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REALIZATION OF AUTOMORPHISMS σ OF ORDER 3 AND G^σ OF COMPACT EXCEPTIONAL LIE GROUPS G, II, $G=E_7$

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Abstract. For the simply connected compact exceptional Lie group E_7, we realize all of automorphisms σ of order 3 and determine the group structures the fixed subgroups $(E_7)^\sigma$ of E_7.

Introduction

J.A. Wolf and A. Gray [3] classified automorphisms σ of order 3 and determined the fixed subgroups G^σ of connected compact simple Lie groups G of centerfree. In the previous paper [5], we found these automorphisms σ and realized G^σ for simply connected compact exceptional Lie groups G of type G_2, F_4 and E_6. In this paper, we consider the case of type E_7. Our result is the second column. The first column is the chart of involutive automorphisms and the fixed subgroups.

ι	$(U(1)\times E_6)/\boldsymbol{Z}_3$	ι_3	$(U(1)\times E_6)/\boldsymbol{Z}_3$
$\lambda\gamma$	$SU(8)/\boldsymbol{Z}_2$	λ_3	$S(U(1)\times U(7))/\boldsymbol{Z}_2$
σ	$(SU(2)\times Spin(12))/\boldsymbol{Z}_2$	σ_3	$(SU(2)\times Spin(2)\times Spin(10))/\boldsymbol{Z}_4$
		${\sigma_3}'$	$(U(1)\times Spin(12))/\boldsymbol{Z}_2$
		w	$(SU(3)\times SU(6))/\boldsymbol{Z}_3$

0. Preliminaries

Let $\mathfrak{C}$ be the division Cayley algebra and let $\mathfrak{J}=\{X\in M(3,\mathfrak{C})\,|\,X^*=X\}$ be the exceptional Jordan algebra with the Jordan multiplication $X\circ Y$, the inner product (X,Y) and the Freudenthal multiplication $X\times Y$ defined by

$$X\circ Y=\tfrac{1}{2}(XY+YX),\quad (X,Y)=\mathrm{tr}(X\circ Y),$$
$$X\times Y=\tfrac{1}{2}(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-(X,Y))E)$$

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(E is the 3×3 unit matrix), respectively. Let $\mathfrak{J}^C$ be the complexification of $\mathfrak{J}$ and in $\mathfrak{J}^C$ we define the Hermitian inner product $\langle X,Y\rangle$ by $\langle X,Y\rangle = (\tau X, Y)$. (The complex conjugation in $C = \boldsymbol{R}^C, \mathfrak{C}^C, \mathfrak{J}^C$ or $\mathfrak{P}^C$ (see below) is always denoted by τ). The simply connected compact Lie group E_6 is defined by

$$E_6 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C)P, \,|\, \tau\alpha\tau(X\times Y) = \alpha X \times \alpha Y, \langle \alpha X, \alpha Y\rangle = \langle X,Y\rangle\}$$

and the Lie algebra $\mathfrak{e}_6$ of the group E_6 is given by

$$\mathfrak{e}_6 = \left\{ \phi \in \mathrm{Hom}_C(\mathfrak{J}^C,\mathfrak{J}^C) \,\middle|\, \begin{array}{l} \tau\phi\tau(X\times Y) = \phi X \times Y + X \times \phi Y, \\ \langle \phi X, Y\rangle + \langle X, \phi Y\rangle = 0 \end{array} \right\}.$$

The C-vector space $\mathfrak{P}^C$ is defined by

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C.$$

For $\phi \in {\mathfrak{e}_6}^C, A, B \in \mathfrak{J}^C$ and $\nu \in C$, we define a C-linear mapping $\Phi(\phi, A, B, \nu)$ of $\mathfrak{P}^C$ by

$$\Phi(\phi,A,B,\nu)\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi X - \frac{1}{3}\nu X + 2B\times Y + \eta A \\ 2A \times X - {}^t\phi Y + \frac{1}{3}\nu Y + \xi B \\ (A,Y) + \nu\xi \\ (B,X) - \nu\eta \end{pmatrix},$$

where ${\mathfrak{e}_6}^C$ is the complexification of $\mathfrak{e}_6$ and ${}^t\phi$ is the transposed mapping of ϕ with respect to the inner product (X,Y) : $({}^t\phi X, Y) = (X, \phi Y), X, Y \in \mathfrak{J}^C$. For $P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, we define a C-linear mapping $P \times Q$ of $\mathfrak{P}^C$ by

$$P \times Q = \Phi(\phi, A, B, \nu), \quad \left\{ \begin{array}{l} \phi = -\frac{1}{2}(X \vee W + Z \vee Y) \\ A = -\frac{1}{4}(2Y \times W - \xi Z - \zeta X) \\ B = \frac{1}{4}(2X \times Z - \eta W - \omega Y) \\ \nu = \frac{1}{8}((X,W) + (Z,Y) - 3(\xi\omega + \zeta\eta)), \end{array} \right.$$

where $X \vee W \in {\mathfrak{e}_6}^C$ is defined by

$$X \vee W = [\tilde{X}, \tilde{W}] + (X \circ W - \frac{1}{3}(X,W)E)^\sim,$$

here $\tilde{X} : \mathfrak{J}^C \to \mathfrak{J}^C$ is defined by $\tilde{X}Z = X \circ Z$, $Z \in \mathfrak{J}^C$. Finally we define a Hermitian inner product $\langle P, Q\rangle$ in $\mathfrak{P}^C$ by

$$\langle P, Q\rangle = \langle X, Z\rangle + \langle Y, W\rangle + (\tau\xi)\zeta + (\tau\eta)\omega,$$

where $P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega)$. Now the simply connected compact Lie group E_7 is defined by

$$\begin{aligned} E_7 &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\} \\ &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \tau\lambda\alpha = \alpha\tau\lambda\}, \end{aligned}$$

where λ is the C-linear transformation of $\mathfrak{P}^C$ defined by $\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi)$. The Lie algebra $\mathfrak{e}_7$ of the group E_7 is given by

$$\mathfrak{e}_7 = \{\Phi(\phi, A, -\tau A, \nu) \mid \phi \in \mathfrak{e}_6, A \in \mathfrak{J}^C, \nu \in i\boldsymbol{R}\}.$$

The group E_7 contains E_6 as a subgroup by

$$E_6 = \{\alpha \in E_7 \mid \alpha(0, 0, 1, 0) = (0, 0, 1, 0)\}.$$

1. Automorphism ι_3 of order 3 and subgroup $(\boldsymbol{U}(1) \times \boldsymbol{E}_6)/\boldsymbol{Z}_3$ of $\boldsymbol{E}_7$

Let $U(1) = \{\theta \in C \mid (\tau\theta)\theta = 1\}$ and we define an embedding $\phi : U(1) \to E_7$ by

$$\phi(\theta)(X, Y, \xi, \eta) = (\theta^{-1}X, \theta Y, \theta^3\xi, \theta^{-3}\eta).$$

For $i \in U(1)$ and $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \in U(1)$, let

$$\iota = \phi(i), \quad \iota_3 = \phi(\omega).$$

Then $\iota, \iota_3 \in E_7$ and $\iota^2 = -1, \iota_3{}^3 = 1$. Let $(E_7)^\iota = \{\alpha \in E_7 \mid \iota\alpha = \alpha\iota\}$.

