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THREE NONTRIVIAL SOLUTIONS FOR A QUASILINEAR ELLIPTIC DIFFERENTIAL EQUATION AT RESONANCE WITH DISCONTINUOUS RIGHT HAND SIDE

By

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Abstract. We study a quasilinear elliptic equation at resonance with discontinuous right hand side. To have an existence theory, we pass to a multivalued version of the problem by filling in the gaps at the discontinuity points. Using the nonsmooth critical point theory of Chang for lically Lipschitz functionals and the Ekeland variational principle, we show that the resulting elliptic inclusion has three distinct nontrivial solutions.

1. Introduction

In a recent paper we studied quasilinear elliptic problems at resonance with a discontinuous right hand side (see Kourogenis-Papageorgiou [11]). Using a variational approach we proved the existence of a nontrivial solution. In this paper we establish the existence of multiple nontrivial solutions for the same problem. Again we assume that the potential function $F(z,x)=\int_{0}^{x}f(z,r)dr$ goes to infinity as $x\rightarrow\pm\infty$ for almost all $z\in E\subseteq Z$, with E having positive Lebesgue measure. In this respect our work is similar to that of Ahmad-Lazer-Paul [3] and Rabinowitz [14]. The case where the potential function has a finite limit as $x\rightarrow\pm\infty$ for almost all $z\in Z$, known in the litterature as "strongly resonant case", was studied by Thews [15], Bartolo-Benci-Fortunato [5] and Ward [17]. All these works deal with semilinear problems which have a continuous right hand side. The problem of multiple solutions for the semilinear, "continuous" resonant problem was investigated by Ahmad [2], Goncalves-Miyagaki [8], [9] and Landesman-Robinson-Rumbos [12].

Our approach is based on the critical point theory for nonsmooth locally

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Lipschitz functionals developed by Chang [6]. Using concepts and results from this theory, we show that our problem has at least three nontrivial solutions. In the next section, for the convenience of the reader, we fix our notation and recall some basic definitions and facts from Chang's critical point theory.

2. Preliminaries

The critical point theory developed by Chang [6] is based on the subdifferential theory of Clarke [7] which is developed for locally Lipschitz functions. So let X be a Banach space and $f : X \to \mathbf{R}$ a function. We say that f is "locally Lipschitz", if for every $x \in X$, there exists a neighbourhood U of x and a constant $k > 0$ depending on U such that $|f(y) - f(z)| \leq k\|y - z\|$ for all, $y, z \in U$. Given $h \in X$, we define the "generalized directional derivative" $f^0(x;h)$ by

$$f^0(x;h) = \overline{\lim_{\substack{x' \to x \\ \lambda \downarrow 0}}} \frac{f(x' + \lambda h) - f(x')}{\lambda}.$$

It is easy to see that for every $x \in X$, $f^0(x;\cdot)$ is sublinear and continuous (in fact $|f^0(x;h)| \leq k\|h\|$, hence $f^0(x;\cdot)$ is Lipschitz continuous). Thus from the Hahn-Banach theorem we infer that $f^0(x;\cdot)$ is the support function of a nonempty, convex and w^*-compact set

$$\partial f(x) = \{x^* \in X^* : (x^*, h) \leq f^0(x;h) \text{ for all } h \in X\}.$$

This set is known as the "generalized (or Clarke) subdifferential" of $f(\cdot)$ at $x \in X$. If $f, g : X \to \mathbf{R}$ are both locally Lipschitz functions, then $\partial(f+g)(x) \subseteq \partial f(x) + \partial g(x)$ and $\partial(\lambda f)(x) = \lambda \partial f(x)$ for all $x \in X$ and all $\lambda \in \mathbf{R}$. Moreover, if $f : X \to \mathbf{R}$ is also convex, then it is well-known that $f(\cdot)$ is locally Lipschitz and the subdifferential in the sense of convex analysis (see for example Hu-Papageorgiou [10], section III.4), coincides with the generalized subdifferential defined above. If f is strictly differentiable at x (in particular if f is continuously Gateaux differentiable at x), then $\partial f(x) = \{f'(x)\}$.

