega) = (\partial_\zeta f)\omega + (\partial_{\bar\zeta} f)\bar\omega, \quad \zeta, \omega \in \mathbb{C}.$$

Define the real bilinear form H_f by

$$H_f(\zeta)(z,w) = (\partial_\zeta\partial_\zeta f)zw + (\partial_\zeta\partial_{\bar\zeta} f)(z\bar w + \bar z w) + (\partial_{\bar\zeta}\partial_{\bar\zeta} f)\bar z\bar w, \quad \zeta, z, w \in \mathbb{C}.$$

We abbreviate $L^p(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n)$ to L^p and H^k, respectively. We denote usual L^p-norm by $\|\ \|_p$. For $I = [0,T]$, put $L^{p,r} = L^r(I; L^p)$, where $1 < p < \infty, 1 \le r \le \infty$, with its norm denoted by

$$\|f\|_{p,r} \equiv \Big(\int_I \|f(t)\|_p^r dt\Big)^{\frac{1}{r}},$$

We denote various constants by C, M, etc. They may differ from line to line.

We now outline the content of this paper. In Section 2, we introduce some geometric notations which were first used in Kato [5], [6], conveniently to estimate the linear and the nonlinear terms in some function spaces. Here, applying formulation of Sobolev's embedding theorem by these notations, we can easily specify the function spaces which give the regularity of the linear and the nonlinear operators in the proof of the regularity estimates. In Section 3, we prove the regularity of the first and the second derivative in space variable of solutions to Eqn.(1) and, in Section 4, we estimate the regularity of solutions multiplying weighted function by applying the method of the geometrical notations.

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2. Geometric notations and preliminary

To simplify the argument, we use Kato's square notations, introduced in [5], [6].

In a square $\Box \equiv (0,1) \times [0,1]$, consider a point $P = (1/p, 1/r)$, $1 < p < \infty$, $1 \le r \le \infty$, and denote $L^{p,r}$ by $L(P)$, and $\| \quad \|_{p,r}$ by $\| \quad : P\|$ or $\| \quad : L(P)\|$. Of course, $L(P)$ depends on the time interval I, but for simplicity we will omit it. And $1/p$, $1/r$ is denoted by $x(P)$, $y(P)$, respectively.

We regard $P \in \Box$ as 2-vector. For the function space $L(P)$, the following properties(i-v) hold(Kato [5], [6]);

(i) If $P + P' = (1,1), P, P' \in \Box$, then $L(P)^* = L(P')$.

(ii)(Hölder's inequality) If $P, Q, P+Q \in \Box$, then

$$\|fg : P+Q\| \le \|f : P\| \times \|g : Q\|. \tag{4}$$

(iii) If $P, kP \in \Box (k > 0)$, then $\|f^k : kP\| = \|f : P\|^k$.

(iv) If $P, Q \in \Box$, then $L(P) \cap L(Q) \subset L(R) \subset L(P) + L(Q)$, for each R on the segment PQ. In fact, the left inclusion relation holds since $L(P)$-norm has the convex property, i.e.

$$\|f : \lambda P + (1-\lambda)Q\| \le \|f : P\|^{\lambda} \|f : Q\|^{1-\lambda}, (0 < \lambda < 1). \tag{5}$$

The right inclusion relation holds by duality.

(v) When I is finite, i.e. $T < \infty$, let $P = (1/p, 1/r) \in \Box$, $Q = (1/p, 1/s) \in \Box$ and $r > s$, that is, $x(P) = x(Q)$ and $y(P) < y(Q)$, then $L(P) \subset L(Q)$ with $\|f : Q\| \le T^{\theta} \|f : P\|$, for $f \in L(P)$, where $\theta = 1/s - 1/r > 0$.

And we often use the following lemma in the proof of Theorem.

