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NOTE ON THE ASYMPTOTIC BEHAVIOR OF THE EMPIRICAL DISTRIBUTION FUNCTIONS GENERATED BY ABSOLUTELY REGULAR SEQUENCES

By

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Summary. In many problems, we often need the almost sure behavior of the empirical distribution function. In this note, we consider some asymptotic behaviors of the empirical distribution function generated by a strictly stationary absolutely regular sequence of uniform $(0,1)$ random variables.

1. Main results

Let $\{\xi_i; -\infty < i < \infty\}$ be a strictly stationary absolutely regular sequence of uniform $(0,1)$ random variables, i.e., $\{\xi_i\}$ satisfies the condition

$$\beta(n) = E\left\{\sup_{B\in \mathbf{M}_n^\infty} \left|P(B \mid \mathbf{M}_{-\infty}^0) - P(B)\right|\right\} \to 0 \quad (n \to \infty)$$

where $\mathbf{M}_a^b$ denotes the σ-algebra generated by $\xi_a, \ldots, \xi_b$ and each ξ_i is distributed uniformly on $(0,1)$. As usual, we define the empirical distribution function by

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} I\left(\xi_i \le x\right).$$

In the following, $[x]$ denotes the integer r such that $r \le x < r+1$.

Firstly, we prove a theorem which corresponds to a result of [3] in the independent case.

Theorem 1. *Suppose there exist an integer-valued function $m = m_n = m(n)$ of n and a sequence $\{a_n\}$ of positive numbers such that m_n and a_n are*

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monotone nondecreasing, m_{2n}/m_n *is bounded and*

$$m_n = o(n)\ (n \to \infty)\,, \quad \sum_{n=1}^{\infty} n\beta(m_n) < \infty\,, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{m_n}{na_n} < \infty\,. \tag{1}$$

Then

$$P\left(\sup_{0<x<1} \frac{F_n(x)}{x} \geq a_n\ i.o. \right) = 0 \tag{2}$$

and

$$P\left(\sup_{0<x<1} \frac{1-F_n(x)}{1-x} \geq a_n\ i.o. \right) = 0\,. \tag{3}$$

The following theorem corresponds to a results of [1] in the independent case.

Theorem 2. *If there are an integer-valued function* $m = m_n = m(n)$ *of* n *and a sequence* $\{b_n\}$ *of positive numbers such that*

$$\sum_{n=1}^{\infty} \frac{m_n^3}{n^2} < \infty\,, \quad \sum_{n=1}^{\infty} n\beta(m_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} m_n b_{[n/m_n]} < \infty\,, \tag{4}$$

then

$$n\sqrt{b_{[n/m_n]}} \sup_{0<x<1} \left| \frac{x - F_n(x)}{\sqrt{x(1-x)}} \right| \xrightarrow{a.s.} 0\,. \tag{5}$$

The following theorem is an extension of a result in [7].

Theorem 3. *Let* $\{\xi_{i:n}\}$ *be the ordered statistics of* $\xi_1, \ldots, \xi_n$. *Suppose there exist a constant* $\rho (\in (0, \frac{1}{2}))$ *suth that*

$$\sum_{n=1}^{\infty} \beta^{\frac{\rho}{2+\rho}}(n) < \infty\,. \tag{6}$$

Let $\{\lambda_n\}$ *be a sequence of real numbers satisfying*

$$\lambda_n > n^{\gamma} \quad \text{with some } \gamma > \frac{3\rho}{2(1+\rho)} \text{ for large } n\,. \tag{7}$$

Then,

$$P\left(\sup_{0<t\leq\xi_{n-1:n}} \left| \frac{1-t}{1-F_n(t)} \right| > \lambda_n \right) \to 0 \qquad (n \to \infty)\,. \tag{8}$$

2. Proofs

The following is a special case of Lemma in [5].