Proposition 1.1. [7]. $(E_7)^\iota \cong (U(1) \times E_6)/\boldsymbol{Z}_3, \boldsymbol{Z}_3 = \{(1, 1), (\omega, \phi(\omega^2)), (\omega^2, \phi(\omega))\}$. *The isomorphism is induced by the homomorphism* $\varphi : U(1) \times E_6 \to (E_7)^\iota, \varphi(\theta, \beta) = \phi(\theta)\beta$.

We shall determine the group structure of

$$(E_7)^{\iota_3} = \{\alpha \in E_7 \mid \iota_3\alpha = \alpha\iota_3\}.$$

We consider C-eigenspaces $(\mathfrak{P}^C)_{\omega^k}, k = 0, 1, 2,$ of $\mathfrak{P}^C$ with respect to ι_3 :

$$\begin{aligned} (\mathfrak{P}^C)_1 &= \{P \in \mathfrak{P}^C \mid \iota_3 P = P\} = \{(0, 0, \xi, \eta) \in \mathfrak{P}^C \mid \xi, \eta \in C\}, \\ (\mathfrak{P}^C)_\omega &= \{P \in \mathfrak{P}^C \mid \iota_3 P = \omega P\} = \{(0, Y, 0, 0) \in \mathfrak{P}^C \mid Y \in \mathfrak{J}^C\}, \\ (\mathfrak{P}^C)_{\omega^2} &= \{P \in \mathfrak{P}^C \mid \iota_3 P = \omega^2 P\} = \{(X, 0, 0, 0) \in \mathfrak{P}^C \mid X \in \mathfrak{J}^C\}. \end{aligned}$$

These spaces are invariant under the action of the group $(E_7)^{\iota_3}$.

Lemma 1.2. *For* $\alpha \in (E_7)^{\iota_3}$, *there exists* $\xi \in U(1)$ *such that* $\alpha(0, 0, 1, 0) = (0, 0, \xi, 0)$.

Proof. Since $(0,0,1,0) \in (\mathfrak{P}^C)_1$, we have $\alpha(0,0,1,0) \in (\mathfrak{P}^C)_1$. Hence let $\alpha(0,0,1,0) = (0,0,\xi,\eta)$ and suppose that $\eta \neq 0$. $\alpha(0,0,1,0) \times \alpha(0,0,1,0) = \alpha((0,0,1,0) \times (0,0,1,0))\alpha^{-1} = 0$. On the other hand, $\alpha(0,0,1,0) \times \alpha(0,0,1,0) = (0,0,\xi,\eta) \times (0,0,\xi,\eta) = \varPhi(0,0,0,-\frac{3}{4}\xi\eta)$. Hence $\xi\eta = 0$, so $\xi = 0$, that is, $\alpha(0,0,1,0) = (0,0,0,\eta)$. Since $(0,E_1,0,0) \in (\mathfrak{P}^C)_\omega$, we have $\alpha(0,E_1,0,0) \in (\mathfrak{P}^C)_\omega$ (E_1 is the usual notation in $\mathfrak{J}^C$ (e.g.[6])). Hence let $\alpha(0,E_1,0,0) = (0,Y,0,0), Y \neq 0$. $\alpha(0,0,1,0) \times \alpha(0,E_1,0,0) = \alpha((0,0,1,0) \times (0,E_1,0,0))\alpha^{-1} = 0$. On the other hand, $\alpha(0,0,1,0) \times \alpha(0,E_1,0,0) = (0,0,0,\eta) \times (0,Y,0,0) = \varPhi(0,0,-\frac{1}{4}\eta Y,0) \neq 0$. This is a contradiction. Therefore $\eta = 0$, that is, $\alpha(0,0,1,0) = (0,0,\xi,0)$. Finally, $|\xi| = 1$ follows from $|\xi|^2 = \langle (0,0,\xi,0),(0,0,\xi,0)\rangle = \langle \alpha(0,0,1,0), \alpha(0,0,1,0)\rangle = \langle (0,0,1,0),(0,0,1,0)\rangle = 1$.

From Lemma 1.2, we see that C-vector subspaces

$$\{(X,0,0,0) \in \mathfrak{P}^C \,|\, X \in \mathfrak{J}^C\}, \quad \{(0,Y,0,0) \in \mathfrak{P}^C \,|\, Y \in \mathfrak{J}^C\},$$
$$\{(0,0,\xi,0) \in \mathfrak{P}^C \,|\, \xi \in C\}, \quad \{(0,0,0,\eta) \in \mathfrak{P}^C \,|\, \eta \in C\}$$

of $\mathfrak{P}^C$ are invariant under the action of the group $(E_7)^{\iota_3}$.

Theorem 1.3. $(E_7)^{\iota_3} \cong (U(1) \times E_6)/\boldsymbol{Z}_3$, $\boldsymbol{Z}_3 = \{(1,1),(\omega,\phi(\omega^2)),(\omega^2,\phi(\omega))\}$.

Proof. From the fact above, $\alpha \in (E_7)^{\iota_3}$ commutes with ι, that is, $(E_7)^{\iota_3} \subset (E_7)^{\iota}$. Conversely, $(E_7)^{\iota} \subset (E_7)^{\iota_3}$ is obvious, because $(E_7)^{\iota} = \phi(U(1))E_6$ (Proposition 1.1). Hence $(E_7)^{\iota_3} = (E_7)^{\iota} \cong (U(1) \times E_6)/\boldsymbol{Z}_3$.