Lemma 2.1. (Sobolev's embedding theorem) *Let $P = (1/p, 1/r) \in \Box$, $Q = (1/q, 1/r) \in \Box$, and $k \in \mathbb{N} \cup \{0\}$. If $1/p \ge 1/q \ge \max\{1/p - k/n, 0\}$, then*

$$\|f : Q\| \le C \sum_{|\alpha| \le k} \|\partial^{\alpha} f : P\|, \quad \exists C > 0. \tag{6}$$

We introduce some special points in $\Box$.

$$B = (1/2, 0), C = (1/2 - 1/n, 1/2), \text{ if } , n \ne 1, C = (0, 1/4), \text{ if } n = 1,$$
$$B' = (1/2, 1), C' = (1/2 + 1/n, 1/2), \text{ if } n \ne 1, C' = (1, 3/4), \text{ if } n = 1.$$

And we denote the semi-open segment BC, which is close at B and open at C, by l. And similarly, $B'C'$ by l', that is,

$$l = \{(x,y) | \, x + 2y/n = 1/2, \; 1/2 - 1/n < x \le 1/2\}.$$
$$l' = \{(x,y) | \, x + 2y/n = 1/2 + 2/n, \; 1/2 \le x < 1/2 + 1/n\}.$$

Note that if $P = (1/p, 1/r) \in l$, the dual point $P' = (1/p', 1/r')$ is on the dual segment l'.

Next, we define the following linear operators.

$$(\Gamma\phi)(t) = U(t)\phi = e^{it\Delta}\phi, \qquad t \in I, \tag{7}$$

$$(Gf)(t) = \int_0^t U(t-\tau)f(\tau)d\tau, \qquad t \in I. \tag{8}$$

Using the above notations, we can express Strichartz estimates as follows.

Lemma 2.2. (Kato [5]) *For any point P on l, Γ is a bounded operator from L^2 to $L(P)$. The bound is independent of T, and is uniform for P on any compact subset of l. Here $L(B)$ may be replaced by $\bar{L}(B) \equiv C(I; L^2)$.*

Lemma 2.3. (Kato [5]) *For any point P on l and any point Q on l', G is a bounded operator from $L(Q)$ to $L(P)$. The bound is independent of T, and is uniform for any compact subsets of l' and l. Here $L(B)$ may be replaced by $\bar{L}(B)$.*

For the differential of the nonlinear term F, the following key lemma holds.

Lemma 2.4. *Assume* (F1) *and* (F2) *with $1 < p < \infty$. And assume $u \in L(P)$, $\partial_j u \in L(Q)$, $\partial_i\partial_j u \in L(R)$, $j, k = 1, 2, , , n$, $P, Q, R \in \square$. If $(p-1)P + Q \in \square$, then we have*

$$\partial_j(F(u)) = F'(u)(\partial_j u) \in L((p-1)P + Q). \tag{9}$$

And in the case of $2 < p < \infty$, assume in addition (F3). If $(p-1)P + R, (p-2)P + 2Q \in \square$, then we have

$$\begin{aligned} \partial_j\partial_k(F(u)) &= F'(u)(\partial_j\partial_k u) + H_F(u)(\partial_j u, \partial_k u) \\ &\in L((p-1)P + R) + L((p-2)P + 2Q), \end{aligned} \tag{10}$$

for $j, k = 1, 2, .., n$.

Remark 2.5. $F'(u)$ and $H_F(u)$ are the first and the second Gâteaux derivative, respectively.

Proof. The proof of (9) is in Lemma4.5 in Kato [5]. We shall prove (10). Since we assume (F2-3) with $p > 2$, we have

$$\begin{aligned} |||H_F(u)(\partial_j u, \partial_k u) : (p-2)P + 2Q||| &\le C||| |u|^{p-2}(\partial_j u)(\partial_k u) : (p-2)P + 2Q||| \\ &\le C|||u : P|||^{p-2}|||\partial_j u : Q||| |||\partial_k u : Q|||. \end{aligned}$$

Thus $H_F(u)(\partial_j u, \partial_k u) \in L((p-2)P+2Q)$. And we have $F'(u)(\partial_j\partial_k u) \in L((p-1)P+R)$ by the similar argument of (9). It remains to prove the equality.