Lemma A. *Let $\{\xi_i\}$ be a strictly stationary absolutely regular sequence of random variables with mixing coefficient $\beta(n)$. Let $g(x_1,\dots,x_k)$ be a bounded Borel function such that $|g(x_1,\dots,x_k)| \le M$ and let $1 \le i_1 \le \cdots \le i_k$. Let $F^{(1)}$ and $F^{(2)}$ be distribution functions of random vectors $(\xi_{i_1},\dots,\xi_{i_j})$ and $(\xi_{i_{j+1}},\dots,\xi_{j_k})$, respectively. Then*

$$\Big| Eg(\xi_{i_1},\xi_{i_2},\dots,\xi_{i_k}) - \int\cdots\int g(x_1,\dots,x_j,x_{j+1},\dots,x_k)dF^{(1)}(x_1,\dots,x_j)dF^{(2)}(x_{j+1},\dots,x_k)\Big| \le 2M\beta(i_{j+1}-i_j)\,.$$

Proof of Theorem 1. It is enough only to show (2). We use the method of proof in [4], p. 421. Let $p_j = [v^j]$ $(v > 1)$ be fixed. Then, by the third condition in (1) it is easily shown that

$$\sum_{i=1}^{\infty} \frac{m_{p_i}}{a_{p_i}} < \infty\,. \tag{9}$$

Let i be fixed and for each $j (j = 1,\dots,m_{p_i})$ let $k_{i,j}$ be the largest integer such that $j + k_{i,j}m_{p_i} \le p_i$. For the moment to simplify notations we write $m_{p_i} = m_i$ and $p_i = n_i$. Define

$$A_i = \left\{\omega : \max_{n_i<n\le n_{i+1}} \sup_{0<x<1} \frac{F_n(x)}{x} \ge \frac{a_{n_i}}{2}\right\}$$

and note that

$$P\left(\bigcup_{n=1}^{\infty}\left\{\sup_{0<x<1}\frac{F_n(x)}{x} \ge \frac{a_{n_i}}{2}\right\}\right) \le P\left(\bigcup_{i=1}^{\infty}\bigcup_{n=n_{i+1}}^{n_{i+1}}\left\{\sup_{0<x<1}\frac{F_n(x)}{x} \ge \frac{a_{n_i}}{2}\right\}\right) \le \sum_{i=1}^{\infty} P(A_i)\,.$$

Thus, by the Borel-Cantelli lemma, to prove (2) it suffices to show that

$$\sum_{i=1}^{\infty} P(A_i) < \infty\,. \tag{10}$$

Because nF_n is monotone and $\frac{1}{2}m_{i+1}(k_{i+1,j}+1) \le n_{i+1}$, we have

$$\begin{aligned}
&P(A_i)\\
&= P\left(\max_{n_i<n\le n_{i+1}} \sup_{0<x<1} \frac{F_n(x)}{x} \ge \frac{a_{n_i}}{2}\right)\\
&\le P\left(\sup_{0<x<1} \frac{n_{i+1}F_{n_{i+1}}(x)}{x} \ge n_i \frac{a_{n_i}}{2}\right) = P\left(\sup_{0<x<1} \frac{1}{x}\sum_{\ell=1}^{n_{i+1}} I(\xi_\ell \le x) \ge n_i\frac{a_{n_i}}{2}\right)\\
&\le P\left(\sup_{0<x<1} \sum_{j=1}^{m_{i+1}} \frac{k_{i+1,j}+1}{n_{i+1}} \frac{1}{k_{i+1,j}+1} \sum_{r=0}^{k_{i+1,j}} \frac{1}{x} I(\xi_{j+rm_{i+1}} \le x) \ge \frac{n_i}{n_{i+1}}\frac{a_{n_i}}{2}\right)\\
&\le P\left(\frac{2}{m_{i+1}} \sum_{j=1}^{m_{i+1}} \sup_{0<x<1} \frac{1}{k_{i+1,j}+1} \sum_{r=0}^{k_{i+1,j}} \frac{1}{x} I(\xi_{j+rm_{i+1}} \le x) \ge \frac{n_i}{n_{i+1}}\frac{a_{n_i}}{2}\right)\\
&\le \sum_{j=1}^{m_{i+1}} P\left(\sup_{0<x<1} \frac{1}{k_{i+1,j}+1} \sum_{r=0}^{k_{i+1,j}} \frac{1}{x} I(\xi_{j+rm_{i+1}} \le x) \ge \frac{n_i}{n_{i+1}}\frac{a_{n_i}}{4}\right)\\
&\le \sum_{j=1}^{m_{i+1}} K_j\,, \qquad \text{(say)}\,.
\end{aligned}$$

