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A PRIORI ESTIMATE ON THE FREE BOUNDARY PROBLEM AND ITS APPLICATION

By

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Abstract. The free boundary problem with the area functional is investigated. A priori estimate for the measure of the caved part of the set where the function is zero is obtained, so that a convex property of the free boundary is established.

0. Introduction

Let $\mathbf{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space, and let Ω be a bounded domain in $\mathbf{R}^n$ whose boundary $\partial\Omega$ is locally a Lipschitz graph. Given a non-negative bounded function u^0 belonging to $BV(\Omega)$ and a positive $L^2(\Omega)$-function Q, we consider the variational problem of minimizing the energy functional

$$J(w) = \int_\Omega \sqrt{1+|Dw|^2} + \int_\Omega Q^2 \chi_{w>0} d\mathcal{L}^n + \int_{\partial\Omega} |w - u^0| d\mathcal{H}^{n-1}$$

among all $w \in BV(\Omega)$. Here, $\mathcal{L}^n$ is the n-dimensional Lebesgue measure, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and $\chi_{w>0}$ is the characteristic function of the set $\{z \in \Omega \mid w(z) > 0\}$. $BV(\Omega)$ is the space of $\mathcal{L}^n$-integrable functions defined on Ω whose derivatives in the sense of distributions are signed measures with finite total variation in Ω. $\sqrt{1+|Dw|^2}$ denotes the total variation measure of the vector valued measure $(\mathcal{L}^n, D_1 w, \ldots, D_n w)$. The last term of $J(w)$ is defined in the sense of L^1-trace (see [7, Chapter 2]).

Our aim of this paper is to investigate a convex property of the set $\Omega(u = 0) := \{z \in \Omega \mid u(z) = 0\}$ for a minimizer u of J in the case $\inf_\Omega Q > 1$. The regularity of the free boundary $\partial\Omega(u > 0) = \Omega \cap \partial\{z \in \Omega \mid u(z) > 0\}$ for a minimizer u is studied by Caffarelli and Friedman in [6] under the hypothesis $n \leq 6$ and $\sup_\Omega Q < 1$. However, their approach that essentially needs the Lipschitz continuity of minimizers is no longer be useful unless $\sup_\Omega Q < 1$. In fact, we constructed in the case where $n = 2$ and $Q \equiv \text{Const.} > 1$ an example of a radially symmetric minimizer which is not even continuous in Ω (see [14],

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[17]). On the other hand, we pursue in this paper a property of the set $\Omega(u=0)$ without arguing about the regularity of u.

For $u \in BV(\Omega)$ let E_u be an interior of the set $\Omega(u=0)$. Then main result of this paper is described as follows: Suppose $\inf_\Omega Q > 1$ and let u be a minimizer of J. If $\mathcal{L}^n(\partial\,\Omega(u>0)) = 0$, then E_u is convex in $\mathcal{L}^n$-measure (see Theorem 2). In showing this result the following a priori estimate plays an essential role:

$$\int_F Q^2 \chi_{u>0} d\mathcal{L}^n \leq \mathcal{L}^n\Big(F \cap \big[F \cap \partial H_{\nu_0}(z_0)) \times \mathbf{R}\big]\Big)$$

holds for any connected component F of $H_{\nu_0}(z_0)\backslash E_u$ having no points of Ω^c, where $z_0 \in \mathbf{R}^n$ and $\nu_0 \in S^{n-1}$ and $H_{\nu_0}(z_0) = \{z \in \mathbf{R}^n \mid \langle z - z_0, \nu_0\rangle \geq 0\}$ (see Theorem 1). Here $(F \cap \partial H_{\nu_0}(z_0)) \times \mathbf{R} = \{\xi + t\nu_0 \mid \xi \in F \cap \partial H_{\nu_0}(z_0), t \in \mathbf{R}\}$.

For proving the inequality above, it suffices to adopt

$$v(x) = \begin{cases} 0 & \text{if } x \in F\,; \\ u(x) & \text{if } x \in \Omega\backslash F \end{cases}$$

as the comparison function. But, when we strictly accomplish the proof, there will appear the necessity that we should verify the legitimacy of choosing such a comparison function. For this purpose we must show the proposition that v belongs to $BV(\Omega)$. This seems, at first glance, to be generally false because the regularity of the free boundary is unknown. However, due to the property that F is a connected component we are able to demonstrate that u must take the value zero on ∂F (see (7) and (9) in the proof of Lemma 1 in Section 2), so that, it is possible to make sure that the proposition above is true. The practical calculation is established with the help of the theory of 'Fubinization' of BV-functions which is summed up in Section 1.