It is obvious that $C_0^\infty(I\times\mathbb{R}^n)$ is dense in $L(S)$, for any $S \in \Box$. (see Lemma2.1 in Kato [5]) If $y(P), y(Q), y(R) \neq 0$, then by using Friedrichs mollifiers on I and on $\mathbb{R}^n$, we can construct a sequence $\{u_n\} \in C_0^\infty(I \times \mathbb{R}^n)$ such that $u_n \longrightarrow u$, as $n \to \infty$, in $L(P)$. For u_n, it is easily to show that $\partial_j u_n \longrightarrow \partial_j u$, as $n \to \infty$, in $L(Q)$, and that $\partial_j\partial_k u_n \longrightarrow \partial_j\partial_k u$, as $n \to \infty$, in $L(R)$, for $j, k = 1, 2...n$, respectively. And we have

$$\partial_j\partial_k(F(u_n)) = F'(u_n)(\partial_j\partial_k u_n) + H_F(u_n)(\partial_j u_n, \partial_k u_n). \tag{11}$$

Since $F \in C^2$ we have

$$F'(u_n)(\partial_j\partial_k u_n) \longrightarrow F'(u)(\partial_j\partial_k u), \text{ as } n \to \infty, \text{ in } L((p-1)P+R),$$
$$H_F(u_n)(\partial_j u_n, \partial_k u_n) \longrightarrow H_F(u)(\partial_j u, \partial_k u), \text{ as } n \to \infty, \text{ in } L((p-2)P+2Q).$$

Thus, taking the limit of (11), we can obtain (10) when $y(P), y(Q), y(R) \neq 0$.

If $y(P) = 0$, then there exists a subsequence v_n of u_n such that $v_n(t) \longrightarrow u(t)$, as $n \to \infty$, in L^p, for a.e. $t \in I$, and $\|v_n(t)\|_p \leq C$, for all n and t, where $P = (1/p, 1/r)$. Using this subsequence, we repeat the above argument. When $y(Q) = 0$ or $y(R) = 0$, using similar subsequences, we can obtain (10). We completed the proof of Lemma.

3. Proof of the part (i) of Theorem

We divide the proof of Theorem into two parts. In this section, we shall prove the part (i) of Theorem, the regularity of the derivatives of a local solution to Eqn.(1).

Proof of the part (i) of Theorem. We start to prove in the case of $4 \leq n \leq 7$. First we introduce the following points and consider each corresponding space with an interval $I = [0, T]$, which will be specified later.

$$P = \left(\frac{1}{4} + \frac{1}{4p}, \frac{n}{8}\left(1 - \frac{1}{p}\right)\right) \in l,$$

$$X = L(B) \cap L(P), \text{ norm} : \|u\|_X = \max\{\|u : B\|, \|u : P\|\},$$
$$\bar{X} = \bar{L}(B) \cap L(P) \subset X,$$
$$X' = L(B') + L(P'), \text{ where } P + P' = (1,1) \text{ and } P' \in l',$$
$$\text{norm} : \|f\|_{X'} = \inf\{\|f_1 : B'\| + \|f_2 : P'\| \mid f = f_1 + f_2\},$$

$$Y = \{u, \partial u, \in X\}, \text{ norm} : \|u\|_Y = \max\{\|u\|_X, \|\partial u\|_X, \},$$

In Kato [4], [5], the solution to Eqn.(1) was constructed as a unique fixed point of the integral equation, which is equivalent to Eqn.(1),

$$u = \Phi(u) \equiv \Gamma u_0 - iGF(u), \tag{12}$$

in $Z \equiv \{u | u \in X, \partial_t u \in X, \Delta u \in L(B)\}$ with norm $\|u\|_Z \equiv \max\{\|u\|_X, \|\partial_t u\|_X, ||\Delta u||_{2,\infty}\}$. To obtain the desirable estimate, formally we differentiate Eqn.(12) in space variables. Since Γ and G commute with ∂, we obtain the following equalities;

$$\partial_j u = \Gamma \partial_j u_0 - iG\partial_j(F(u)), \ j = 1, 2, .., n, \tag{13}$$

$$\partial_j \partial_k u = \Gamma \partial_j \partial_k u_0 - iG\partial_j \partial_k(F(u)), \ j, k = 1, 2, .., n. \tag{14}$$

If $\partial(F(u)), \partial^2(F(u)) \in X'$ for each $u \in Z$, then Eqn.(13) and (14) hold and we can conclude (i) of Theorem. So, it is sufficient to prove $\partial(F(u)), \partial^2(F(u)) \in X'$ for each $u \in Z$.