Now, noting $k_{i+1,j} \le k_{i+1,1}$ and applying Lemma A to each K_j $(j = 1, \dots, m_{i+1})$, we have

$$\begin{aligned}
K_j &\le P\left(\frac{1}{k_{i+1,j}+1} \sup_{0<x<1} \sum_{r=0}^{k_{i+1,j}} \frac{1}{x} I(X_r \le x) \ge \frac{n_i}{n_{i+1}}\frac{a_{n_i}}{4}\right) + 2k_{i+1,1}\beta(m_{i+1})\\
&= P\left(\sup_{0<x<1} \frac{\tilde{F}_{k_{i+1,j}}(x)}{x} \ge \frac{n_i}{n_{i+1}}\frac{a_{n_i}}{4}\right) + 2k_{i+1,1}\beta(m_{i+1})
\end{aligned}$$

where $\tilde{F}_\ell(\cdot)$ denotes the empirical distribution function defined by i.i.d. uniform $(0,1)$ random variables $\{X_i, 0 \le i \le \ell\}$. Since $(n_{i+1}/n_i) \le v$ for all i, by the Daniels theorem ([4], p. 345) we have

$$K_j \le \frac{4n_{i+1}}{n_i a_{n_i}} + 2k_{i+1,1}\beta(m_{i+1}) \le \frac{4v}{a_{n_i}} + 2k_{i+1,1}\beta(m_{i+1})$$

and consequently

$$P(A_i) \le \sum_{j=1}^{m_{i+1}} \left\{\frac{4v}{a_{n_i}} + 2k_{i+1,1}\beta(m_{i+1})\right\} < \frac{4vm_{i+1}}{a_{n_i}} + 2n_i\beta(m_{i+1}) \tag{11}$$

Now, (10) follows from (9) and (11). □

Proof of Theorem 2. As in the proof of Theorem 1 for each $j (j = 1, \dots, m)$ let $k_j = k_{j,n}$ be the largest integer such that $j + k_j m \le n$.

We note that

$$P\left(\sup_{0<x<1}\left|\frac{x - F_n(x)}{\sqrt{x(1-x)}}\right| > \epsilon\right) \tag{12}$$

$$= P\left(\sup_{0<x<1}\frac{1}{n\sqrt{x(1-x)}}\left|\sum_{i=1}^{n}\{x - I(\xi_i < x)\}\right| > \epsilon\right)$$

$$\le P\left(\sup_{0<x<1}\frac{1}{n}\sum_{j=1}^{m}\left|\frac{1}{\sqrt{x(1-x)}}\sum_{i=0}^{k_j}\{x - I(\xi_{j+mi} < x)\}\right| > \epsilon\right)$$

$$\le P\left(\sup_{0<x<1}\sum_{j=1}^{m}\frac{k_j+1}{n}\left|\frac{1}{(k_j+1)\sqrt{x(1-x)}}\sum_{i=0}^{k_j}\{x - I(\xi_{j+mi} < x)\}\right| > \epsilon\right)$$

$$\le P\left(\sup_{0<x<1}\frac{2}{m}\sum_{j=1}^{m}\left|\frac{1}{(k_j+1)\sqrt{x(1-x)}}\sum_{i=0}^{k_j}\{x - I(\xi_{j+mi} < x)\}\right| > \epsilon\right)$$

$$\le \sum_{j=1}^{m} P\left(\sup_{0<x<1}\left|\frac{1}{(k_j+1)\sqrt{x(1-x)}}\sum_{i=0}^{k_j}\{x - I(\xi_{j+mi} < x)\}\right| > \frac{\epsilon}{2}\right)$$