2. Automorphism λ_3 of order 3 and subgroup $\boldsymbol{S}(\boldsymbol{U}(1) \times \boldsymbol{U}(7))/\boldsymbol{Z}_2$ of $\boldsymbol{E}_7$

Let $\{1,e_1,e_2,\cdots,e_7\}$ be the canonical $\boldsymbol{R}$-basis of $\mathfrak{C}$ and the field $\boldsymbol{C}_1$ of complex numbers is embedded in $\mathfrak{C}$ as $\boldsymbol{C}_1 = \{x + ye_1 \,|\, x,y \in \boldsymbol{R}\}$. Now we define a C-linear isomorphism $\chi : \mathfrak{P}^C \to \mathfrak{S}(8,\boldsymbol{C}_1)^C = \{Q \in M(8,\boldsymbol{C}_1)^C \,|\, {}^tQ = -Q\}$ by

$$\chi(X,Y,\xi,\eta) = (k(gX - \frac{\xi}{2}E))J + e_1(k(g(\gamma Y) - \frac{\eta}{2}E))J.$$

(The definitions of k, g, γ and J are found in [7]). Now, we define a mapping $\varphi : SU(8) \to E_7$ by

$$\varphi(A)P = \chi^{-1}(A\chi(P){}^tA), \quad P \in \mathfrak{P}^C.$$

For $A_\epsilon = \mathrm{diag}\,(\epsilon,\epsilon,\epsilon,\epsilon,\epsilon,\epsilon,\epsilon,\epsilon) \in SU(8)$, $\epsilon = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}e_1$ and $A_{\omega_1} = \mathrm{diag}\,(\omega_1^{-1}, \omega_1,\omega_1,\omega_1,\omega_1,\omega_1,\omega_1,\omega_1) \in SU(8)$, $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1$, let

$$\lambda\gamma = \varphi(A_\epsilon), \quad \lambda_3 = \varphi(A_{\omega_1}).$$

Then $\lambda\gamma, \lambda_3 \in E_7$ and $(\lambda\gamma)^2 = -1$, ${\lambda_3}^3 = 1$. Let $(E_7)^{\lambda\gamma} = \{\alpha \in E_7 \mid \lambda\gamma\alpha = \alpha\lambda\gamma\}$.

Proposition 2.1. [7]. *$(E_7)^{\lambda\gamma} \cong SU(8)/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{E, -E\}$. The isomorphism is induced by the homomorphism $\varphi : SU(8) \to (E_7)^{\lambda\gamma}$.*

We shall determine the group structure of

$$(E_7)^{\lambda_3} = \{\alpha \in E_7 \mid \lambda_3\alpha = \alpha\lambda_3\}.$$

The explicit form of λ_3 is given by

$$\lambda_3 \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} {\gamma_3}^{-1}(-\frac{1}{2}X + \frac{\sqrt{3}}{2}Y) + \frac{3}{8}(\mathrm{tr}(X) - \xi)E - \frac{\sqrt{3}}{8}(\mathrm{tr}(Y) - \eta)E \\ {\gamma_3}^{-1}(-\frac{1}{2}Y - \frac{\sqrt{3}}{2}X) + \frac{3}{8}(\mathrm{tr}(Y) - \eta)E + \frac{\sqrt{3}}{8}(\mathrm{tr}(X) - \xi)E \\ -\frac{1}{2}\xi - \frac{3}{8}(\mathrm{tr}(X) - \xi) + \frac{\sqrt{3}}{2}\eta + \frac{\sqrt{3}}{8}(\mathrm{tr}(Y) - \eta) \\ -\frac{1}{2}\eta - \frac{3}{8}(\mathrm{tr}(Y) - \eta) - \frac{\sqrt{3}}{2}\xi - \frac{\sqrt{3}}{8}(\mathrm{tr}(X) - \xi) \end{pmatrix},$$

where γ_3 of the right side is the C-linear transformation of $\mathfrak{J}^C$ defined by

$$\gamma_3 X = \gamma_3 \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \gamma_3(x_3) & \overline{\gamma_3(x_2)} \\ \overline{\gamma_3(x_3)} & \xi_2 & \gamma_3(x_1) \\ \gamma_3(x_2) & \overline{\gamma_3(x_1)} & \xi_3 \end{pmatrix},$$

moreover γ_3 of the right side is the C-linear transformation of $\mathfrak{C}^C = \boldsymbol{H}^C \oplus \boldsymbol{H}^C e_4$ defined by $\gamma_3(a + be_4) = a + (\omega_1 b)e_4$, $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1$. ($\boldsymbol{H} = \{1, e_1, e_2, e_3\}_{\boldsymbol{R}} \subset \mathfrak{C}$ is the field of quaternion numbers).

In the following Lemma, for $S \in M(3, \mathfrak{C})$ such that $\mathrm{tr}(S) = 0$ and $S^* = -S$, $\tilde{S} \in \mathfrak{e}_6$ is defined by $\tilde{S}X = \frac{1}{2}[S, X] = \frac{1}{2}(SX - XS)$, $X \in \mathfrak{J}^C$ and $(\mathfrak{d}_4)^{\gamma_3} = ((\mathfrak{e}_6)^{\gamma_3})_{E_1,E_2,E_3}$ is $\{\delta \in \mathfrak{e}_6 \mid \gamma_3\delta = \delta\gamma_3,\ \delta E_k = 0,\ k = 1, 2, 3\}$. ($E_k, k = 1, 2, 3$ are the usual notations in $\mathfrak{J}^C$ (e.g. [6])).

Lemma 2.2. *The Lie algebra $(\mathfrak{e}_7)^{\lambda_3}$ of the group $(E_7)^{\lambda_3}$ is given by*

$$(\mathfrak{e}_7)^{\lambda_3} = \{\Phi \in \mathfrak{e}_7 \mid \lambda_3\Phi = \Phi\lambda_3\}$$

$$= \left\{ \begin{array}{l} \Phi(\phi, A, -\tau A, 0) \mid \\ \phi = \delta + \begin{pmatrix} 0 & s_3 & -\overline{s_2} \\ -\overline{s_3} & 0 & s_1 \\ s_2 & -\overline{s_1} & 0 \end{pmatrix}^{\sim} + 2i \begin{pmatrix} 0 & (e_1b_3)e_4 & \overline{(e_1b_2)e_4} \\ \overline{(e_1b_3)e_4} & 0 & (e_1b_1)e_4 \\ (e_1b_2)e_4 & \overline{(e_1b_1)e_4} & 0 \end{pmatrix}^{\sim}, \\ A = \begin{pmatrix} \mu_1 & a_3 + ib_3e_4 & \overline{a_2 + ib_2e_4} \\ \overline{a_3 + ib_3e_4} & \mu_2 & a_1 + ib_1e_4 \\ a_2 + ib_2e_4 & \overline{a_1 + ib_1e_4} & \mu_3 \end{pmatrix}, \\ \delta \in (\mathfrak{d}_4)^{\gamma_3}, \quad s_k, a_k, b_k \in \boldsymbol{H},\ \mu_k \in \boldsymbol{R} \end{array} \right\}$$

In particular, the dimension of the Lie algebra $(\mathfrak{e}_7)^{\lambda_3}$ *is* 49.