The variational problem of the kind treated in this paper was firstly introduced by Alt and Caffarelli in the article [1]. Their results have good applications for solving jet and cavitational flow problem (see [2], and refer to [3]). In [10], [11], [12], the free boundary problem for quasi linear equations is investigated. Their problem can be related to the experiment of peeling off the charged film in the electric field (see [12, Appendix]). The surface area type problem is dealt with in [6], [14]–[17]. In [6] the regularity result of the free boundary is shown in the case $n \leq 6$ and $\sup_\Omega Q < 1$, and the result is applied to the capillary drop problem. The author investigated in [14]–[17] the regularity of a minimizer in the radially symmetric case according to the value of the constant Q. In particular, a 'peeling off' and a 'soap film' experiments which can be tied in with the variational problem of the present paper are stated in [15].

Notation We sum up notation used throughout this paper: $\mathbf{N}$ is the set of all positive integers, and $\mathbf{R}$ is the set of all real numbers. We set $\mathbf{R}^+ =$

$\{t \in \mathbf{R} \mid t > 0\}$. For subsets A and B of $\mathbf{R}^n$, $A \backslash B := A \cap B^c$, where B^c is the complement of B. For a subset A of $\mathbf{R}^n$, we indicate with ∂A and A^o the boundary and interior of A respectively. For $x \in \mathbf{R}^n$ and a positive number r, $B_r(x)$ is the n-dimensional open ball of radius r with the center x. We denote for a subset A of $\mathbf{R}^n$ by χ_A the characteristic function of A. Let $i \in \mathbf{N}$, then $\mathcal{L}^i$ is the i-dimensional Lebesgue measure. For an open subset W of $\mathbf{R}^i$, we denote by $L^1(W)$ is the space of all $\mathcal{L}^i$-integrable functions defined on W, and $BV(W)$ is the space of $L^1(W)$-functions whose derivatives are measures with finite total variation. For a function f defined on $\mathbf{R}^i$ and a subset A of $\mathbf{R}^i$, $f|_A$ is the function obtained by restricting the domain of definition of f to A. Let $\zeta \in BV(\mathbf{R})$ and $t_0 \in \mathbf{R}$. When there exists a $\mathcal{L}^1$-null set N such that

$$\lim_{\substack{\varepsilon \in (t_0, \infty) \backslash N \\ \varepsilon \downarrow t_0}} \zeta(t_0 + \varepsilon)$$

exists, we say ζ has the right-trace at t_0 and denote by $\zeta^+(t_0)$ its value. Similarly we define the left-trace of ζ at t_0 denote by $\zeta^-(t_0)$. It is well-known that $\zeta^\pm(t_0)$ are definite for any $\zeta \in BV(\mathbf{R})$ and $t_0 \in \mathbf{R}$.

1. Summary on Fubinization of BV-functions

The theory of Fubinization is useful for showing properties of BV-functions (see for instance [4], [5]). The results stated in this section can be demonstrated by referring the proofs of [18, Theorem 5.3.5] and theorems in [9, Chapter 2 & 3] (see also [13, Theorem in Page 160]). The following theorem is shown by taking advantage of the theory of the decomposition of measures (refer to [8]). In the statement, for $w \in BV(\mathbf{R}^n)$ and an integer k, $1 \leq k \leq n$, $|D_k w|$ is the total variation measure of $D_k w$.

Theorem A. *Let $w \in BV(\mathbf{R}^n)$, and let $1 \leq k \leq n$. Then there exists a $\mathcal{L}^{n-1}$-null set $\mathcal{N} \subset \mathbf{R}^{n-1}_k \equiv \{(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n) \mid (z_1, \ldots, z_n) \in \mathbf{R}^n\}$ such that for any fixed $\xi = (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n) \in \mathbf{R}^{n-1}_k \backslash \mathcal{N}$, the function $w_\xi(t) \equiv w(\xi_1, \ldots, \xi_{k-1}, t, \xi_{k+1}, \ldots, \xi_n)$ $(t \in \mathbf{R})$ belongs to $BV(\mathbf{R})$. Furthermore, for any Borel measurable and $|D_k w|$-integrable function ψ on $\mathbf{R}^n$ and for any fixed $\xi \in \mathbf{R}^{n-1}_k \backslash \mathcal{N}$, the function $\psi_\xi(t) \equiv \psi(\xi_1, \ldots, \xi_{k-1}, t, \xi_{k+1}, \ldots, \xi_n)$ $(t \in \mathbf{R})$ is $|Dw_\xi|$-integrable on $\mathbf{R}$, and the function defined on $\mathbf{R}^{n-1}_k$ by*

$$\xi \mapsto \begin{cases} \displaystyle\int_{\mathbf{R}} \psi_\xi |Dw_\xi| & \text{if } \xi \in \mathbf{R}^{n-1}_k \backslash \mathcal{N}; \\ 0 & \text{if } \xi \in \mathcal{N} \end{cases}$$

is $\mathcal{L}^{n-1}$-integrable and the following equality holds:

$$\int_{\mathbf{R}^n} \psi |D_k w| = \int_{\mathbf{R}_k^{n-1}} d\,\mathcal{L}^{n-1}(\xi) \int_{\mathbf{R}} \psi_\xi |Dw_\xi| .$$