Given a locally Lipschitz function $f : X \to \mathbf{R}$, a point $x \in X$ is a "critical point" of f if $0 \in \partial f(x)$. It is easy to check that if $x \in X$ is a local extremum of f, then x is a critical point. We say that f satisfies the "nonsmooth Palais-Smale condition" (nonsmooth (PS)-condition for short), if for any sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $|f(x_n)| \leq M$ for all $n \geq 1$ and $m(x_n) = \min\{\|x^*\| : x^* \in \partial f(x_n)\} \xrightarrow{n \to \infty} 0$, has a strongly convergent subsequence. If $f \in C^1(X)$, then $\partial f(x_n) = \{f'(x_n)\}$, $n \geq 1$, and so the above definition of the (PS)-condition coincides with the classical one (see Rabinowiz [14]).

Consider the negative p-Laplacian ($2 \leq p < \infty$) differential operator $-\Delta_p x = -div(\|Dx\|^{p-2}Dx)$ with Dirichlet boundary conditions (i.e. $(-\Delta_p, W_0^{1,p}(Z))$). The first eigenvalue λ_1 of this operator is the least real number λ for which the following eigenvalue problem

$$\left\{\begin{array}{l} -div\big(\|Dx(z)\|^{p-2}Dx(z)\big) = \lambda|x(z)|^{p-2}x(z) \text{ a.e. on } Z \\ x_{|\Gamma} = 0\,. \end{array}\right\} \tag{1}$$

has a nontrivial solution. The first eigenvalue λ_1 is positive, isolated and simple (i.e. the associated eigenfunctions are constant multiples of each other). Furthermore we have the following variational characterization of λ_1 (Rayleigh quotient):

$$\lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z)\right]\,. \tag{2}$$

The minimum in (2) is realized at the normalized eigenfunction u_1. Note that if u_1 minimizes the quotient in (2), then so does $|u_1|$ and so we infer that first eigenfunction u_1 does not change its sign on Z. In fact we can show that $u_1 \neq 0$ a.e. on Z and so we may assume that $u_1(z) > 0$ a.e. on Z. Moreover, from nonlinear elliptic regularity (see Tolksdorf [16]), we have that $u \in C^{1,a}(Z)$ for some $a > 0$. For details we refer to Lindqvist [13] and the references therein. The Ljusternik-Schnirelmann theory gives, in addition to λ_1, a whole strictly increasing sequence of positive numbers $\{\lambda_n\}_{n\geq 1}$ for which there exist nontrivial solutions of the eigenvalue problem (1). In other words the spectrum $\sigma(-\Delta_p)$ of $(-\Delta_p, W_0^{1,p}(Z))$ contains at least these points. However, nothing in general is known about the possible existence of other points in $\sigma(-\Delta_p) \subseteq [\lambda_1, \infty) \subseteq \mathbf{R}_+$. Nevertheless we can define

$$\mu = \inf\Big\{\lambda > 0 : \lambda \text{ is an eigenvalue of } \big(-\Delta_p, W_0^{1,p}(Z)\big), \lambda \neq \lambda_1\Big\}\,.$$

Since $\lambda_1 > 0$ is isolated, we have $\mu > \lambda_1 > 0$. Moreover, if V is a topological complement of $\langle u_1 \rangle = \mathbf{R}u_1$ (= the eigenspace of u_1), then $\mu_v = \Big\{\frac{\|Dv\|_p^p}{\|u\|_p^p} : v \in V,$ $v \neq 0\Big\} > \lambda_1$, $\mu = \sup_v \mu_v$.

The next theorem is due to Chang [6] and is a nonsmooth version of he well-known "Mountain Pass Theorem" of Ambrosetti-Rabinowitz [4].

Theorem 1.

If *X is a reflexive Banach space, $R : X \to \mathbf{R}$ is a locally Lipschitz functional which satisfies the nonsmooth (PS)-condition and for some $\rho > 0$ and $y \in X$ with $\|y\| > \rho$ we have*

$$\max\{R(0), R(y)\} < \inf\left[R(x) : \|x\| = \rho\right] = a$$

then *there exists a nontrivial critical point* $x \in X$ *of* R *such that* $c = R(x) \geq a$ *and* c *is characterized by the following minimax principle*

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} R(\gamma(t)),$$
$$\textit{where } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = y\}.$$

3. Auxiliary results

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^1-boundary Γ. We consider the following quasilinear elliptic problem:

$$\left\{\begin{array}{l} -\, div(\|Dx(z)\|^{p-1} Dx(z)) - \lambda_1 |x(z)|^{p-2} x(z) = f(z, x(z)) \text{ a.e. on } Z \\ x_{|\Gamma} = 0, 2 \leq p < \infty \,. \end{array}\right\} \quad (3)$$