Proof. We obtain this result from direct calculations. (Note that we have

$$((E_6)^{\gamma_3})_{E_1,E_2,E_3} = \{\alpha \in (E_6)^{\gamma_3} \mid \alpha E_k = E_k,\ k = 1,2,3\}$$
$$\cong (U(1) \times Sp(1) \times Sp(1) \times Sp(1))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(1,1,1,1)),(-1,-1,-1,-1)\},$$

hence the dimension of $(\mathfrak{d}_4)^{\gamma_3} = ((\mathfrak{e}_6)^{\gamma_3})_{E_1,E_2,E_3}$ is 10).

Theorem 2.3. $(E_7)^{\lambda_3} \cong S(U(1) \times U(7))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{E, -E\}$.

Proof. We define a homomorphism $\varphi : S(U(1) \times U(7)) \to (E_7)^{\lambda_3}$ by the restriction mapping of φ in Proposition 2.1. Obviously φ is well-defined. Since $(E_7)^{\lambda_3}$ is connected and $\dim((\mathfrak{e}_7)^{\lambda_3}) = 49$ (Lemma 2.2) $= \dim(\mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(7)))$, φ is onto. $\operatorname{Ker}\varphi = \{E, -E\}$ is easily obtained from Proposition 2.1. Thus we have the required isomorphism.

3. Automorphism σ_3 of order 3 and subgroup $(\boldsymbol{SU}(2) \times \boldsymbol{Spin}(2) \times \boldsymbol{Spin}(10))/\boldsymbol{Z}_4$ of $\boldsymbol{E}_7$

Let $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e_4$ and the field $\boldsymbol{C}_4$ of complex numbers is embedded in $\mathfrak{C}$ as $\boldsymbol{C}_4 = \{x + ye_4 \mid x, y \in \boldsymbol{R}\}$. Let $Spin(2) = \{a \in \boldsymbol{C}_4 \mid \overline{a}a = 1\} (\cong U(1))$ and we define an embedding $D : Spin(2) \to E_7$ by

$$D_a(X, Y, \xi, \eta) = (D_a X, D_a Y, \xi, \eta),$$

where D_a of the right side is the C-linear transformation of $\mathfrak{J}^C$ defined by

$$D_a X = D_a \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & x_3\overline{a} & \overline{x_2}a \\ a\overline{x_3} & \xi_2 & ax_1a \\ \overline{a}x_2 & \overline{ax_1a} & \xi_3 \end{pmatrix}.$$

For $-1 \in Spin(2)$ and $\omega_4 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_4 \in Spin(2)$, let

$$\sigma = D_{-1}, \quad \sigma_3 = D_{\omega_4}.$$

Then $\sigma, \sigma_3 \in E_7$ and $\sigma^2 = 1$, $\sigma_3{}^3 = 1$. Let $SU(2) = \{A \in M(2, C) \mid (\tau\,{}^t A)A = E, \det A = 1\}$. We define an embedding $\phi : SU(2) \to E_7$ by

$$\phi(A)\Big(\begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \overline{y_2} \\ \overline{y_3} & \eta_2 & y_1 \\ y_2 & \overline{y_1} & \eta_3 \end{pmatrix} \xi, \eta\Big)$$
$$= \Big(\begin{pmatrix} \xi_1{}' & x_3{}' & \overline{x_2{}'} \\ \overline{x_3{}'} & \xi_2{}' & x_1{}' \\ x_2{}' & \overline{x_1{}'} & \xi_3{}' \end{pmatrix}, \begin{pmatrix} \eta_1{}' & y_3{}' & \overline{y_2{}'} \\ \overline{y_3{}'} & \eta_2{}' & y_1{}' \\ y_2{}' & \overline{y_1{}'} & \eta_3{}' \end{pmatrix} \xi', \eta'\Big),$$

$$\begin{pmatrix}\xi_1{}'\\ \eta'\end{pmatrix} = A\begin{pmatrix}\xi_1\\ \eta\end{pmatrix}, \begin{pmatrix}\xi'\\ \eta_1{}'\end{pmatrix} = A\begin{pmatrix}\xi\\ \eta_1\end{pmatrix}, \begin{pmatrix}\eta_2{}'\\ \xi_3{}'\end{pmatrix} = A\begin{pmatrix}\eta_2\\ \xi_3\end{pmatrix}, \begin{pmatrix}\eta_3{}'\\ \xi_2{}'\end{pmatrix} = A\begin{pmatrix}\eta_3\\ \xi_2\end{pmatrix},$$

$$\begin{pmatrix}x_1{}'\\ y_1{}'\end{pmatrix} = (\tau A)\begin{pmatrix}x_1\\ y_1\end{pmatrix}, \begin{pmatrix}x_2{}'\\ y_2{}'\end{pmatrix} = \begin{pmatrix}x_2\\ y_2\end{pmatrix}, \begin{pmatrix}x_3{}'\\ y_3{}'\end{pmatrix} = \begin{pmatrix}x_3\\ y_3\end{pmatrix}.$$

The group $Spin(12)$ and $Spin(10)$ are defined by

$$Spin(12) = \{\alpha \in E_7 \mid \kappa\alpha = \alpha\kappa,\ \mu\alpha = \alpha\mu\},$$

where $\kappa = \Phi(-2E_1 \vee E_1, 0, 0, -1)$, $\mu = \Phi(0, E_1, E_1, 0)$ and

$$\begin{aligned} Spin(10) &= \{\alpha \in Spin(12) \mid \alpha(F_1(s), 0, 0, 0) = (F_1(s), 0, 0, 0),\ s \in \boldsymbol{C}_4\} \\ &= \left\{\alpha \in Spin(12) \,\middle|\, \begin{array}{l} \alpha(F_1(1), 0, 0, 0) = (F_1(1), 0, 0, 0), \\ \alpha(F_1(e_4), 0, 0, 0) = (F_1(e_4), 0, 0, 0) \end{array}\right\} \end{aligned}$$

($F_k(x)$, $k = 1, 2, 3$ are the usual notations in $\mathfrak{J}^C$ (e.g. [6]) and $\boldsymbol{C}_4{}^{\perp}$ is the orthogonal complement of $\boldsymbol{C}_4$ in $\mathfrak{C}$ with respect to the inner product $(x, y) = \frac{1}{2}(x\overline{y} + y\overline{x})$) which is the covering group $SO(10) = SO(V^{10})$, where

$$\begin{aligned} V^{10} &= \left\{P \in \mathfrak{P}^C \,\middle|\, \begin{array}{l} \kappa P = P,\ \mu\tau\lambda P = P, \\ P \times (F_1(1), 0, 0, 0) = 0,\ P \times (F_1(e_4), 0, 0, 0) = 0 \end{array}\right\} \\ &= \left\{\left(\begin{pmatrix}0 & 0 & 0\\ 0 & \xi & t\\ 0 & \overline{t} & -\tau\xi\end{pmatrix}, \begin{pmatrix}\eta & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, 0, \tau\eta\right) \mid t \in \boldsymbol{C}_4{}^{\perp},\ \xi, \eta \in C\right\}. \end{aligned}$$