The following theorem is directly proved by applying Fubini-Tonelli theorem. The notations $\mathbf{R}_k^{n-1}$, w_ξ and $|D_k w|$ appeared in the statement are defined in the same way as in the preceding theorem:

Theorem B. *Let $w \in L^1(\mathbf{R}^n)$, V be an open set in $\mathbf{R}^n$, and let $1 \leq k \leq n$. Assume that there exists a $\mathcal{L}^{n-1}$-null set $\mathcal{N} \subset \mathbf{R}_k^{n-1}$ such that $\int_{V_\xi} |Dw_\xi| < \infty$ for any fixed $\xi = (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n) \in \mathbf{R}_k^{n-1} \backslash \mathcal{N}$, where $V_\xi = \Phi_\xi(V \cap \mathbf{R}_\xi)$, $\mathbf{R}_\xi = \{(\xi_1, \ldots, \xi_{k-1}, t, \xi_{k+1}, \ldots, \xi_n) \mid t \in \mathbf{R}\}$ and $\Phi_\xi(\xi_1, \ldots, \xi_{k-1}, t, \xi_{k+1}, \ldots, \xi_n) = t$ for each $t \in \mathbf{R}$, and assume that the function on $\mathbf{R}_k^{n-1}$ defined by*

$$\xi \mapsto \begin{cases} \displaystyle\int_{V_\xi} |Dw_\xi| & \text{if } \xi \in \mathbf{R}_k^{n-1} \backslash \mathcal{N}\,; \\ 0 & \text{if } \xi \in \mathcal{N} \end{cases}$$

is $\mathcal{L}^{n-1}$-integrable on $\mathbf{R}_k^{n-1}$. Then

$$\int_V |D_k w| < \infty .$$

2. Lemmas

Let u be a minimizer for the functional J in the class $BV(\Omega)$ (see [14] for the existence of such a minimizer) and let E_u be the interior of the set $\{z \in \Omega \mid u(z) = 0\}$. Since the regularity of u is not obtained, u is nothing but the equivalence class of functions, where two functions are equivalent if they agree everywhere on Ω except possibly for a set of $\mathcal{L}^n$-measure zero. In the following, nevertheless, u signifies an arbitrarily fixed representative in the class. As a result, the value of u is definite everywhere on Ω, and we may assume

$$0 \leq u(z) \leq \operatorname*{essup}_{\Omega} u^0 \qquad \text{for each } z \in \Omega \tag{1}$$

by taking into account the maximum principle described as in [15]. Moreover, we emphasize that the set E_u is well-defined for such u. Let $z_0 \in \mathbf{R}^n$ and $\nu_0 \in \mathbf{S}^{n-1}$, then $H_{\nu_0}(z_0)$ and $\mathcal{F}_{\nu_0}(z_0)$ be the family of connected component of $H_{\nu_0}(z_0) \backslash E_u$ containing no point of Ω^c. In the present section, we designate by F an arbitrarily fixed element of $\mathcal{F}_{\nu_0}(z_0)$. To simplify notation, we will take $z_0 = 0$

and $\nu_0 = \vec{e}_1 = (1, 0, \ldots, 0)$, and moreover, the notaions $H_{\vec{e}_1}(0)$ and $\mathcal{F}_{\vec{e}_1}(0)$ are abbreviated to H and $\mathcal{F}$ respectively. We start from the following remark:

Remark. F is a compact set contained in Ω.

In fact, F is closed since F is a connected component of the closed set $H \backslash E_u$. Furthermore, F does not have any point belonging to Ω^c, and so $F \subset \Omega$. Thus, the desired result follows from the boundedness assumption on Ω.

Lemma 1. *The function v defined by*

$$v(x) = \begin{cases} 0 & \text{if } x \in F\,; \\ u(x) & \text{if } x \in \Omega \backslash F \end{cases}$$

belongs to $BV(\Omega)$.