To use the notations introduced in Section 2, we choose two points in $\Box$.

$$P_0 = (\frac{1}{2p}, 0), \quad P_1 = (\frac{1}{4} + \frac{1}{4p}, \frac{n}{8}(1 - \frac{1}{p})) \\ = (p-1)P_0 + P.$$

By Sobolev's embedding theorem, the following inclusion relation hold.

$$Z \subset L^\infty(I; H^2) \subset L(P_0).$$

Since $x(P_1) = x(P')$ and $y(P') > y(P_1)$, there exists a constant $\theta_1 \equiv y(P') - y(P_1) > 0$, then $L(P_1) \subset L(P')$, with $\|f : P'\| \leq T^{\theta_1} \|f : P_1\|$, for $f \in L(P_1)$.

Then we obtain the following lemma.

Lemma 3.1. *Let F be satisfied with the assumptions of Theorem. If $u \in Z$, then $F(u) \in X'$ and $F'(u) \in L(X; X')$.*

Proof. We follow the argument in the proof of Lemma 6.3 in Kato [5].

We begin by writing F in the form

$$F = F_1 + F_p, \quad F_1, F_p \in C^1(\mathbb{C}; \mathbb{C}), \quad F_1(0) = F_p(0) = 0,$$

$$|F_1(\zeta)| \leq M_1 |\zeta|, \qquad |F_1'(\zeta)| \leq M_1,$$

$$|F_p(\zeta)| \leq M_p |\zeta|^p, \qquad |F_p'(\zeta)| \leq M_p |\zeta|^{p-1}.$$

This may be achieved by multiplying F with smooth cut-off function.

We start to estimate $F_1(u)$. Since $x(B) = x(B')$ and $y(B') - y(B) = 1$, we have

$$\| F_1(u) : B' \| \leq T \| F_1(u) : B \| \leq M_1 T \| u : B \| \leq M_1 T \| u \|_X \leq M_1 T \| u \|_Z,$$

for any $u \in Z$. Here we use property(v). For the nonlinear term $F_p(u)$, we have, for $u \in Z$,

$$\begin{aligned} \| F_p(u) : P' \| &\leq T^{\theta_1} \| F_p(u) : P_1 \| \leq M_p T^{\theta_1} \| u^p : (p-1)P_0 + P \| \\ &\leq M_p T^{\theta_1} \| u : P_0 \|^{p-1} \| u : P \| \leq M_p T^{\theta_1} \| u \|_Z^{p-1} \| u \|_X \leq M_p T^{\theta_1} \| u \|_Z^p, \end{aligned}$$

where we use the properties(ii),(iii),(v). Thus we have $F(u) \in X'$.

It is obvious that $F'(u)$ is a real linear map. And we have, similarly as above,

$$\| F_1'(u)v : B' \| \leq T \| F_1'(u)v : B \| \leq M_1 T \| v : B \| \leq M_1 T \| v \|_X, \ u \in Z, \ v \in X,$$

$$\begin{aligned} \| F_p'(u)v : P' \| &\leq T^{\theta_1} \| F_p'(u)v : P_1 \| \leq M_p T^{\theta_1} \| u^{p-1} v : (p-1)P_0 + P \| \\ &\leq M_p T^{\theta_1} \| u : P_0 \|^{p-1} \| v : P \| \leq M_p T^{\theta_1} \| u \|_Z^{p-1} \| v \|_X, \ u \in Z, \ v \in X. \end{aligned}$$

Here we use the properties(ii),(iii),(v). Thus we can prove Lemma 3.1.

Lemma 3.2. *Suppose $4 \leq n \leq 7$. Let F be satisfied with the assumptions of Theorem. If $u \in Z$, then $H_F(u)$ is a real continuous bilinear form from $Y \times Y$ to X'.*

Proof. By definition, it is clear that $H_F(u)$ is real bilinear on $Y \times Y$. We will show that $H_F(u) : Y \times Y \to X'$. By definition of H_F and (F3), the following inequality holds;

$$\begin{aligned} |H_F(u)(v, w)| &\leq M|v||w| + M|u|^{p-2}|v||w| \\ &\equiv A_1 + A_2 \end{aligned} \tag{15}$$

Hence it suffices to estimate A_1 and A_2.