This group $Spin(10)$ is isomorphic to the usual spinor group $Spin_1(10)$ [7] :

$$Spin_1(10) = \left\{\alpha \in Spin(12) \,\middle|\, \begin{array}{l} \alpha(0, E_1, 0, 1) = (0, E_1, 0, 1), \\ \alpha(0, -E_1, 0, 1) = (0, -E_1, 0, 1) \end{array}\right\}.$$

In fact, for $\rho = \delta(\frac{\pi}{4}e_4)\gamma(\frac{\pi}{4}) \in Spin(12) \subset E_7$, where, for $a \in \mathfrak{C}$,

$$\delta(a) = \exp \Phi(0, iF_1(a), iF_1(a), 0), \quad \gamma(a) = \exp \Phi(0, F_1(a), -F_1(a), 0)$$

[2], we have $Spin(10) = \rho^{-1}Spin_1(10)\rho$.

Let $(E_7)^\sigma = \{\alpha \in E_7 \mid \sigma\alpha = \alpha\sigma\}$.

Proposition 3.1. [7]. $(E_7)^\sigma \cong (SU(2) \times Spin(12))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(E, 1), (-E, -\sigma)\}$. *The isomorphism is induced by the homomorphism* $\varphi : SU(2) \times Spin(12) \to (E_7)^\sigma$, $\varphi(A, \beta) = \phi(A)\beta$.

We shall determine the group structure of

$$(E_7)^{\sigma_3} = \{\alpha \in E_7 \mid \sigma_3\alpha = \alpha\sigma_3\}.$$

We consider two C-vector subspaces $(\mathfrak{P}^C)_{\sigma_3}$ and $((\mathfrak{P}^C)_{\sigma_3})^\perp$ of $\mathfrak{P}^C$:

$$(\mathfrak{P}^C)_{\sigma_3} = \{P \in \mathfrak{P}^C \mid \sigma_3 P = P\}$$
$$= \left\{ \left(\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & s \\ 0 & \overline{s} & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & t \\ 0 & \overline{t} & \eta_3 \end{pmatrix}, \xi, \eta \right) \middle| \begin{array}{l} \xi_k, \eta_k, \xi, \eta \in C, \\ s, t \in (\boldsymbol{C}_4{}^\perp)^C \end{array} \right\}$$

($(\boldsymbol{C}_4{}^\perp)^C$ is the complexification of $\boldsymbol{C}_4{}^\perp$)

$$((\mathfrak{P}^C)_{\sigma_3})^\perp = \{P \in \mathfrak{P}^C \mid \langle P, P' \rangle = 0 \text{ for any } P' \in (\mathfrak{P}^C)_{\sigma_3}\}$$
$$= \left\{ \left(\begin{pmatrix} 0 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & m \\ x_2 & \overline{m} & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_3 & \overline{y_2} \\ \overline{y_3} & 0 & n \\ y_2 & \overline{n} & 0 \end{pmatrix}, 0, 0 \right) \mid x_k, y_k \in \mathfrak{C}^C,\ m, n \in \boldsymbol{C}_4{}^C \right\}$$

Then $\mathfrak{P}^C = (\mathfrak{P}^C)_{\sigma_3} \oplus ((\mathfrak{P}^C)_{\sigma_3})^\perp$ and $(\mathfrak{P}^C)_{\sigma_3}$, $((\mathfrak{P}^C)_{\sigma_3})^\perp$ are invariant under the action of the group $(E_7)^{\sigma_3}$.

We define a C-vector subspace $(\mathfrak{P}^C)_0$ of $\mathfrak{P}^C$:

$$(\mathfrak{P}^C)_0 = \{(F_1(m), F_1(n), 0, 0) \mid m, n \in \boldsymbol{C}_4{}^C\}.$$

Lemma 3.2. *$(\mathfrak{P}^C)_0$ is invariant under the action of the group $(E_7)^{\sigma_3}$.*

Proof. The Lie algebra $(\mathfrak{e}_7)^{\sigma_3}$ of $(E_7)^{\sigma_3}$ is

$$(\mathfrak{e}_7)^{\sigma_3} = \{\Phi \in \mathfrak{e}_7 \mid \sigma_3 \Phi = \Phi \sigma_3\}$$
$$= \{\Phi(\phi, A, -\tau A, \nu) \in \mathfrak{e}_7 \mid \phi \in (\mathfrak{e}_6)^{\sigma_3},\ A \in (\mathfrak{J}^C)_{\sigma_3},\ \nu \in i\boldsymbol{R}\}$$

($(\mathfrak{e}_6)^{\sigma_3} = \{\phi \in \mathfrak{e}_6 \mid \sigma_3 \phi = \phi \sigma_3\}$ and $(\mathfrak{J}^C)_{\sigma_3} = \{A \in \mathfrak{J}^C \mid \sigma_3 A = A\}$). Now, for $\Phi(\phi, A, -\tau A, \nu) \in (\mathfrak{e}_7)^{\sigma_3}$ and $(F_1(m), F_1(n), 0, 0) \in (\mathfrak{P}^C)_0$, we have $\Phi(\phi, A, -\tau A, \nu)(F_1(m), F_1(n), 0, 0) = (F_1(m'), F_1(n'), 0, 0) \in (\mathfrak{P}^C)_0$ by direct calculations. Hence, for $\alpha \in (E_7)^{\sigma_3}$, $\alpha(F_1(m), F_1(n), 0, 0)$ has also the form of $(F_1(m''), F_1(n''), 0, 0)$, because $(E_7)^{\sigma_3}$ is connected.

From Lemma 3.2, we see that two C-vector subspaces

$$\{(F_2(x_2) + F_3(x_3), F_2(y_2) + F_3(y_3), 0, 0) \mid x_k, y_k \in \mathfrak{C}^C\}, \quad (\mathfrak{P}^C)_0$$

of $((\mathfrak{P}^C)_{\sigma_3})^\perp$ are invariant under the action of the group $(E_7)^{\sigma_3}$.