Since we do not assume that $f(z, \cdot)$ is continuous, problem (3) need not have a solution. To develop a reasonable existence theory, we pass to a multivalued version of (3) by, roughly speaking, filling in the gaps at the discontinuity points of $f(z, \cdot)$. For this purpose we introduce the following two functions:

$$f_1(z, x) = \underline{\lim}_{x' \to x} f(z, x') = \lim_{\delta \downarrow 0} \operatorname{ess\,inf}_{|x' - x| < \delta} f(z, x')$$
$$\text{and} \quad f_2(z, x) = \overline{\lim}_{x' \to x} f(z, x') = \lim_{\delta \downarrow 0} \operatorname{ess\,sup}_{|x' - x| < \delta} f(z, x').$$

Clearly $f_1(z, \cdot)$ is lower semicontinuous and $f_2(z, \cdot)$ is upper semicontinuous. Set $\hat{f}(z, x) = [f_1(z, x), f_2(z, x)]$. Then instead of (3) we study the following quasilinear elliptic inclusion:

$$\left\{\begin{array}{l} -\, div(\|Dx(z)\|^{p-2} Dx(z)) - \lambda_1 |x(z)|^{p-2} x(z) \in \hat{f}(z, x(z)) \text{ a.e. on } Z \\ x_{|\Gamma} = 0, 2 \leq p < \infty \,. \end{array}\right\} \quad (4)$$

We will show that under certain hypotheses on $f(z, x)$, problem (4) has at least three nontrivial solutions. The hypotheses on $f(z, x)$ are the following:
$\mathbf{H(f)}$: $f : Z \times \mathbf{R} \to \mathbf{R}$ is a measurable function such that

(i) f_1, f_2 are both N-measurable functions (i.e. for every $x : Z \to \mathbf{R}$ measurable function, $z \to f_i(z, x(z))$ is measurable, $i = 1, 2$);
(ii) for every $r > 0$, there exists $a_r \in L^\infty(Z)$ such that for almost all $z \in Z$ and all $|x| \leq r$ we have $|f(z, x)| \leq a_r(z)$;
(iii) there exist functions $\eta_\pm \in L^\infty(Z)$ such that $\eta_\pm(z) \leq 0$ a.e. on Z and $\eta_\pm(z) < 0$ for all $z \in E \subseteq Z$ with $|E| > 0$ (here by $|\cdot|$ we denote the

Lebesgue measure on $\mathbf{R}^N$) and uniformly for almost all $z \in Z$ we have $\lim_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = \eta_\pm(z)$;

(iv) if $F(z,x) = \int_0^x f(z,r)dr$, then for almost all $z \in Z$ and all $x \in \mathbf{R}$, $pF(z,x) \leq (\mu - \lambda_1)|x|^p$;

(v) there exists $\beta > \lambda_1$ such that $\overline{\lim_{|x| \to 0}} \frac{pF(z,x)}{|x|^p} \leq -\beta$ uniformly for almost all $z \in Z$.

Remarks: Hypothesis H(f)(i) is satisfied if f is independent of $z \in Z$ or if for almost all $z \in Z$, $f(z,\cdot)$ is monotone nondecreasing. Indeed, in the first case the N-measurability of f_1 and f_2 follows from the fact that f_1 is lower semicontinuous, while f_2 is upper semicontinuous. For the second case, note that $f_1(z,x) = \lim_{n\to\infty} f(z, x - \frac{1}{n})$ and $f_2(z,x) = \lim_{n\to\infty} f(z, x + \frac{1}{n})$, hence both functions f_1 and f_2 are measurabel, thus N-measurable too. Hypothesis H(f)(iii) implies that for all $z \in Z$ in a set of positive measure, $|F(z,x)| \xrightarrow{|x|\to\infty} +\infty$. Hypothesis H(f)(iv) is analogous to hypothesis H_∞ of Goncalves-Miyagaki [9]. Hypothesis H(f)(v) is needed in order to be able to apply theorem 1 and have a third nontrivial solution. Without it, we can not guarantee that the third solution (which in this case is obtained via the Mountain Pass Theorem) is nontrivial (note that $f(z,0) = 0$ a.e. on Z and so $x = 0$ is a solution of (4)). Finally by virtue of hypothesis H(f)(iii), given $\xi > 0$ we can find $M(\xi) > 0$ such that $\frac{f(z,x)}{|x|^{p-2}x} \leq \xi$ for almost all $z \in Z$ and $|x| \geq M(\xi)$, while for almost all $z \in Z$. and all $|x| \leq M(\xi)$, by hypothesis H(f)(ii) we have $|f(z,x)| \leq \hat{a}(\xi)(z)$, where $\hat{a}(\xi)(\cdot) = a_{M(\xi)}(\cdot) \in L^\infty(Z)$. Therefore finally we have that for almost all $z \in Z$ and all $x \in \mathrm{R}$, $|f(z,x)| \leq \hat{a}(\xi)(z) + \xi|x|^{p-1}$.