First we estimate A_1. Since $4 \leq n \leq 7$, we can introduce the following points in $\square$.

$$P_2 = (\frac{1}{4}, \frac{n}{2}(\frac{1}{4} - \frac{1}{n})), \quad P_3 = (\frac{1}{4} + \frac{1}{n}, \frac{n}{2}(\frac{1}{4} - \frac{1}{n})) \in \overline{BP},$$

$$\begin{aligned} P_4 &= (\frac{1}{2}, n(\frac{1}{4} - \frac{1}{n})) \\ &= 2P_2. \end{aligned}$$

In particular, $P_4 = B$ when $n = 4$. Then, by Sobolev's embedding theorem, we have, for some $C, C' > 0$,

$$|||v : P_2||| \leq C \sum_{|\alpha|\leq 1} |||\partial^\alpha v : P_3||| \leq C' |||v|||_Y, \; v \in Y, \tag{16}$$

since $P_3 \in \overline{BP}$, that is, $L(P_3) \supset X$. And $x(P_4) = x(B')$ and $y(B') > y(P_4)$ since $4 \leq n \leq 7$. Thus there exists a constant $\theta_2 \equiv y(B') - y(P_4) > 0$, which satisfies

$$\begin{aligned} |||A_1 : B'||| &\leq T^{\theta_2} |||A_1 : P_4||| \leq MT^{\theta_2} |||\,|v||w| : 2P_2||| \\ &= MT^{\theta_2} |||v : P_2||| \, |||w : P_2||| \\ &\leq C'MT^{\theta_2} |||v|||_Y |||w|||_Y, \; v, w \in Y. \end{aligned}$$

It follows that $A_1 \in L(B') \subset X'$ in the case of $4 \leq n \leq 7$.

Next we estimate A_2. First we consider the case of $n = 4, 5$ with $2 < p < 3n/(3n-8)$. Since $1/2 > 1/8 + 3/8p > 1/2 - 1/n$, Sobolev's embedding theorem implies that

$$||v||_{8p/(p+3),\infty} \leq C||v||_{L^\infty(I;H^1)} \leq C|||v|||_Y, \; v \in Y,$$

for some $C > 0$. Now we set the points in $\Box$,

$$P_5 = (\frac{1}{8} + \frac{3}{8p}, 0), \quad P_6 = (\frac{3}{4} - \frac{1}{4p}, 0)$$
$$= (p-2)P_0 + 2P_5.$$

Then $||v||_{8p/(p+3),\infty} = |||v : P_5|||$, and $x(P') = x(P_6)$. Hence there exists a constant $\theta_3 \equiv y(P') - y(P_6) > 0$, which satisfies

$$\begin{aligned} |||A_2 : P'||| &\leq T^{\theta_3} |||A_2 : P_6||| \leq MT^{\theta_3} |||\,|u|^{p-2}|v||w| : (p-2)P_0 + 2P_5||| \\ &\leq MT^{\theta_3} |||u : P_0|||^{p-2} |||v : P_5||| \, |||w : P_5||| \\ &\leq C'MT^{\theta_3} |||u|||_Z^{p-2} |||v|||_Y |||w|||_Y, \; u \in Z, \; v, w \in Y, \end{aligned}$$

for some $C' > 0$.

When $n = 4, 5$ with $3n/(3n-8) \leq p < n/(n-4)$ or $n = 6, 7$, we introduce the following points in $\Box$.

$$P_7 = (\frac{1}{8} + \frac{3}{8p}, \frac{n}{2}(\frac{3}{8} - \frac{3}{8p} - \frac{1}{n})),$$

$$P_8 = (\frac{1}{8} + \frac{3}{8p} + \frac{1}{n}, \frac{n}{2}(\frac{3}{8} - \frac{3}{8p} - \frac{1}{n})) \in \overline{BP},$$

$$\begin{aligned} P_9 &= (\frac{3}{4} - \frac{1}{4p}, n(\frac{3}{8} - \frac{1}{8p} - \frac{1}{n})) \\ &= (p-2)P_0 + 2P_7. \end{aligned}$$