Proof. We shall extend the domain of definition of u to $\mathbf{R}^n$ by defining $u = 0$ in $\mathbf{R}^n \backslash \Omega$. We here notice that $u \in BV(\mathbf{R}^n)$ because $\partial\Omega$ is assumed to be Lipschitz continuous (see [18, Lemma 5.10.4 and Remark 5.10.2]). In order to show the assertion of Lemma 1, it is enough to prove that the function

$$x \mapsto \begin{cases} 0 & \text{if } x \in F\,; \\ u(x) & \text{if } x \in \mathbf{R}^n \backslash F \end{cases}$$

which we again denote by v belongs to $BV(\mathbf{R}^n)$. For this purpose, we shall show for each $i = 1, \ldots, n$ that

$$\int_{H^o} |D_i v| < \infty\,. \tag{2}$$

If (2) will be shown to be true, we have $v|_{H^o} \in BV(H^o)$. Moreover, $v|_{H^c} = u|_{H^c} \in BV(H^c)$, and hence due to the facts that v is bounded in $\mathbf{R}^n$ and the support of v is compact we are able to reach $v \in BV(\mathbf{R}^n)$ (see [7, Proposition 2.8]).

Let us demonstrate (2). We prove only the case $i = 1$ since the other cases are similarly shown. Let $\mathcal{N} \subset \partial H$ be a $\mathcal{L}^{n-1}$-null set obtained by applying Theorem A to $w = u$ and $k = 1$. Then, by adopting Theorem B to $w = v$, $W = H^o$ and $k = 1$, the proof of (2) with $i = 1$ will be achieved if we show

- $\displaystyle\int_{\mathbf{R}+} |Dv_\xi| < +\infty$ *holds for each* $\xi \in \partial H \backslash \mathcal{N}$; (3)
- *The function* $\xi \mapsto \begin{cases} \displaystyle\int_{\mathbf{R}+} |Dv_\xi| & \text{if } \xi \in \partial H \backslash \mathcal{N}\,; \\ 0 & \text{if } \xi \in \mathcal{N} \end{cases}$

  is $\mathcal{L}^{n-1}$-integrable on ∂H, where $v_\xi(t) = v(t, \xi)$ for $t \in \mathbf{R}$. (4)

Before showing (3) and (4), we verify a property of u_ξ, $u_\xi(t) = u(t,\xi)$, which will play a key role in the proof below. Let us fix $\xi \in \partial H\backslash\mathcal{N}$ arbitrarily. We then note that u_ξ belongs to $BV(\mathbf{R})$ and

$$u_\xi(t) = \begin{cases} 0 & \text{if } (t,\xi) \in F \cap \mathbf{R}_\xi^+ \,; \\ u_\xi(t) & \text{if } (t,\xi) \in U \cap \mathbf{R}_\xi^+ \,, \end{cases} \tag{5}$$

where $U = F^c$ and $\mathbf{R}_\xi^+ = \{(t,\xi) \mid t \in \mathbf{R}^+\}$. In the sequel, for a subset C of $\mathbf{R}$ we denote by C_ξ the set $\{(t,\xi \mid t \in C\}$. Let $C_R(0) = \{x \in \mathbf{R}^n \mid |x^j| < R,\ j = 1,2,\ldots,n\}$ be an n-dimensional cube such $\Omega \subset\subset C_R(0)$. $U \cap \mathbf{R}_\xi^+ \cap C_R(0)$ is a 1-dimensional open set, it is, as well-known, described as the union of at most countable and pairwise disjoint open intervals. The following three cases can occur:

$$U \cap \mathbf{R}_\xi^+ \cap C_R(0) = (0,R)\,; \tag{6a}$$

$$U \cap \mathbf{R}_\xi^+ \cap C_R(0) = (0,b_0) \cup \bigcup_{l=1}^{\infty}(a_l,b_l) \cup (a_0,R) \quad \text{(disjoint union)}\,; \tag{6b}$$

$$U \cap \mathbf{R}_\xi^+ \cap C_R(0) = \bigcup_{l=1}^{\infty}(a_l,b_l) \cup (a_0,R) \quad \text{(disjoint union)}\,, \tag{6c}$$

where $a_l, b_l \in (0,R)$, $l = 0,1,2,\ldots$. Hereafter, for notational simplicity, we identify subsets of $\mathbf{R}$ and the corresponding subsets of $\mathbf{R}_\xi$. In the case (6b,c) we shall show

$$u_\xi^+(a_l) = 0 \quad \text{for } l = 0,1,2,\ldots, \tag{7}$$

where $u^+(a_l)$ is the right-trace of u_ξ at a_l (see Notation of Introduction). To do this, we have only to show that $u_\xi^+(a_l)$ is not positive because $u_\xi^+(a_l)$ is known by (1) to be non-negatime. Suppose on the contrary that $u_\xi^+(a_l) > 0$. Then there exists a positive number δ and a $\mathcal{L}^1$-null set Λ such that

$$u_\xi(t) > 0 \quad \text{for any } t \in (a_l, a_l+\delta)_\xi\backslash\Lambda\,. \tag{8}$$