Our final hypothesis on $F(z,x)$ is the following:

$\mathbf{H_1}$: There exist $\xi_- < 0 < \xi_+$ such that $\int_Z F(z, \xi_\pm u_1(z))dz > 0$.

We introduce the energy functional $R : W_0^{1,p}(Z) \to \mathbf{R}$ defined by

$$R(x) = \frac{1}{p}||Dx||_p^p - \frac{\lambda_1}{p}||x||_p^p - \int_Z F(z, x(z))dz\,.$$

Clearly $R(\cdot)$ is locally Lipschitz.

Proposition 2.

If *hypotheses $H(f)$ hold,*
then *$R(\cdot)$ is coercive.*

Proof. Suppose not. We can find $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ such that

$\|x_n\|_{1,p} \xrightarrow{n\to\infty} \infty$ and $R(x_n) \le M$ for all $n \ge 1$. So we have

$$R(x_n) = \frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z F(z,x(z))dz \le M, \; n \ge 1 .$$

Since $|f(z,x)| \le a(z) + c|x|^{p-1}$ for almost all $z \in Z$ and all $x \in \mathbf{R}$, with $a \in L^\infty(Z)$ and $c > 0$, we have

$$|F(z,x)| \le \int_0^{|x|} |f(z,r)|dr \le a(z)|x| + \frac{c}{p}|x|^p .$$

$$\text{Hence} \quad \frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p - \|a\|_q\|x_n\|_p - \frac{c}{p}\|x_n\|_p^p \le M$$
$$\Rightarrow \quad \frac{1}{p}\|Dx_n\|_p^p \le M + \frac{1}{p}(\lambda_1 + c)\|x_n\|_p^p + \|a\|_q\|x_n\|_p \text{ for all } n \ge 1 .$$

Since $\|x_n\|_{1,p} \to \infty$, by virtue of Poincaré's inequality, we have $\|Dx_n\|_p \to \infty$ and so from the last inequality it follows that $\|x_n\|_p \xrightarrow{n\to\infty} \infty$. Let $y_n = \frac{x_n}{\|x_n\|_p}$, $n \ge 1$. Dividing the last inequality with $\|x_n\|_p^p$, we obtain

$$\frac{1}{p}\|Dy_n\|_p^p \le \frac{M}{\|x_n\|_p^p} + \frac{1}{p}(\lambda_1 + c) + \|a\|_p \frac{1}{\|x_n\|_p^{p-1}}$$
$$\Rightarrow \quad \{y_n\}_{n\ge1} \subseteq W_0^{1,p}(Z) \text{ is bounded (by Poincaré's inequality).}$$

Thus by passing to a subsequence if necessary, we may assume that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(Z)$, $y_n \to y$ in $L^p(Z)$, $y_n(z) \to y(z)$ a.e. on Z as $n \to \infty$ and $|y_n(z)| \le h(z)$ a.e. on Z with $h \in L^p(Z)$. Note that $\|y\|_p = 1$ and so $y \ne 0$. Also from the choice of the sequence $\{x_n\}_{n\ge1}$ we have that

$$\frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p \le M + \int_Z F(z,x_n(z))dz$$
$$\Rightarrow \quad \frac{1}{p}\|Dy_n\|_p^p - \frac{\lambda_1}{p} \le \frac{M}{\|x_n\|_p^p} + \int_Z \frac{F(z,x_n(z))}{\|x_n\|_p^p}dz . \tag{5}$$

By virtue of hypothesis H(f)(ii), given $\varepsilon > 0$, we can find $M_1 = M_1(\varepsilon) > 0$ such that

$$\eta_+(z) - \varepsilon \le \frac{f(z,x)}{|x|^{p-2}x} \le \eta_+(z) + \varepsilon \text{ for almost all } z \in Z \text{ and all } x \ge M$$

$$\text{and} \quad \eta_-(z) - \varepsilon \le \frac{f(z,x)}{|x|^{p-2}x} \le \eta_-(z) + \varepsilon \text{ for almost all } z \in Z \text{ and all } x \le -M .$$