Note that we can define $L(P_8)$ when $n = 6, 7$, since $3n/(3n-8) < 2$. Then, by Sobolev's embedding theorem, we have, for some $C > 0$,

$$\|v : P_7\| \leq C \sum_{|\alpha| \leq 1} \|\partial^\alpha v : P_8\| \leq C' \|v\|_Y, \ v \in Y,$$

since $P_8 \in \overline{BP}$, that is $L(P_8) \supset X$. And $x(P') = x(P_9)$ and $y(P') > y(P_9)$ under the above condition. Thus there exists a constant $\theta_4 \equiv y(P') - y(P_9) > 0$, which satisfies

$$\begin{aligned} \|A_2 : P'\| &\leq T^{\theta_4} \|A_2 : P_9\| \leq M T^{\theta_4} \||u|^{p-2}|v||w| : (p-2)P_0 + 2P_7\| \\ &\leq M T^{\theta_4} \|u : P_0\|^{p-2} \|v : P_7\| \|w : P_7\| \\ &\leq C' M T^{\theta_4} \|u\|_Z^{p-2} \|v\|_Y \|w\|_Y, \ u \in Z, \ v, w \in Y, \end{aligned}$$

for some $C' > 0$. It follows that $A_2 \in L(P') \subset X'$ in the case of $n = 4, 5$ with $3n/(3n-8) \leq p$ or $n = 6, 7$. Thus we can prove Lemma 3.2.

(*Back to the proof of Theorem in the case of* $4 \leq n \leq 7$.) Applying Lemma 2.4 to Eqn.(13) and (14), we have

$$\partial_j u = \Gamma \partial_j u_0 - iGF'(u)(\partial_j u), \ j = 1, 2, .., n, \tag{17}$$

$$\partial_j \partial_k u = \Gamma \partial_j \partial_k u_0 - iG(F'(u)(\partial_j \partial_k u) + H_F(u)(\partial_j u, \partial_k u)), \quad j, k = 1, 2, .., n. \tag{18}$$

By Lemma 3.1-2, we obtain that $\partial u \in Y$ i.e. $\partial u \in X$ and $\partial^2 u \in X$.

(*Proof of Theorem in the case of* $1 \leq n \leq 3$.) We prove Theorem to use the following lemmas instead of Lemma 3.1 and 3.2.

Lemma 3.3. *Let F be satisfied with the assumptions of Theorem. If $u \in L^\infty(I; H^2)$, then $F(u) \in L^{2,1}$ and $F'(u)$ is a real continuous linear map from $L^{2,\infty}$ to $L^{2,1}$.*

Proof. It is a well-known result that $F(u) \in L^{2,1}$ for $u \in L^\infty(I; H^2) \subset L^{2,\infty}$. (see, for example, Lemma4.1 in T.Kato [5].) It is obvious that $F'(u)$ is a linear map by definition. And (F1') implies that there exists a $M > 0$ such that $|DF(u)| \leq M$ for $u \in L^\infty(I; H^2)$. We have

$$||F'(u)v||_{2,1} \leq T||F'(u)v||_{2,\infty} \leq MT||v||_{2,\infty},$$

from which we obtain Lemma 3.3.

Lemma 3.4. *Suppose $1 \le n \le 3$. Let F be satisfied with the assumptions of Theorem. If $u \in L^\infty(I; H^2)$, then $H_F(u)$ is a real continuous bilinear form from $L^\infty(I; H^1) \times L^\infty(I; H^1)$ to $L^{2,1} = L(B')$.*

Proof. As Lemma3.2, it is clear that $H_F(u)$ is real bilinear on $L^\infty(I; H^1) \times L^\infty(I; H^1)$ by definition. Since $1 \le n \le 3$, $L^\infty(I; H^2) \subset L^\infty(I \times \mathbb{R}^n)$ by Sovolev's embedding theorem. Hence there exists a $M > 0$ such that $|D^2 F(u)| \le M$ for $u \in L^\infty(I; H^2)$ as above. Then, by the definition of H_F and (F3), we have the following inequality.