Therefore, we particularly have $(a_l, a_l+\delta)_\xi\backslash\Lambda \subset H\backslash E_u$. On the other hand, any point $x \in (a_l, a_l+\delta)_\xi \cap \Lambda$ also belongs to $H\backslash E_u$. Otherwise, $x \in E_u = \{z \in \Omega \mid u(z) =\}^o$ must hold, and hence there exists a positive number ε such that $u_\xi = 0$ in $A_\varepsilon := B_\varepsilon(x) \cap (a_l, a_l+\delta)_\xi$. We thus have from (8) the inclusion $A_\varepsilon \subset \Lambda$, which yields the contradiction $0 < \mathcal{L}^1(A_\varepsilon) \le \mathcal{L}^1(\Lambda)$. Subsequently, we obtain $(a_l, a_l+\delta)_\xi \subset H\backslash E_u$. Now, by noticing that the point $(a_l,\xi) \in \mathbf{R}^n$ belongs to F, $F \cup (a_l, a_l+\delta)_\xi$ is turned out to be a connected subset of $H\backslash E_u$ which strictly contains F. This contradicts to the maximality of F. We thus have proved (7). Similarly, it holds that

$$u_\xi^-(b_l) = 0 \quad \text{for } l = 0, 1, 2, \dots. \tag{9}$$

We are now in a position to prove (3). In the case (6a), since $v_\xi = u_\xi$ in $\mathbf{R}$, the assertion holds. We only show the case (6b) because (6c) is done in the same way. Set for $j \in \mathbf{N}$,

$$f_j(t) := \begin{cases} u_\xi(t) & \text{if } t \in \mathbf{R}\setminus\left[\bigcup_{l=1}^j (a_l, b_l) \cup (0, b_0) \cup (a_0, R)\right]; \\ 0 & \text{if } t \in \bigcup_{l=1}^j (a_l, b_l) \cup (0, b_0) \cup (a_0, R), \end{cases}$$

For $\zeta \in BV(\mathbf{R})$ and $t_0 \in \mathbf{R}$ the equality

$$\int_{\{t_0\}} |D\zeta| = |\zeta^+(t_0) - \zeta^-(t_0)|$$

is known (refer to [7, Proposition 2.8]). Taking this equality and (7), (9), we infer

$$\begin{aligned} \int_{\mathbf{R}^+} |Df_j| &= \int_{\mathbf{R}^+\setminus[\cup_{l=1}^j [a_l, b_l] \cup (0, b_0] \cup [a_0, R)]} |Du_\xi| + \sum_{l=1}^j \left(|u_\xi^-(a_l)| + |u_\xi^+(b_l)|\right) \\ &= \int_{\mathbf{R}^+\setminus[\cup_{l=1}^j [a_l, b_l] \cup (0, b_0] \cup [a_0, R)]} |Du_\xi| + \sum_{l=1}^j \left(\int_{\{a_l\}} |Du_\xi| + \int_{\{b_l\}} |Du_\xi|\right) \\ &= \int_{\mathbf{R}^+\setminus[\cup_{l=1}^j (a_l, b_l) \cup (0, b_0) \cup (a_0, R)]} |Du_\xi| \le \int_{\mathbf{R}^+} |Du_\xi|. \end{aligned} \tag{10}$$

Since from (5) $\lim_{j\to\infty} f_j = u_\xi - v_\xi$ pointwise in $\mathbf{R}^+$ and $|f_j| \le u_\xi$ in $\mathbf{R}^+$ for each $j \in \mathbf{N}$, with the aid of the dominated convergence theorem we have $\lim_{j\to\infty} f_j = u_\xi - v_\xi$ in $L^1(\mathbf{R}^+)$. Thus, by the lower-semicontinuity of the total variation measure (see [18, Theorem 5.2.1]), we obtain by letting $j \to \infty$ in (10) that

$$\int_{\mathbf{R}^+} |D(u_\xi - v_\xi)| \le \int_{\mathbf{R}^+} |Du_\xi|. \tag{11}$$

Consequently, (11) implies

$$\int_{\mathbf{R}^+} |Dv_\xi| \le \int_{\mathbf{R}^+} |Du_\xi| + \int_{\mathbf{R}^+} |D(u_\xi - v_\xi)| \le 2\int_{\mathbf{R}^+} |Du_\xi| < +\infty.$$