If $x_n(z) \xrightarrow{n\to\infty} +\infty$, then for $n \geq 1$ large enough we have $x_n(z) > 0$ and so

$$\begin{aligned}\frac{F(z,x_n(z))}{|x_n(z)|^p} &\geq \frac{1}{|x_n(z)|^p}F(z,M_1) + \frac{1}{|x_n(z)|^p}\int_{M_1}^{x_n(z)}(\eta_+(z)-\varepsilon)|r|^{p-2}r\,dr \\ &= \frac{1}{|x_n(z)|^p}F(z,M_1) + \frac{1}{|x_n(z)|^p}(\eta_+(z)-\varepsilon)\frac{1}{p}(|x_n(z)|^p - M_1^p) \\ &\Rightarrow \varliminf_{n\to\infty}\frac{F(z,x_n(z))}{|x_n(z)|^p} \geq \frac{1}{p}(\eta_+(z)-\varepsilon)\,. \end{aligned} \tag{6}$$

Similarly we can show that

$$\varlimsup_{n\to\infty}\frac{F(z,x_n(z))}{|x_n(z)|^p} \leq \frac{1}{p}(\eta_+(z)+\varepsilon)\,. \tag{7}$$

From (6) and (7) and since $\varepsilon > 0$ was arbitrary, we infer that if $x_n(z) \xrightarrow{n\to\infty} -\infty$, then

$$\lim_{n\to\infty}\frac{F(z,x_n(z))}{|x_n(z)|^p} = \frac{1}{p}\,\eta_+(z)\,. \tag{8}$$

If $x_n(z) \xrightarrow{n\to\infty} -\infty$, then through a similar reasoning we obtain

$$\lim_{n\to\infty}\frac{F(z,x_n(z))}{|x_n(z)|^p} = \frac{1}{p}\,\eta_-(z)\,. \tag{9}$$

Let $g_n^+(z) = \begin{cases}\frac{F(z,x_n(z))}{|x_n(z)|^p} & \text{if } x_n(z) > 0 \\ 0 & \text{otherwise}\end{cases}$ and $g_n^-(z) = \begin{cases}\frac{F(z,x_n(z))}{|x_n(z)|^p} & \text{if } x_n(z) < 0 \\ 0 & \text{otherwise}\end{cases}$.
Note that since $y_n(z) \to y(z)$ a.e. on Z and $y \neq 0$, we deduce that $|x_n(z)| \to \infty$ for all $z \in E_1 \subseteq Z$, with $|E_1| > 0$. we have

$$\begin{aligned}\int_Z \frac{F(z,x_n(z))}{|x_n(z)|^p}\,|y_n(z)|^p dz &= \int_{\{x_n>0\}}\frac{F(z,x_n(z))}{|x_n(z)|^p}\,|y_n(z)|^p dz \\ &\quad + \int_{\{x_n<0\}}\frac{F(z,x_n(z))}{|x_n(z)|^p}\,|y_n(z)|^p dz \\ &\quad \text{(note that for all } z \in Z,\ F(z,0)=0) \\ &= \int_Z g_n^+(z)y_n^+(z)^p dz + \int_Z g_n^-(z)y_n^-(z)^p dz\end{aligned}$$

From (8), (9) and the dominated convergence theorm, we have

$$\int_Z g_n^+(z)y_n^+(z)^p dz \xrightarrow{n\to\infty} \frac{1}{p}\int_Z \eta_+(z)y^+(z)^p dz$$

and

$$\int_Z g_n^-(z)y_n^-(z)^p dz \xrightarrow{n\to\infty} \frac{1}{p}\int_Z \eta_-(z)y^-(z)^p dz\,.$$

Passing to the limit as $n \to \infty$ in (5), using these convergences and the fact that $||Dy||_p \leq \underline{\lim}_{n\to\infty} ||Dy_n||_p$ (from the weak lower semicontinuity of the norm functional, recall that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(Z)$), we obtain

$$\frac{1}{p}||Dy||_p^p \leq \frac{1}{p}\int_Z (\eta_+(z)y^+(z)^p + \eta_-(z)y^-(z)^p)dz \leq 0$$

(see hypothesis H(f)(iii))

$\Rightarrow \quad y = 0$ a contradiciton.

This proves the coercivity of $R(\cdot)$. Q.E.D.

Now Let $W_0^{1,p}(Z) = X \oplus V$, where $X = \mathbf{R}u_1$ and V is its topological complement.

Proposition 3.