$$|H_F(u)(v, w)| \le M|v||w|, \ u, v \in L^\infty(I; H^1).$$

We estimate $|v||w|$. By Sobolev's embedding theorem, we have the following inequality;

$$||v||_{4,\infty} \le C||v||_{L^\infty(I;H^1)},$$

for some $C > 0$. Then we have

$$\begin{aligned} |||v||w|||_{2,1} &\le T|||v||w|||_{2,\infty} \\ &= T||v||_{4,\infty}||w||_{4,\infty} \\ &\le C^2 T||v||_{L^\infty(I;H^1)}||w||_{L^\infty(I;H^1)}, \ v, w \in L^\infty(I; H^1). \end{aligned}$$

We obtained that $H_F(u)(v, w) \in L^{2,1} = L(B')$ for $u \in L^\infty(I; H^2)$, $v, w \in L^\infty(I; H^1)$ in the case of $1 \le n \le 3$.

(*Back to the proof of Theorem in the case of* $1 \le n \le 3$.) As in the case of $4 \le n \le 7$, we have (13) and (14). By Lemma 3.3-4, we conclude $\partial u, \partial^2 u \in \mathcal{X}$. This completes the proof of the part(i) of Theorem.

4. Proof of the part (ii) of Theorem

In this section, for $\varphi \in \mathcal{A}$ and the solution u of Eqn.(1) we will investigate the regularity of φu.

Proof of the part (ii) of Theorem. Let $\varphi \in \mathcal{A}$. Then $\varphi u = \varphi \Gamma u_0 - i\varphi GF(u)$. Since $||\varphi \Gamma u_0||_{L^2(I;H^{\frac{5}{2}})} \le ||u_0||_{H^2}$ by (3) with $\overline{H} = -\Delta$, $r = 0$, $\rho = 5/2$, it suffices to estimate $||\varphi GF(u)||_{L^2(I;H^{\frac{5}{2}})}$.

It is obvious that if $f \in L^1(I; L^2)$, then $Gf = \Gamma \int_0^t U(-\tau)f(\tau)d\tau$, since $U(t)$ is unitary on L^2. (see, for example, Ginible and Velo [2]) Then, using the following lemma, we have

$$\begin{aligned}||\varphi GF(u)||_{L^2(I;H^{5/2})} &= ||\varphi\Gamma \int_0^t U(-\tau)F(u)(\tau)d\tau||_{L^2(I;H^{5/2})} \\ &\leq ||\varphi\Gamma\Gamma^* F(u)||_{L^2(I;H^{5/2})} \\ &\leq C_{\varphi,T}||\Gamma^* F(u)||_{H^2} \\ &\leq C_{\varphi,T}||F(u)||_{L^1(I;H^2)},\end{aligned}$$

where $\Gamma^* f = \int_0^T U(-\tau)f(\tau)d\tau$, for any $f \in L^{2,1}$, and $\Gamma^* : L^{2,1} \to L^2$ is bounded linear. Note that $U(t)$ is unitary on L^2 and that ∂ commutes with Γ^*. Thus we can obtain the conclusion of Theorem.

It remains to prove the following lemma.

Lemma 4.1. *Let F be satisfied with the assumptions of Theorem. If u is solutions of Eqn.*(1) *as in Theorem,* $F(u) \in L^1(I; H^2)$.

Proof of Lemma 4.1. In the case of $1 \leq n \leq 3$, we have already proved this lemma in the previous section, and we obtained that $F_1(u), F_1'(u)v, A_1 \in L(B') = L^1(I; L^2)$, for $u \in Z$, $v \in Y$, in the previous section. It suffices to prove $F_p(u), F_p'(u)v, A_2 \in L(B') = L^1(I; L^2)$, for $u \in Z$, $v \in Y$ in the case of $4 \leq n \leq 7$. Here A_1 and A_2 is defined in (15).

In the case of $4 \leq n \leq 7$, we concluded that $u \in W \equiv \{\partial u, \partial^2 u \in X\} \cup Z$. Thus we may use $\|u\|_W \equiv \max\{\|u\|_Z, \|\partial u\|_X, \|\partial^2 u\|_X\}$ instead of $\|u\|_Z$.