$$= \sum_{j=1}^{m} K_j \quad \text{(say)}.$$

Now, we estimate $K_j (j = 1, \dots, m)$. By applying Lemma A we have

$$K_j \le P\left(\sup_{0<x<1}\left|\frac{1}{(k_j+1)\sqrt{x(1-x)}}\sum_{i=0}^{k_j}\{x - I(X_i < x)\}\right| > \frac{\epsilon}{2}\right) + 2k_j\beta(m) \tag{13}$$

$$\le P\left(\sup_{0<x<1}\left|\frac{x - \tilde{F}_{k_j+1}(x)}{\sqrt{x(1-x)}}\right| > \frac{\epsilon}{2}\right) + 2k_j\beta(m)$$

where $\tilde{F}_j(\cdot)$ denotes the empirical distribution function defined by i.i.d. uniform $(0, 1)$ random variables $\{X_i, 0 \le i \le j\}$.

We consider

$$U_i = \sup_{0<x\le\frac{1}{2}}\frac{\tilde{F}_i(x) - x}{\sqrt{x}} \tag{14}$$

and prove that

$$\sum_{n=1}^{\infty}\sum_{j=1}^{m} P\left(k_j\sqrt{b_{k_j}}U_{k_j} > \epsilon\right) < \infty \quad \text{for every } \epsilon (> 0). \tag{15}$$

We use the Csáki method in [1]. Let $k = k_1$ or $k_1 + 1$ and define the following event:

$$C_k = \left\{ U_i \le \frac{\epsilon}{i\sqrt{b_i}},\ i = 1, 2, \ldots, k-1;\ U_k > \frac{\epsilon}{k\sqrt{b_k}} \right\}$$

Then, to prove (15) it suffices to show that

$$\sum_{n=1}^{\infty} \sum_{j=1}^{m_n} P(C_{k_j}) < \infty \quad \text{for every } \epsilon (> 0)\,. \tag{16}$$

Let

$$B_{k,j} = \left\{ {}^{\exists}x(x_{j-1} \le x < x_j) : k\tilde{F}_k(x) > kx + \frac{\epsilon\sqrt{x}}{\sqrt{b_k}},\ (k-1)\tilde{F}_{k-1}(x) < kx + \frac{\epsilon\sqrt{x}}{\sqrt{b_k}} \right\}$$

where x_j is the solution of the equation

$$kx + \frac{\epsilon\sqrt{x}}{\sqrt{b_k}} = j$$

that is,

$$x_j = \frac{4j^2}{4kj + \frac{\epsilon^2}{b_k}\left(1 + \sqrt{1 + \frac{4kjb_k}{\epsilon^2}}\right)}\,. \tag{17}$$

Then

$$\begin{aligned} C_k &\subset \left\{ {}^{\exists}x\left(0 < x \le \frac{1}{2}\right) : \tilde{F}_k(x) > x + \frac{\epsilon\sqrt{x}}{k\sqrt{b_k}},\ \tilde{F}_{k-1}(x) < x + \frac{\epsilon\sqrt{x}}{(k-1)\sqrt{b_k}} \right\} \\ &\subset \left\{ {}^{\exists}x\left(0 < x \le \frac{1}{2}\right) : k\tilde{F}_k(x) > kx + \frac{\epsilon\sqrt{x}}{\sqrt{b_k}},\ (k-1)\tilde{F}_{k-1}(x) < kx + \frac{\epsilon\sqrt{x}}{\sqrt{b_k}} \right\} \\ &= \bigcup_{j=1}^{[\frac{k}{2}]} B_{k,j}\,. \end{aligned}$$

Hence

$$P(C_k) \le \sum_{j=1}^{[\frac{k}{2}]} P(B_{k,j}) = \sum_{j=1}^{L_k} P(B_{k,j}) + \sum_{j=L_k+1}^{[\frac{k}{2}]} P(B_{k,j})$$

where

$$L_k = \left[\frac{\epsilon^2}{64kb_k} \right]\,.$$

Obviously

$$B_{k,j} = \left\{ {}^{\exists}x(x_{j-1} \le x < x_j) : (k-1)\tilde{F}_{k-1}(x) = j-1;\ X_k \le x \right\}\,.$$