Lemma 3.3. *$D_a, a \in Spin(2)$ and $\delta \in Spin(10)$ commute with each other.*

Proof. Since δD_a and $\tau\lambda$ commute, to show that $D_a \delta = \delta D_a$, it suffices to prove

$$D_a \delta P = \delta D_a P, \quad P = (X, 0, 0, 0), (0, 0, 1, 0).$$

Now, $\delta D_a(E_2,0,0,0) = \delta(E_2,0,0,0) = (\xi E_2 - \tau\xi E_3 + F_1(t), \eta E_1, 0, \tau\eta)(\xi, \eta \in C, t \in \boldsymbol{C_4}^{\perp})$. On the other hand, $D_a\delta(E_2,0,0,0) = D_a(\xi E_2 - \tau\xi E_3 + F_1(t), \eta E_1, 0, \tau\eta) = (\xi E_2 - \tau\xi E_3 + F_1(ata), \eta E_1, 0, \tau\eta) = (\xi E_2 - \tau\xi E_3 + F_1(t), \eta E_1, 0, \tau\eta)$. Hence we have $\delta D_a(E_2,0,0,0) = D_a\delta(E_2,0,0,0)$. Similaly $\delta D_a(E_3,0,0,0) = D_a\delta(E_3,0,0,0)$. Next,

$$\begin{aligned}\delta D_a(iE_1,0,-i,0) &= \delta(iE_1,0,-i,0) = \delta\tau\lambda(0,-iE_1,0,i)\\ &= \tau\lambda\delta(0,-iE_1,0,i) = \tau\lambda(\xi E_2 - \tau\xi E_3 + F_1(t), \eta E_1, 0, \tau\eta)\end{aligned}$$

$(\xi, \eta \in C,\ t \in \boldsymbol{C_4}^{\perp})$. On the other hand,

$$\begin{aligned}D_a\delta(iE_1,0,-i,0) &= D_a\delta\tau\lambda(0,-iE_1,0,i) = \tau\lambda D_a\delta(0,-iE_1,0,i)\\ &= \tau\lambda D_a(\xi E_2 - \tau\xi E_3 + F_1(t), \eta E_1, 0, \tau\eta)\\ &= \tau\lambda(\xi E_2 - \tau\xi E_3 + F_1(ata), \eta E_1, 0, \tau\eta)\\ &= \tau\lambda(\xi E_2 - \tau\xi E_3 + F_1(t), \eta E_1, 0, \tau\eta).\end{aligned}$$

Hence $\delta D_a(iE_1,0,-i,0) = D_a\delta(iE_1,0,-i,0)$, that is, $\delta D_a(E_1,0,-1,0) = D_a\delta(E_1,0,-1,0)$. Similarly $\delta D_a(E_1,0,1,0) = D_a\delta(E_1,0,1,0)$. Thus we have

$$\delta D_a(E_1,0,0,0) = D_a\delta(E_1,0,0,0) \quad \text{and} \quad \delta D_a(0,0,1,0) = D_a\delta(0,0,1,0).$$

For $z \in \mathfrak{C}^C$, we have

$$\begin{aligned}\delta D_a(F_1(z),0,0,0) &= \delta(F_1(aza),0,0,0)\\ &= \delta(F_1(a^2s+t),0,0,0) \quad (z = s+t \in \boldsymbol{C_4}^C \oplus (\boldsymbol{C_4}^{\perp})^C = \mathfrak{C}^C)\\ &= (F_1(a^2s),0,0,0) + \delta(F_1(t),0,0,0)\\ &= (F_1(a^2s),0,0,0) + \delta(F_1(t_1+it_2),0,0,0) \quad (t_k \in \boldsymbol{C_4}^{\perp})\\ &= (F_1(a^2s),0,0,0) + \delta(F_1(t_1),0,0,0) + i\delta(F_1(t_2),0,0,0)\\ &= (F_1(a^2s),0,0,0) + (\xi_2E_2 + \xi_3E_3 + F_1(t'), \eta_1E_1, 0, \eta)\end{aligned}$$

$(\xi_2, \xi_3, \eta_1, \eta \in C, t' \in (\boldsymbol{C_4}^{\perp})^C)$. On the other hand

$$\begin{aligned}D_a\delta(F_1(z),0,0,0) &= D_a\delta(F_1(s+t),0,0,0)\\ &= D_a((F_1(s),0,0,0)) + \delta(F_1(t),0,0,0))\\ &= (F_1(a^2s),0,0,0) + D_a(\xi_2E_2 + \xi_3E_3 + F_1(t'), \eta_1E_1, 0, \eta)\\ &= (F_1(a^2s),0,0,0) + (\xi_2E_2 + \xi_3E_3 + F_1(t'), \eta_1E_1, 0, \eta).\end{aligned}$$

$$\begin{aligned}
&\delta D_a(F_2(z),0,0,0) = \delta(F_2(\overline{a}z),0,0,0)\\
&= \delta(((F_1(1),0,0,0) \times (F_2(z),0,0,0))(0,4F_1(a),0,0))\\
&= \delta((F_1(1),0,0,0) \times (F_2(z),0,0,0))\delta^{-1}\delta(0,4F_1,(a),0,0)\\
&= (\delta(F_1(1),0,0,0) \times \delta(F_2(z),0,0,0))(0,4F_1(a),0,0)\\
&= ((F_1(1),0,0,0) \times (F_2(x_2)+F_3(x_3),F_2(y_2)+F_3(y_3),0,0))(0,4F_1(a),0,0)\\
&= \Phi(-\tfrac{1}{2}(F_1(1) \vee F_2(y_2) + F_1(1) \vee F_3(y_3)),0,\\
&\qquad\qquad \tfrac{1}{4}(F_3(\overline{x_2}) + F_2(\overline{x_3})),0)(0,4F_1(a),0,0)\\
&= (F_2(\overline{a}x_2) + F_3(x_3\overline{a}), F_2(\overline{a}y_2) + F_3(y_3\overline{a}),0,0) \quad (x_k, y_k \in \mathfrak{C}^C).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
D_a\delta(F_2(z),0,0,0) &= D_a(F_2(x_2)+F_3(x_3),F_2(y_2)+F_3(y_3),0,0)\\
&= (F_2(\overline{a}x_2)+F_3(x_3\overline{a}),F_2(\overline{a}y_2)+F_3(y_3\overline{a}),0,0).
\end{aligned}$$

Hence we have $\delta D_a(F_k(z),0,0,0) = D_a\delta(F_k(z),0,0,0)$, $k = 1,2$. Similarly $\delta D_a(F_3(z),0,0,0) = D_a\delta(F_3(z),0,0,0)$. Thus we have $\delta D_a P = D_a \delta P$ for any $P \in \mathfrak{P}^C$, that is, $\delta D_a = D_a \delta$.