And we introduce the following points in $\square$.

$$P_{10} = (\frac{1}{4p}, \frac{n}{2}(\frac{1}{4} - \frac{1}{n})).$$

By Sobolev's embedding theorem, we have, for some $C, C' > 0$,

$$\begin{aligned}\|u : P_{10}\| &\leq C \sum_{|\alpha| \leq 1} \|\partial^\alpha u : P_2\| \\ &\leq C' \sum_{|\alpha| \leq 2} \|\partial^\alpha u : P_3\| \leq C'\|u\|_W,\ u \in W,\end{aligned} \tag{19}$$

By $1/4p \geq 1/4 - 1/n$, the first inequality holds. The second inequality holds by (16).

Next, to estimate $F_p(u)$, $F'(u)v$, we set the following point in $\square$;

$$\begin{aligned}P_{11} &= (\frac{1}{2}, \frac{n-4}{8}p + \frac{1}{2} - \frac{n}{8p}) \\ &= (p-1)P_{18} + P \in \square.\end{aligned}$$

By $x(B') = x(P_{11})$ and $y(B') > y(P_{11})$, there exists a constant $\theta_5 \equiv y(B') - y(P_{11}) > 0$, which satisfies

$$\begin{aligned}\|F_p(u) : B'\| &\le T^{\theta_5}\|F_p(u) : P_{11}\| \le M_p T^{\theta_5}\|u^p : (p-1)P_{10} + P\| \\ &\le M_p T^{\theta_5}\|u : P_{10}\|^{p-1}\|u : P\| \\ &\le C' M_p T^{\theta_5}\|u\|_W^{p-1}\|u\|_X \le C' M_p T^{\theta_5}\|u\|_W^p,\ u \in W,\end{aligned}$$

$$\begin{aligned}\|F_p'(u)v : B'\| &\le T^{\theta_5}\|F_p'(u)v : P_{11}\| \le M_p T^{\theta_5}\|u^{p-1}v : (p-1)P_{10} + P\| \\ &\le M_p T^{\theta_5}\|u : P_{10}\|^{p-1}\|v : P\| \\ &\le C' M_p T^{\theta_5}\|u\|_W^{p-1}\|v\|_Y,\ u \in W, v \in Y,\end{aligned}$$

for some $C' > 0$. We obtain $F_p(u), F_p'(u)v \in L(B') = L^1(I; L^2)$, for $u \in W$, $v \in Y$.

To estimate A_2, we introduce the following points in $\Box$.

$$P_{12} = (\frac{1}{8} + \frac{1}{4p}, \frac{n}{2}(\frac{3}{8} - \frac{1}{4p} - \frac{1}{n})),$$

$$P_{13} = (\frac{1}{8} + \frac{1}{4p} + \frac{1}{n}, \frac{n}{2}(\frac{3}{8} - \frac{1}{4p} - \frac{1}{n})) \in \overline{BP},$$

$$\begin{aligned}P_{14} &= (\frac{1}{2}, \frac{n-4}{8}p + \frac{n}{8} - \frac{n}{4p}) \\ &= (p-2)P_{18} + P_{12} \in \Box.\end{aligned}$$

By Sobolev's embedding theorem, we have, for some $C > 0$,

$$\|v : P_{12}\| \le C \sum_{|\alpha|\le 1} \|\partial^\alpha v : P_{13}\| \le C'\|v\|_Y,\ v \in Y,$$

since $P_{13} \in \overline{BP}$, that is, $L(P_{13}) \supset X$. And $x(B') = x(P_{14})$ and $y(B') > y(P_{14})$ under the above condition. Thus there exists a constant $\theta_6 \equiv y(B') - y(P_{14}) > 0$, which satisfies

$$\begin{aligned}\|A_2 : B'\| &\le T^{\theta_6}\|A_2 : P_{14}\| \le M T^{\theta_6}\||u|^{p-2}|v||w| : (p-2)P_{10} + 2P_{12}\| \\ &\le M T^{\theta_6}\|u : P_{10}\|^{p-2}\|v : P_{12}\|\|w : P_{12}\| \\ &\le C' M T^{\theta_6}\|u\|_W^{p-2}\|v\|_Y\|w\|_Y,\ u \in W,\ v, w \in Y,\end{aligned}$$

for some $C' > 0$. Hence we have $A_2 \in L(B') = L^1(I; L^2)$, for $u \in W$, $v, w \in Y$.

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