We now turn to the proof of (4). Denote for each $\xi \in \partial H$ and $t \in \mathbf{R}$, $(\chi_{U\cap H^\circ})_\xi(t) = \chi_{U\cap H^\circ}(t, \xi)$. Then our aim is established if the equality

$$\int_{\mathbf{R}^+} |Dv_\xi| = \int_{\mathbf{R}} (\chi_{U\cap H^\circ})_\xi |Du_\xi| \quad \text{for } \xi \in \partial H\setminus\mathcal{N} \tag{12}$$

will be shown, because by Theorem A the function

$$\xi \mapsto \begin{cases} \int_{\mathbf{R}} (\chi_{U \cap H^\circ})_\xi |Du_\xi| & \text{if } \xi \in \partial H \backslash \mathcal{N}\,; \\ 0 & \text{if } \xi \in \mathcal{N} \end{cases}$$

is known to be $\mathcal{L}^{n-1}$-integrable on ∂H. Let us show (12). Fix $\xi \in \partial \backslash \mathcal{N}$ arbitrarily. As usual we only show the case (6b). Let $\{(a_l, b_l)\}_{l=0}^{\infty}$ be as in (6b). Set for $j \in \mathbf{N}$

$$g_j(t) = \begin{cases} u_\xi(t) & \text{if } t \in \bigcup_{l=1}^{j}(a_l, b_l) \cup (0, b_0) \cup (a_0, R)\,; \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\lim_{j\to\infty} g_j = u_\xi$ pointwise in $U \cup \mathbf{R}_\xi^+$, and therefore by noticing that $0 \le g_j \le \operatorname{essup}_\Omega u^0$ in $U \cap \mathbf{R}_\xi^+$ we have $\lim_{j\to\infty} g_j = u_\xi$ in $L^1(U \cap \mathbf{R}_\xi^+)$. Hence, from the lower semi-continuity of the total variation measure

$$\int_{U \cap \mathbf{R}_\xi^+} |Du_\xi| \le \varliminf_{j\to\infty} \int_{U \cap \mathbf{R}_\xi^+} |Dg_j|\,. \tag{13}$$

On the other hand, for each $j \in \mathbf{N}$

$$\int_{U \cap \mathbf{R}_\xi^+} |Dg_j| = \int_{\cup_{l=1}^{j}(a_l,b_l)\cup(0,b_0)\cup(a_0,R)} |Du_\xi| \le \int_{U \cap \mathbf{R}_\xi^+} |Du_\xi|\,. \tag{14}$$

From (13) and (14)

$$\lim_{j\to\infty} \int_{U \cap \mathbf{R}_\xi^+} |Dg_j| = \int_{U \cap \mathbf{R}_\xi^+} |Du_\xi|\,. \tag{15}$$

Since by make use of (7) and (9)

$$\begin{aligned} \int_{\mathbf{R}^+} |Dg_j| &= \int_{U \cap \mathbf{R}_\xi^+} |Dg_j| + \sum_{l=0}^{j} \Big(|u_\xi^+(a_l)| + |u_\xi^-(b_l)| \Big) \\ &= \int_{U \cap \mathbf{R}_\xi^+} |Dg_j| \\ &= \int_{\cup_{l=1}^{j}(a_l,b_l)} |Du_\xi| = \int_{\cup_{l=1}^{j}(a_l,b_l)} |Dv_\xi| \le \int_{\mathbf{R}^+} |Dv_\xi| \end{aligned} \tag{16}$$

and $\lim_{j\to\infty} g_j = v_\xi$ pointwise in $\mathbf{R}^+$, we are able to conclude by the same argument leading to (15) that

$$\lim_{j\to\infty} \int_{\mathbf{R}^+} |Dg_j| = \int_{\mathbf{R}^+} |Dv_\xi|\,. \tag{17}$$

(16) also tells us that

$$\int_{\mathbf{R}^+} |Dg_j| = \int_{U\cap\mathbf{R}^+_\xi} |Dg_j|,$$

and therefore from (15) and (17) we arrive at our desired result (12). □

Lemma 2. *With the hypothesis of Lemma 1, we have the equality*

$$\int_{F\cap H^\circ} |Dv| = 0.$$

Proof. Let the notation U, u_ξ, v_ξ and $(\chi_{F\cap H^\circ})_\xi$ be defined as in Lemma 1. It suffices to show

$$\int_{F\cap H^\circ} |D_i v| = 0 \quad (i = 1, \dots, n).$$

For each $\xi \in \partial H$,

$$\begin{aligned}\int_{\mathbf{R}^+} |Dv_\xi| &= \int_{\mathbf{R}^+_\xi\cap U} |Dv_\xi| + \int_{\mathbf{R}^+_\xi\cap F} |Dv_\xi| \\ &= \int_{\mathbf{R}^+_\xi\cap U} |Du_\xi| + \int_{\mathbf{R}^+_\xi\cap F} |Dv_\xi|.\end{aligned} \tag{18}$$