Lemma 3.4. *For $\beta \in (Spin(12))^{\sigma_3} = \{\alpha \in Spin(12) \mid \sigma_3\alpha = \alpha\sigma_3\}$, there exists $s \in \boldsymbol{C}_4$, $|s| = 1$ such that*

$$\beta(F_1(1),0,0,0) = (F_1(s),0,0,0), \quad \beta(F_1(e_4),0,0,0) = (F_1(e_4 s),0,0,0).$$

Proof. Consider the 2 dimensional $\boldsymbol{R}$-vector space

$$\begin{aligned}
V^2 &= \{P \in \mathfrak{P}^C \mid \kappa P = P,\ \mu\tau\lambda P = P,\ \langle P, P'\rangle = 0 \text{ for any } P' \in (\mathfrak{P}^C)_{\sigma_3}\}\\
&= \{(F_1(s),0,0,0) \in \mathfrak{P}^C \mid s \in \boldsymbol{C}_4\}.
\end{aligned}$$

Since $\beta \in (Spin(12))^{\sigma_3}$ acts on V^2, there exists $s \in \boldsymbol{C}_4$ such that $\beta(F_1(1),0,0,0) = (F_1(s),0,0,0)$. Then for $\omega_4 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_4$. we have

$$\begin{aligned}
\beta(F_1(\omega_4),0,0,0) &= \beta(F_1({\omega_4}^2 1 {\omega_4}^2),0,0,0) = \beta\sigma_3\sigma_3(F_1(1),0,0,0)\\
&= \sigma_3\sigma_3\beta(F_1(1),0,0,0) = \sigma_3\sigma_3(F_1(s),0,0,0)\\
&= (F_1(\omega_4 s),0,0,0) \cdots (1).
\end{aligned}$$

Similarly $\beta(F_1(\overline{\omega_4}),0,0,0) = (F_1(\overline{\omega_4}s),0,0,0) \cdots (2)$. Subtract (1)–(2), then we have $\beta(F_1(e_4),0,0,0) = (F_1(e_4 s),0,0,0)$. The fact that $|s| = 1$ follows from $\langle\beta P, \beta P\rangle = \langle P, P\rangle$, $P = (F_1(1),0,0,0) \in \mathfrak{P}^C$.

Lemma 3.5. *$\phi(SU(2))$, $Spin(2)$ and $Spin(10)$ are contained in $(E_7)^{\sigma_3}$.*

Proof. By direct calculations $\phi(SU(2))$ is contained in $(E_7)^{\sigma_3}$, and since $\sigma_3 = D_{\omega_4} \in Spin(2)$, $Spin(2)$ is contained in $(E_7)^{\sigma_3}$. Next, $\delta \in Spin(10)$ satisfied $\sigma_3\delta = \delta\sigma_3$ (Lemma 3.3). Hence $Spin(10)$ is contained in $(E_7)^{\sigma_3}$.

Theorem 3.6. $(E_7)^{\sigma_3} \cong (SU(2)\times Spin(2)\times Spin(10))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4 = \{(E,1,1), (E,-1,\sigma), (-E,e_4,\phi(-E)D_{-e_4}), (-E,-e_4,\phi(-E)D_{e_4})\}$.

Proof. We define a mapping $\varphi : SU(2) \times Spin(2) \times Spin(10) \to (E_7)^{\sigma_3}$ by

$$\varphi(A,a,\delta) = \phi(A)D_a\delta.$$

φ is well-defined from Lemma 3.5. From Proposition 3.1, $\phi(A)$, $A \in SU(2)$ and $\delta \in Spin(10) \subset Spin(12)$ commute with each other, and $D_a, a \in Spin(2)$ and $\delta \in Spin(10)$ also commute with each other from Lemma 3.3. Moreover $\phi(A)\delta = \delta\phi(A)$ by direct calculations. Hence φ is a homomorphism. We shall show that φ is onto. Since $(\mathfrak{P}^C)_{\sigma_3}$, $(\mathfrak{P}^C)_0$ and $\{(F_2(x_2)+F_3(x_3), F_2(y_2)+F_3(y_3),0,0 \mid x_k, y_k \in \mathfrak{C}^C\}$ are invariant under the action of the group $(E_7)^{\sigma_3}$, we have $(E_7)^{\sigma_3} \subset (E_7)^{\sigma}$. Hence, for $\alpha \in (E_7)^{\sigma_3}$, there exist $A \in SU(2)$ and $\beta \in Spin(12)$ such that $\alpha = \phi(A)\beta$ (Proposition 3.1). From $\sigma_3\alpha = \alpha\sigma_3$, we have $\beta \in (Spin(12))^{\sigma_3}$. Hence from Lemma 3.4, there exists $s \in \boldsymbol{C}_4$, $|s| = 1$ such that $\beta(F_1(1),0,0,0) = (F_1(s),0,0,0)$ and $\beta(F_1(e_4),0,0,0) = (F_1(e_4s),0,0,0)$. Choose $a \in \boldsymbol{C}_4$ such that $a^2 = s$ and let $\delta = {D_a}^{-1}\beta$, then $\delta(F_1(1),0,0,0) = (F_1(1),0,0,0)$ and $\delta(F_1(e_4),0,0,0) = (F_1(e_4),0,0,0)$, that is, $\delta \in Spin(10)$. Hence we have a representation $\alpha = \phi(A)D_a\delta$, $A \in SU(2)$, $a \in Spin(2)$, $\delta \in Spin(10)$. Therefore φ is onto. $\operatorname{Ker}\varphi = \boldsymbol{Z}_4$ is easily obtained. Thus we have the required isomorphism.

4. Automorphism σ_3' of order 3 and subgroup $(\boldsymbol{U}(1)\times \boldsymbol{Spin}(12))/\boldsymbol{Z}_2$ of $\boldsymbol{E}_7$

Let $U(1) = \{\theta \in C \mid (\tau\theta)\theta = 1\}$ and we define an embedding $\phi : U(1) \to E_7$ by

$$\phi(\theta)\left(\begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \overline{y_2} \\ \overline{y_3} & \eta_2 & y_1 \\ y_2 & \overline{y_1} & \eta_3 \end{pmatrix}, \xi, \eta\right)$$
$$= \left(\begin{pmatrix} \theta\xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \theta^{-1}\xi_2 & \theta^{-1}x_1 \\ x_2 & \theta^{-1}\overline{x_1} & \theta^{-1}\xi_3 \end{pmatrix}, \begin{pmatrix} \theta^{-1}\eta_1 & y_3 & \overline{y_2} \\ \overline{y_3} & \theta\eta_2 & \theta y_1 \\ y_2 & \theta\overline{y_1} & \theta\eta_3 \end{pmatrix}, \theta\xi, \theta^{-1}\eta\right).$$

ϕ is well-defined, that is, $\phi(\theta) \in E_7$. In fact, since $U(1)$ is contained in $SU(2)$ as $\left\{\begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix} \mid \theta \in C,\ (\tau\theta)\theta = 1\right\}$, this ϕ is the restriction mapping of ϕ :

$SU(2) \to E_7$ of Section 3. For $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \in U(1)$, let

$$\sigma_3{}' = \phi(\omega).$$

Then $\sigma_3{}' \in E_7$ and $(\sigma_3{}')^3 = 1$.