If *hypotheses* $H(f)$ *hold,*
then $R_{|V} \geq 0$.

Proof. Recall (see section 2) that for all $v \in V$, we have that

$$\mu||v||_p^p \leq ||Dv||_p^p \tag{10}$$

Then using (10) and hypothesis H(f)(iv), we have that

$$R(v) \geq \frac{1}{p}||Dv||_p^p - \frac{\lambda_1}{p}||v||_p^p - \frac{\mu - \lambda_1}{p}||v||_p^p \geq 0, \text{ for all } v \in V\,.$$

Q.E.D.

Since $||Du_1||_p^p = \lambda_1||u_1||_p^p$ and using hypothesis H_1, we have at once the following proposition

Proposition 4.

If *hypothesis* H_1 *and* $H(f)$ *hold,*
then $R(\xi_\pm u_1) < 0$.

The next proposition shows that $R(\cdot)$ satisfies a kind of nonsmooth (PS)-condition over closed and convex subsets of $W_0^{1,p}(Z)$.

Proposition 5.

If *hypotheses* $H(f)$ *hold,* $K \subseteq W_0^{1,p}(Z)$ *is a nonempty, closed and convex and* $\{x_n\}_{n\geq 1} \subseteq K$, $\varepsilon_n > 0$, $\varepsilon_n \downarrow 0$ *satisfy* $|R(x_n)| \leq M$, $0 \leq R^0(x_n; y - x_n) + \varepsilon_n||y - x_n||$ *for all* $y \in K$,

then $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ *has a strongly convergent subsequence.*

Proof. Since by hypothesis $\{R(x_n)\}_{n\geq 1}$ is bounded and because $R(\cdot)$ is coercive (see proposition 2) we infer that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$ as $n \to \infty$. Recall that $R^0(x_n; x - x_n) = \sup\{< x^*, x - x_n >: x^* \in \partial R(x_n)\}$, $n \geq 1$ (here by $< \cdot, \cdot >$ we denote the duality brackets of the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$). Since $\partial R(x_n) \subseteq W^{-1,q}(Z)$ is weakly compact, we can find $x_n^* \in \partial R(x_n)$ such that $R^0(x_n; x - x_n) =< x_n^*, x - x_n >$, $n \geq 1$. Note that $x_n^* = A(x_n) - v_n$ where $A : W^{1,p}(Z) \to W^{-1,q}(Z)$ is defined by $< A(x), y >= \int_Z \|Dx(Z)\|^{p-2}(Dx(z), Dy(z))_{\mathbf{R}^N} dz$ for all $x, y \in W_0^{1,p}(Z)$ and $v_n \in \partial J(x_n)$, $n \geq 1$, with $J(x_n) = \int_Z F(z, x_n(z))dz$. We know that $f_1(z, x_n(z)) \leq v_n(z) \leq f_2(z, x_n(z))$ a.e. on Z (see Chang [6] and Kourogenis-Papageorgiou [11]). It is easy to scheck that A is monotone, demicontinuous, hence maximal monotone. Evidently $\{v_n\}_{n\geq 1} \subseteq L^q(Z)$ is bounded. We have

$$
\begin{aligned}
& 0 \leq < x_n^*, x - x_n > + \varepsilon_n \|x - x_n\| \\
& =< A(x_n), x - x_n > - \int_Z v_n(z)(x - x_n)(z)dz + \varepsilon_n \|x - x_n\| \\
\Rightarrow \quad & \overline{\lim} < A(x_n), x_n - n > \leq 0 \\
& (\text{since} \int_Z v_n(z)(x - x_n)(z)dz \xrightarrow{n\to\infty} 0 \text{ and } \varepsilon_n \|x - x_n\| \xrightarrow{n\to\infty} 0).
\end{aligned}
$$

Because A is maximal monotone, is generalized pseudomonotone (see Hu-Papageorgiou [10], remark III 6.3, p. 365). So we have $< A(x_n), x_n - x \xrightarrow{n\to\infty} < A(x), x > \Rightarrow \|Dx_n\|_p \xrightarrow{n\to\infty} \|Dx\|_p$. Recall that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbf{R}^N)$ and that $L^p(Z, \mathbf{R}^N)$ is uniformly convex, thus it has the Kadec-Klee property (see Hu-Papageorgiou [10], Definition I.1.72, p. 28). Therefore $Dx_n \to Dx$ in $L^p(Z, \mathbf{R}^N) \Rightarrow x_n \xrightarrow{n\to\infty} x$ in $W_0^{1,p}(Z)$. Q.E.D.