Thus

$$P(B_{k,j}) \leq x_j \binom{k-1}{j-1} x_j^{j-1} = \frac{j}{k} \binom{k}{j} x_j^j \leq \frac{j}{k} \cdot \frac{(kx_j)^j}{j!} \, .$$

It can be seen from (17) that

$$x_j < \frac{2j^2 b_j}{\epsilon^2} \, ,$$

and hence

$$P(B_{k,j}) \leq \frac{j}{k} \left(\frac{2kj^2 b_j}{\epsilon^2} \right)^j \frac{1}{j!}$$

and that for $j \leq L_n$

$$\frac{j}{k} \left(\frac{2kj^2 b_k}{\epsilon^2} \right)^j \frac{1}{j!} \leq \frac{2b_k}{\epsilon^2} \left(\frac{1}{2} \right)^{j-1} \, .$$

Therefore

$$\sum_{j=1}^{L_n} P(B_{k,j}) \leq \frac{2b_k}{\epsilon^2} \sum_{j=1}^{L_n} \left(\frac{1}{2} \right)^{j-1} < \frac{4b_k}{\epsilon^2} \, . \tag{18}$$

For $j > L_k$ we use the estimation

$$P(B_{k,j}) \leq \binom{k}{j} x_j^j (1-x_j)^{k-j} < M \exp\left\{ -\frac{1}{2} \frac{(j-kx_j)^2}{j} \right\} \, . \tag{19}$$

It is easy to see that for n large enough

$$\frac{1}{2} \frac{(j-kx_j)^2}{j} > 3 \log k \, ,$$

which implies that for $j > L_k$

$$P(B_{j,k}) < \frac{M}{k^3} \, , \qquad \sum_{j=L_k+1}^{[\frac{k}{2}]} P(B_{k,j}) < \frac{M}{k^2} \, .$$

This, together with (19), gives

$$P(C_k) < \frac{4}{\epsilon^2} b_k + \frac{M}{k^2} \, . \tag{20}$$

Hence, we obtain from (4) that for every $\epsilon (> 0)$

$$\sum_{n=1}^{\infty} \sum_{j=1}^{m_n} P(C_{k_j}) \leq \sum_{n=1}^{\infty} \left\{ \frac{4}{\epsilon^2} m_n b_{k_{1,n}} + \frac{M m_n}{k^2} \right\} < \infty \, . \tag{21}$$

Now, let

$$V_n = \sup_{0<x\le\frac{1}{2}} \frac{x - F_n(x)}{\sqrt{x}}\,. \tag{22}$$

We can see similarly that

$$P\left(V_k > \frac{\epsilon}{k\sqrt{b_k}}\right) \le \sum_{j=0}^{[\frac{k}{2}]} \binom{k}{j} t_j^j (1-t_j)^{k-j} < \frac{M}{k^2} \tag{23}$$

where

$$t_j = \frac{j}{k} + \frac{\epsilon^2}{2k^2 b_k} + \frac{\epsilon}{k\sqrt{b_k}}\sqrt{\frac{\epsilon^2}{k^2 b_k} + \frac{4j}{k}}\,.$$

It follows from (23) that

$$\sum_{n=1}^{\infty}\sum_{j=1}^{m} P(k_j\sqrt{b_k}V_k > \epsilon) < \infty\,. \tag{24}$$

From (12), (13), (15), (24) and the obvious inequality

$$\sup_{0<x\le\frac{1}{2}} \frac{|x - F_n(x)|}{\sqrt{x(1-x)}} \le \sqrt{2} \sup_{0<x\le\frac{1}{2}} \frac{|x - F_n(x)|}{\sqrt{x}}$$

we obtain that

$$\lim_{n\to\infty} n\sqrt{b_k} \sup_{0<x\le\frac{1}{2}} \frac{|x - F_n(x)|}{\sqrt{x(1-x)}} = 0 \quad a.s. \tag{25}$$

The same reasoning yields

$$\lim_{n\to\infty} n\sqrt{b_k} \sup_{\frac{1}{2}\le x<1} \frac{|x - F_n(x)|}{\sqrt{x(1-x)}} = 0 \quad a.s. \tag{26}$$

Hence, (5) is obtained from (25) and (26). □

To prove Theorem 3 we need the following lemma which is a special case of Theorem 5 in [2].