We shall determine the group structure of

$$(E_7)^{\sigma_3{}'} = \{\alpha \in E_7 \mid \sigma_3{}'\alpha = \alpha\sigma_3{}'\}.$$

Lemma 4.1. *$\phi(U(1))$ and $Spin(12)$ are contained in $(E_7)^{\sigma_3{}'}$.*

Proof. Since $\sigma_3{}' = \phi(\omega)$, it is clear that $\phi(U(1))$ is contained in $(E_7)^{\sigma_3{}'}$. Next, from Proposition 3.1, $\phi(A)$, $A \in SU(2)$ and $\beta \in Spin(12)$ commute with each other. Hence $\phi(\theta)\beta = \beta\phi(\theta)$, $\theta \in U(1) \subset SU(2)$. Therefore $\sigma_3{}'\beta = \beta\sigma_3{}'$, that is, $Spin(12) \subset (E_7)^{\sigma_3{}'}$.

We consider two C-vector subspaces $(\mathfrak{P}^C)_{\sigma_3{}'}$ and $((\mathfrak{P}^C)_{\sigma_3{}'})^\perp$ of $\mathfrak{P}^C$:

$$(\mathfrak{P}^C)_{\sigma_3{}'} = \{P \in \mathfrak{P}^C \mid \sigma_3{}'P = P\}$$
$$= \left\{\left(\begin{pmatrix} 0 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_3 & \overline{y_2} \\ \overline{y_3} & 0 & 0 \\ y_2 & 0 & 0 \end{pmatrix}, 0, 0\right) \mid x_k, y_k \in \mathfrak{C}^C\right\},$$

$$((\mathfrak{P}^C)_{\sigma_3{}'})^\perp = \{P \in \mathfrak{P}^C \mid \langle P, P'\rangle = 0 \text{ for any } P' \in (\mathfrak{P}^C)_{\sigma_3{}'}$$
$$= \left\{\left(\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x_1} & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & y_1 \\ 0 & \overline{y_1} & \eta_3 \end{pmatrix}, \xi, \eta\right) \,\middle|\, \begin{matrix} \xi_k, \eta_k, \xi, \eta \in C, \\ x_1, y_1 \in \mathfrak{C}^C \end{matrix}\right\}.$$

Then $\mathfrak{P}^C = (\mathfrak{P}^C)_{\sigma_3{}'} \oplus ((\mathfrak{P}^C)_{\sigma_3{}'})^\perp$ and $(\mathfrak{P}^C)_{\sigma_3{}'}$, $((\mathfrak{P}^C)_{\sigma_3{}'})^\perp$ are invariant under the action of the group $(E_7)^{\sigma_3{}'}$.

Theorem 4.2. $(E_7)^{\sigma_3{}'} \cong (U(1) \times Spin(12))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1,1), (-1,-\sigma)\}$.

Proof. We define a mapping $\varphi : U(1) \times Spin(12) \to (E_7)^{\sigma_3{}'}$ by

$$\varphi(\theta, \beta) = \phi(\theta)\beta.$$

φ is well-defined from Lemma 4.1. Since $\phi(\theta)$, $\theta \in U(1)$ and $\beta \in Spin(12)$ commute (Lemma 4.1), φ is a homomorphism. We shall show that φ is onto. Since $(\mathfrak{P}^C)_{\sigma_3{}'}$ and $((\mathfrak{P}^C)_{\sigma_3{}'})^\perp$ are invariant under the action of the group $(E_7)^{\sigma_3{}'}$, $\alpha \in (E_7)^{\sigma_3{}'}$ commutes with σ, that is, $(E_7)^{\sigma_3{}'} \subset (E_7)^\sigma$. Hence, for $\alpha \in (E_7)^{\sigma_3{}'}$, there exists $A \in SU(2)$ and $\beta \in Spin(12)$ such that $\alpha = \phi(A)\beta$ (Proposition

3.1). From $\sigma_3'\alpha = \alpha\sigma_3'$, that is, $\sigma_3'\phi(A) = \phi(A)\sigma_3'$, we have $A = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$, $\theta \in U(1)$. Therefore φ is onto. $\operatorname{Ker}\varphi = \boldsymbol{Z}_2$ is easily obtained. Thus we have the required isomorphism.

5. Automorphism w of order 3 and subgroup $(\boldsymbol{SU}(3) \times \boldsymbol{SU}(6))/\boldsymbol{Z}_3$ of $\boldsymbol{E}_7$

A. Borel and J. de Siebenthal [1] classified the maximal subgroups of maximal rank of compact simple Lie groups G and showed that the group E_7 has a maximal subgroup of rank 7 which is the fixed subgroup of an automorphism of E_7 of order 3 and whose type is $A_2 \oplus A_5$. In the previous paper [4], we realized this group. The result is as follows. We define a C-linear transformation w of $\mathfrak{P}^C$ by

$$w(X, Y, \xi, \eta) = (wX, wY, \xi, \eta),$$

here w of the right side in the C-linear transformation of $\mathfrak{J}^C$ defined by

$$wX = w\begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & w(x_3) & \overline{w(x_2)} \\ \overline{w(x_3)} & \xi_2 & w(x_1) \\ w(x_2) & \overline{w(x_1)} & \xi_3 \end{pmatrix},$$

moreover w of the right side is the C-linear transformation of $\mathfrak{C}^C = \boldsymbol{C}_4{}^C \oplus (\boldsymbol{C}_4{}^3)^C$ [5] defined by $w(a+\boldsymbol{m}) = a + \omega_4\boldsymbol{m}$ ($\omega_4 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_4$). Then $w \in E_7$ and $w^3 = 1$.

Let $(E_7)^w = \{\alpha \in E_7 \mid w\alpha = \alpha w\}$.

Theorem 5.1. [4]. $(E_7)^w \cong (SU(3) \times SU(6))/\boldsymbol{Z}_3$, $\boldsymbol{Z}_3 = \{(E, E), (\omega_4 E, \omega_4 E), (\omega_4{}^2 E, \omega_4{}^2 E)\}$.

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