Because $\partial R(x) \subseteq W^{-1,q}(Z)$ is weakly compact, we can find $x^* \in \partial R(x)$ such that $\|x^*\|_* = m(x) = \inf\{\|y^*\| : y^* \in \partial R(x)\}$ (by $\|\cdot\|_*$ we denote the norm of $W^{-1,q}(Z)$). Then the same proof as for proposition 5 show that $R(\cdot)$ satisfies the nonsmooth (PS)-condition (see section 2).

Proposition 6.

If *hypotheses* $H(f)$ *hold,*
then $R(\cdot)$ *satisfies the nonsmooth (PS)-conditon.*

4. Existence of three nontrivial solutions

In this section we state and prove the main result of this work, which says that under the hypotheses introduced in section 3, problem (4) has at least three nontrivial solutions.

Theorem 7.

If *hypotheses* $H(f)$ *and* H_1 *hold,*
then *problem (4) has at least three nontrivial solutions.*

Proof. Let $U^{\pm} = \{x \in W_0^{1,p}(Z) : x = \pm tu_1 + v, t > 0, v \in V\}$. We show that $R(\cdot)$ attains its infimum on both open sets U^+ and U^-. To this end let $m_+ = \inf[R(x) : x \in U^+] = \inf[R(x) : x \in \overline{U}^+]$ (since $R(\cdot)$ is locally Lipschitz on $W_0^{1,p}(Z)$). Let

$$\overline{R}(x) = \begin{cases} R(x) & \text{if } x \in \overline{U}^+ \\ +\infty & \text{otherwise.} \end{cases}$$

Evidently $\overline{R}(\cdot)$ is lower semicontinuous on the Banach space $W_0^{1,p}(Z)$ and is bounded below (see proposition 2). By Ekeland's variational principle (see Hu-Papageorgiou [10], corollary V.1.2, p. 520), we can find $\{x_n\}_{n\geq 1} \subseteq U^+$ such that $R(x_n) \downarrow m_+$ as $n \to \infty$ and

$$\begin{aligned} & \overline{R}(x_n) \leq \overline{R}(y) + \varepsilon_n \|y - x_n\| \text{ for all } y \in W_0^{1,p}(Z) \\ \Rightarrow \quad & R(x_n) \leq R(y) + \varepsilon_n \|y - x_n\| \text{ for all } y \in \overline{U}^+ . \end{aligned}$$

Because $\overline{U}^+$ is convex, for every $t \in (0,1)$ and every $w \in \overline{U}^+$, $y_n = (1-t)x_n + tw \in \overline{U}^+$ for all $n \geq 1$. So we have

$$\begin{aligned} & -\varepsilon_n \|w - x_n\| \leq \frac{R(x_n + t(w - x_n)) - R(x_n)}{t} \\ \Rightarrow \quad & 0 \leq R^0(x_n; w - x_n) + \varepsilon_n \|w - x_n\| \text{ for all } w \in \overline{U}^+ . \end{aligned}$$

By virtue of proposition 5 and by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{n\to\infty} y_1$ in $W_0^{1,p}(Z)$. If $y_1 \in \partial U^+ = V$, then by virtue of proposition 3, we have $\lim R(x_n) = R(y_1) = m_+ > 0$. On the other hand proposition 4 implies that $0 > m_+$. Hence we have a contradiction, from which we infer that $y_1 \in int\overline{U}^+ = U^+$, $y_1 \neq 0$. Thus y_1 is a local minimum of R and so $0 \in \partial R(y_1)$ (see section 2). Similarly working on U^-, we obtain $y_2 \in U^-$, $y_1 \neq y_2 \neq 0$ such that $0 \in \partial R(y_2)$.

Next we will produce a third distinct, nontrivial critical point of $R(\cdot)$. By virtue of hypotheses H(f)(v), given $\varepsilon > 0$ we can find $\delta > 0$ such that for almost all $z \in Z$ and all $|x| \leq \delta$ we have

$$F(z,x) \leq \frac{1}{p}(-\beta+\varepsilon)|x|^p \,.$$

On the other hand recall that $|F(z,x)| \leq a(z)|x| + \frac{\varepsilon}{p}|x|^p$ for almost all $z \in Z$ and all $x \in \mathbf{R}$. Using Young's inequality on the term $a(z)|x|$ and since $a \in L^\infty(Z)$, we deduce that for almost all $z \in Z$ and all $x \in \mathbf{R}$, we have $|F(z,x)| \leq c_1 + c_2|x|^p$ for some $c_1, c_2 > 0$. Thus we can find $c_3 > 0$ such that for almost all $z \in Z$ and all $|x| > \delta$ we have $F(z,x) \leq c_3|x|^\theta$ with $\theta \in (p, p^* = \frac{Np}{N-p}]$. therefore finally we can say that for almost all $z \in Z$ and all $x \in \mathbf{R}$ we have

$$F(z,x) = \frac{1}{p}(-\beta+\varepsilon)|x|^p + c_3|x|^\theta, \; p < \theta \leq p^* = \frac{Np}{N-p} \,.$$