Lemma B. *Let $\{\eta_i\}$ be an absolutely regular sequence of real-valued random variables with mixing coefficients $\beta(n)$. Suppose that $E\eta_i = 0$ and $|\eta_i| \le M_0 < \infty$. Then, for every $\epsilon(> 4mM_0)$*

$$P\left(\left|\sum_{i=1}^{n} \eta_i\right| > \epsilon\right) \le 4\exp\left\{-\frac{\epsilon^2}{64\frac{n}{m}D(n,m) + \frac{8}{3}\epsilon m M_0}\right\} + \frac{4n\beta(m)}{m}$$

where

$$D(n,m) = \sup_{0\le j\le n-1} E\left(\sum_{i=j+1}^{(j+m)\wedge n} \eta_i\right)^2 .$$

Proof of Theorem 3. It suffices to prove (8) assuming $\lambda_n = n^\gamma$, where $\gamma \in (3\rho/\{2(2+\rho)\}, 1/2)$. For each positive integer n, let $m = m_n = [\lambda_n \log^{-2} n]$. Then, we see that

$$\frac{n^2\beta(m)}{m} \le \frac{n^2(n^\gamma \log^{-2} n)^{-\frac{2+\rho}{\rho}}}{n^\gamma \log^{-2} n} = n^2(n^\gamma \log^{-2} n)^{-\frac{2(1+\rho)}{\rho}} \tag{27}$$

$$\le n^2\left(\frac{\log^2 n}{n^{\frac{3\rho}{2(1+\rho)}}}\right)^{\frac{2(1+\rho)}{\rho}} = n^{-1}\log^{\frac{2(1+\rho)}{\rho}} n \to 0 \qquad (n\to\infty),$$

since by (6) $\beta(m) = o(m^{-\frac{2+\rho}{\rho}})$. Set $i_n = [\lambda_n/4] + 2$ and $c_n = (i_n\lambda_n)/n$.

Firstly, we note that for any $i (1 \le i \le n-1)$

$$\sup_{\xi_{n-i-1:n} < t \le \xi_{n-i:n}} \frac{1-t}{1-F_n(t)} \le \frac{1-\xi_{n-i-1:n}}{1-F_n(\xi_{n-i:n})} = \left(\frac{i}{n}\right)^{-1}(1-\xi_{n-i-1:n})$$

and so

$$\left\{\omega : \sup_{\xi_{n-i-1:n} < t \le \xi_{n-i:n}} \frac{1-t}{1-F_n(t)} > \lambda_n\right\}$$

$$\subset \left\{\omega : \left(\frac{i}{n}\right)^{-1}(1-\xi_{n-i-1:n}) > \lambda_n\right\} = \left\{\omega : \xi_{n-i-1:n} < 1 - \frac{i\lambda_n}{n}\right\}.$$

Further, since

$$\{\omega : \xi_{n-i:n} = \xi_k\} = \left\{\omega : \sum_{j=1}^{n} I(\xi_j < \xi_k) = n-i-1\right\},$$

for all $k (1 \le k \le n)$,

$$\bigcup_{i=1}^{i_n-1} \{\omega : \xi_{n-i:n} = \xi_k\} = \left\{\omega : n - i_n \le \sum_{j=1}^{n} I(\xi_j < \xi_k) \le n-2\right\}.$$