Using this growth of F, we obtain

$$\begin{aligned} R(x) &= \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z F(z,x(z))dz \\ &\geq \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p + \frac{1}{p}(\beta-\varepsilon)\|x\|_p^p - c_3\|x\|_\theta^\theta \\ &= \frac{1}{p}\|Dx\|_p^p - \frac{(\lambda_1-\beta+\varepsilon)}{p}\|x\|_p^p - c_3\|x\|_\theta^\theta \,. \end{aligned}$$

From hypothesis H(f)(v) we know that $\beta > \lambda_1$. So we can choose $\varepsilon > 0$ such that $\lambda_1 + \varepsilon < \beta$. Also because $\theta \leq p^* = \frac{Np}{N-p}$, $W_0^{1,p}(Z)$ is embedded continuously in $L^\theta(Z)$ (Sobolev embedding theorem). Thus we can find $c_4 > 0$ such that $\|x\|_\theta \leq c_4\|Dx\|_p$. Hence for $c_5 = \frac{c_4^\theta}{c_3} > 0$, we have

$$R(x) \geq \frac{1}{p}\|Dx\|_p^p - c_5\|Dx\|_p^\theta \,.$$

By Poincaré's inequality we can find $c_6, c_7 > 0$ such that

$$R(x) \geq c_6\|x\|_{1,p}^p - c_7\|x\|_{1,p}^\theta \text{ for all } x \in W_0^{1,p}(Z) \,.$$

This last inequality implies that there exists $0 < \rho < \min\{\xi_+, \xi_-\}$ such that $R(x) > 0$ for all $\|x\|_{1,p} = \rho$. Because $R(\xi_\pm u_1) < 0 = R(0)$, we can apply theorem 1 and obtain $y_3 \in W_0^{1,p}(Z)$, $y_3 \neq 0$, $y_3 \neq y_1$, $y_3 \neq y_2$ (since $R(y_3) > 0$) such that $0 \in \partial R(y_3)$.

Finally let $y = y_k$, $k = \{1,2,3\}$. From our previous considerations we know that $0 \in \partial R(y)$. Hence $A(y) - \lambda_1|y|^{p-2}y = v$ with $v \in L^q(Z)$, $f_1(z,y(z)) \leq v(z) \leq f_2(z,y(z))$ a.e. on Z (i.e. $u \in \partial J(y)$). Then for any $\theta \in C_0^\infty(Z)$ we have

$$< A(y), \theta > -\lambda_1 \int_Z |y(z)|^{p-2}y(z)\theta(z)dz = \int_Z v(z)\theta(z)dz$$

$$\Rightarrow \quad \int_Z \|Dy(z)\|^{p-2}(Dy(z), D\theta(z))_{\mathbf{R}^N}dz = \int_Z (v(z) + \lambda_1|y(z)|^{p-2}y(z))\theta(z)dz$$

Note that $D \in \mathbf{L}(W_0^{1,p}(Z), L^p(Z, \mathrm{R}^N))$ and $D^* = -div \in \mathbf{L}(L^q(Z, \mathbf{R}^N), W^{-1,q}(Z))$. Thus we have

$$< -div(\|Dy\|^{p-2}Dy), \theta >= (v + \lambda_1 |y|^{p-2}y, \theta)_{pq} \, .$$

Since $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$, we deduce that

$$\left\{ \begin{array}{l} -div(\|Dy(z)\|^{p-2}Dy(z)) - \lambda_1 |y(z)|^{p-2}y(z) = v(z) \text{ a.e. on } Z \\ y_{|\Gamma} = 0, 2 \leq p < \infty \end{array} \right\} \qquad (11)$$

and $v(z) \in \widehat{f}(z, x(z))$ a.e. on Z. This proves that y_1, y_2, y_3 are three distinct nontrivial solutions of (4). Q.E.D.

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