Hence, we have

$$P\left(\sup_{\xi_{n-i_n:n}\le t\le \xi_{n-1:n}}\left|\frac{1-t}{1-F_n(t)}\right|>\lambda_n\right) \tag{28}$$
$$\le P\left(\bigcup_{i=1}^{i_n-1}\left\{\xi_{n-i:n}<1-\frac{i\lambda_n}{n}\right\}\right)$$
$$\le P\left(\bigcup_{k=1}^{n}\left\{\xi_k<1-\frac{\lambda_n}{n},\ n-i_n-1\le\sum_{j=1}^{n}I\left(\xi_j<\xi_k\right)\le n-2\right\}\right)$$
$$\le\sum_{k=1}^{n}P\left(\xi_k<1-\frac{\lambda_n}{n},\ n-i_n-p_k-1\le\sum_{|j-k|>m}I\left(\xi_j<\xi_k\right)\le n-2\right)$$

where $p_k=\text{card}\{j:|j-k|\le m\}$. Applying Lemma A

$$L.H.S.\text{ of }(28) \tag{29}$$
$$\le\sum_{k=1}^{n}P\left(X_k<1-\frac{\lambda_n}{n},\ n-i_n-p_k-1\le\sum_{|j-k|>m}I\left(\bar{\xi}_j<X_k\right)\le n-2\right)$$
$$+2n\beta(m)\,.$$

where $\{\bar{\xi}_i\}$ is a copy of $\{\xi_i\}$ and $\{X_k\}$ is a sequence of i.i.d. uniform $(0,1)$ random variables, independent of $\{\bar{\xi}_i\}$. Put

$$S_{n,k}(z)=\begin{cases}\displaystyle\sum_{j=2m+1}^{n}I\left(\bar{\xi}_j<x\right) & (1\le k\le m)\,,\\ \displaystyle\sum_{|j-k|>m}I\left(\bar{\xi}_j<z\right) & (m+1\le k\le n-m)\,,\\ \displaystyle\sum_{j=1}^{n-2m-1}I\left(\bar{\xi}_j<z\right) & (n-m+1\le k\le n)\,,\end{cases}$$

We note that for each fixed k a family of random variables $\{\bar{\xi}_{1\vee(k-m)},\ldots,\bar{\xi}_{(k+m+1)\wedge n}\}$ satisfies the absolute regularity condition with mixing coefficients $\beta_{\bar{\xi}}(j)\le\beta(j)$ and that $ES_{n,k}(z)=(n-2m-1)z$. Further, since

$$\left|\,I\left(\bar{\xi}_i<z\right)-z\right|\le 1\quad(i=1,\ldots,n)\,,$$

it is easily shown that for arbitrary numbers $z\ (\in(0,1))$, $\ell\ (\in\{2m+1,\ldots,n\})$ and $k\ (\in\{1,\ldots,n\})$

$$\text{Var}\,S_{\ell,k}(z)\le M_1\left\|I\left(\xi_1<z\right)-z\right\|_{\frac{2+\rho}{2}}$$
$$\le M_1(\ell-2m)\left\{E|I\left(\xi_1<z\right)-z|\right\}^{\frac{2}{2+\rho}}\le M_1\ell\left\{2z(1-z)\right\}^{\frac{2}{2+\rho}}\,.$$

where M_1 is some positive constant and so

$$D(n,m)(z) = \sup_{0\le j\le n-1} E\left(\sum_{i=j+1}^{(j+m)\wedge n} I(\xi_i < z)\right)^2 \le M_1 m\{2z(1-z)\}^{\frac{2}{2+\rho}}. \quad (30)$$

Put

$$\epsilon(z) = (n-2m-1)(1-z) - i_n, \quad (z \in (0,1)).$$

Then

$$\epsilon(z) > \max\left\{\frac{1}{2}(n-2m-1)(1-z), 4m\right\} \quad \text{for all } z(\in (0, 1-\lambda_n n^{-1})). \quad (31)$$

Hence, using Lemma B (with $M_0 = 1$) and (30) we have

$$\begin{aligned}
&P\left(S_{n,k}(z) \ge n - i_n - p_k - 1\right)\\
&\le P\left(S_{n,k}(z) - ES_{n,k}(z) \ge n - i_n - 2m - 1 - ES_{n,k}(z)\right)\\
&\le P\left(|S_{n,k}(z) - ES_{n,k}(z)| \ge (n-2m-1)(1-z) - i_n\right)\\
&\le 4\exp\left\{-\frac{\epsilon^2(z)}{64\frac{n}{m}D(n,m)(z) + \frac{8}{3}\epsilon(z)m}\right\} + \frac{4(n-2m)\beta(m)}{m}\\
&\le 4\exp\left\{-\frac{\epsilon^2(z)}{64M_1 n\{z(1-z)\}^{\frac{2}{2+\rho}} + \frac{8}{3}\epsilon(z)m}\right\} + \frac{4n\beta(m)}{m}.
\end{aligned}$$

Further, by (7) and (31) we see that for some $\tau(>2)$ and all n sufficiently large

$$\begin{aligned}
&\inf_{0<z<1-\lambda_n n^{-1}} \frac{\epsilon^2(z)}{64M_1 n\{z(1-z)\}^{\frac{2}{2+\rho}} + \frac{8}{3}\epsilon(z)m}\\
&\ge \inf_{0<z<1-\lambda_n n^{-1}} \min\left\{\frac{\epsilon^2(z)}{128M_1 n\{z(1-z)\}^{\frac{2}{2+\rho}}}, \frac{\epsilon(z)}{\frac{16}{3}m}\right\}\\
&\ge \inf_{0<z<1-\lambda_n n^{-1}} \min\left\{\frac{\left(\frac{1}{2}(n-2m-1)(1-z)\right)^2}{128M_1 n\{z(1-z)\}^{\frac{2}{2+\rho}}}, \frac{\frac{1}{4}\lambda_n}{\frac{16}{3}m}\right\} \ge \tau\log n
\end{aligned}$$

and so

$$\sup_{0<z<1-\lambda_n n^{-1}} P\left(S_{n,k}(z) \ge n - i_n - p_k - 1\right) \le 4n^{-\tau} + \frac{4n\beta(m)}{m}. \quad (32)$$

Hence, from (27), (29) and (32) we obtain

$$\begin{aligned}
&L.H.S.\text{ of }(28) && (33)\\
&\le nP\Big(X_1 < 1-\frac{\lambda_n}{n},\ S_{n,k}(X_1)-(n-2m-1)X_1\\
&\qquad \ge (n-2m-1)(1-X_1)-i_n\Big)+\frac{4n^2\beta(m)}{m}\\
&\le n\int^{(1-\frac{\lambda_n}{n})-} P\big(S_{n,k}-(n-2m-1)X_1\\
&\qquad \ge (n-2m-1)(1-X_1)-i_n \mid X_1=z\big)dz+o(1)\\
&\le 4n^{-\tau+1}+o(1)\to 0 \quad \text{as } n\to\infty .
\end{aligned}$$

It remains to show that

$$P\left(\sup_{0<t\le\xi_{n-i_n:n}}\left|\frac{1-t}{1-F_n(t)}\right|>\lambda_n\right)\to 0. \tag{34}$$

Since $nc_n - i_n + 1 \le i_n(\lambda_n - 1)$, by Lemma B we have

$$\begin{aligned}
&P\left(\xi_{n-i_n+1:n}<1-c_n\right) && (35)\\
&=P\left(\sum_{j=1}^{n} I\left(\xi_j<1-c_n\right)\ge n-i_n+1\right)\\
&=P\left(\sum_{j=1}^{n}\left(I\left(\xi_j<1-c_n\right)-(1-c_n)\right)\ge nc_n-i_n+1\right)\to 0,
\end{aligned}$$

(cf. the proof of (32)), and so from (35) we have that as $n\to\infty$

$$\begin{aligned}
L.H.S.\text{ of }(34) &\le P\left(\max_{i_n-1\le i<n}\left(\frac{i}{n}\right)^{-1}(1-\xi_{n-i:n}>\lambda_n\right)\\
&\le P\left(\max_{i_n-1\le i<n}(1-\xi_{n-i:n})>(i_n-1)\frac{\lambda_n}{n}\right)\\
&\le P\left(\max_{i_n-1\le i<n}(1-\xi_{n-i:n})>c_n\right)\\
&\le P\left(1-\xi_{n-i_n+1:n}>c_n\right)\to 0.
\end{aligned}$$

Therefore, the desired conclusion follows. □

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