Let $\mathcal{N}$ be a $\mathcal{L}^{n-1}$-null set as in the proof of Lemma 1. Then by recalling (12) in the proof of Lemma 1, we infer from (18) that

$$\int_{\mathbf{R}^+_\xi\cap F} |Dv_\xi| = 0 \quad \text{for } \xi \in \partial H \setminus \mathcal{N}. \tag{19}$$

Since from Lemma 1 $v \in BV(\mathbf{R}^n)$, we are able to apply Theorem A in order to obtain

$$\begin{aligned}\int_{F\cap H^\circ} |D_1 v| &= \int_{\mathbf{R}^n} \chi_{F\cap H^\circ} |D_1 v| \\ &= \int_{\partial H} d\mathcal{L}^{n-1}(\xi) \int_{\mathbf{R}} (\chi_{F\cap H^\circ})_\xi |Dv_\xi| \\ &= \int_{\partial H} d\mathcal{L}^{n-1}(\xi) \int_{\mathbf{R}^+_\xi\cap F} |Dv_\xi|.\end{aligned}$$

Thus, we establish by (19) our assertion for $i = 1$. The result follows in the corresponding way for $i = 2, \dots, n$. □

Let $\mathcal{A} = \{\xi \in \partial H \mid \xi + t\vec{e}_1 \in F \text{ holds for a certain } t \geq 0\}$, and for each $\xi \in \mathcal{A}$ we set $l_\xi = \min\{t \in \mathbf{R} \mid \xi + t\vec{e}_1 \in F\}$. Notice here that the minimum is attained because the set $\{t \in \mathbf{R} \mid \xi + t\vec{e}_1 \in F\}$ is compact (see Remark). Let $\mathcal{A}_+ = \{\xi \in \mathcal{A} \mid l_\xi > 0\}$ and $\mathcal{A}_0 = \{\xi \in \mathcal{A} \mid l_\xi = 0\}$. Then, we in particular notice that $\mathcal{A}_0 = \partial H \cap F$.

The last lemma of this section asserts an inequality which will be a key-estimate in the proof of main theorem of this paper:

Lemma 3. *Let v be the function defined as in Lemma 1. Then we have*

$$\int_{F\cap[\mathcal{A}_0\times\mathbf{R}]} |D_1 u| \geq \int_{\mathcal{A}_0} \sqrt{1+|Dv|^2}\,.$$

Proof. Let $\mathcal{N} \subset \partial H$ be the $\mathcal{L}^{n-1}$-null set which is obtained by applying Theorem A to $w = u$ and $k = 1$. Fix $\xi \in \mathcal{A}_0\backslash\mathcal{N}$ arbitrarily. In particular, from the proof of Lemma 1 it holds that u_ξ and v_ξ belong to $BV(\mathbf{R})$. Let us define

$$\delta_\xi := \sup\{t \geq 0 \mid [0,t]_\xi \subset F\}\,.$$

We here remark that δ_ξ is definite since $\xi \in \mathcal{A}_0$, and that $[0,\delta_\xi]_\xi \subset F$ for the compactness of F. We shall assert

$$u_\xi^+(\delta_\xi) = 0\,. \tag{20}$$

Assume that $u_\xi^+(\delta_\xi)$ is positive. Then, by the same argument as for the proof of (7), there exists a positive number δ such that $(\delta_\xi, \delta_\xi+\delta]_\xi \subset H\backslash E_u$. By the maximality of F it must hold that $[0,\delta_\xi+\delta]_\xi \subset F$, which contradicts to the definition of δ_ξ. We thus have $u_\xi^+(\delta_\xi) \leq 0$. Taking the non-negativity of u (see (1)) into account, we are led to (20).

Now, we shall show the inequality

$$\int_{F\cap\mathbf{R}_\xi} |Du_\xi| \geq u_\xi^-(0)\,. \tag{21}$$

In case $\delta_\xi = 0$, we are able to lead (21) as follows:

$$\int_{F\cap\mathbf{R}_\xi} |Du_\xi| \geq \int_{\{0\}} |Du_\xi| = |u_\xi^+(0) - u_\xi^-(0)| = u_\xi^-(0)\,,$$

where the last equality follows from (20) with $\delta_\xi = 0$. On the other hand, in case $\delta_\xi > 0$, we establish (21) with the aid of the fundamental theorem of the calculus for $BV(\mathbf{R})$-functions in the following manner:

$$\begin{aligned}\int_{F\cap\mathbf{R}_\xi}|Du_\xi| &\geq \int_{[0,\delta_\xi]}|Du_\xi| \\ &= |u_\xi^+(0)-u_\xi^-(0)| + \int_{(0,\delta_\xi)}|Du_\xi| + |u_\xi^+(\delta_\xi)-u_\xi^-(\delta_\xi)| \\ &\geq |u_\xi^+(0)-u_\xi^-(0)| + \left|\int_{(0,\delta_\xi)}Du_\xi\right| + u_\xi^-(\delta_\xi) \\ &\geq \left|-u_\xi^-(0) + \left(u_\xi^+(0) + \int_{(0,\delta_\xi)}Du_\xi\right)\right| + u_\xi^-(\delta_\xi) \\ &= |-u_\xi^-(0)+u_\xi^-(\delta_\xi)| + u_\xi^-(\delta_\xi) \geq u_\xi^-(0)\,,\end{aligned}$$

where we used (20).

Since $u=v$ in H^c,

$$u_\xi^-(0) = v_\xi^-(0)\,. \tag{22}$$

Moreover,

$$v_\xi^+(0) = 0 \tag{23}$$

is valid. In fact, when $\delta_\xi = 0$, we infer from (20) and the inequality $u \leq v$ in Ω the desired result (23). When $\delta_\xi > 0$, because $(0,\delta_\xi)_\xi \subset F$ holds, we have $v=0$ in $(0,\delta_\xi)_\xi$, and therefore we also in this case reach (23) by the definition of v_ξ^+. Coupling (22) and (23), we deduce from (21) that

$$\int_{F\cap\mathbf{R}_\xi}|Du_\xi| \geq |v_\xi^+(0)-v_\xi^-(0)| = \int_{\mathbf{R}}\chi_{\{0\}}(t)|Dv_\xi|\,. \tag{24}$$

Due to Theorem A, we are able to integrate the both sides of (24) on $\mathcal{A}_0$ by $d\mathcal{L}^{n-1}$, so that we obtain

$$\int_{F\cap[\mathcal{A}_0\times\mathbf{R}]}|D_1u| \geq \int_{\mathcal{A}_0}|D_1v|\,. \tag{25}$$

If we denote by $P_i (i=2,\ldots,n)$ the orthogonal projection: $P_i(z) = (z_1,\ldots,z_{i-1}, z_{i+1},\ldots,z_n)$ for $z=(z_1,\ldots,z_n)\in\mathbf{R}^n$, it holds that $\mathcal{L}^{n-1}(P_i(\mathcal{A}_0)) = 0$ for $i=2,\ldots,n$. Therefore, with the help of Theorem A, we derive

$$\int_{\mathcal{A}_0}|D_iv| = 0 \quad \text{for } i=2,\ldots,n\,.$$

Hence, we have $\int_{\mathcal{A}_0}|D_1v| = \int_{\mathcal{A}_0}|Dv|$. Moreover, since $\mathcal{L}^n(A_0)=0$, we have $\int_{\mathcal{A}_0}|Dv| = \int_{\mathcal{A}_0}\sqrt{1+|Dv|^2}$. Combining these equality with (25), we are able to derive our desired result. □

3. A priori estimate

For $u \in BV(\Omega)$, we denote by E_u the interior of the set $\Omega(u = 0)$. For $z_0 \in \mathbf{R}^n$ and an unitary vector $\nu_0 \in \mathbf{S}^{n-1}$, let $H_{\nu_0}(z_0) := \{z \in \mathbf{R}^n \mid \langle z - z_0, \nu_0\rangle \geq 0\}$. We designate by $\mathcal{F}_{\nu_0}(z_0)$ the family of all connected components of the set $H_{\nu_0}(z_0)\backslash E_u$ which have no points belonging to Ω^c. If we denote by $\mathcal{G}_{\nu_0}(z_0)$ the family of all connected components of $H_{\nu_0}(z_0)\backslash E_u$, $\mathcal{F}_{\nu_0}$ can be described in the following way:

$$\mathcal{F}_{\nu_0}(z_0) = \mathcal{G}_{\nu_0}(z_0)\backslash \bigcup\{\text{connected components of } H_{\nu_0}(z_0)\backslash E_u \text{containing } x\},$$

where the union is taken over all $x \in [H_{\nu_0}(z_0)\backslash E_u] \cap \Omega^c$.

Theorem 1. (A priori estimate) *Let u be a minimizer for J in the class $BV(\Omega)$. Let $z_0 \in \mathbf{R}^n$ and $\nu_0 \in \mathbf{S}^{n-1}$. Then for each $F \in \mathcal{F}_{\nu_0}(z_0)$ the following estimate holds:*

$$\int_F Q^2\chi_{u>0}d\mathcal{L}^n \leq \mathcal{L}^n\Big(F \cap [F \cap \partial H_{\nu_0}(z_0)) \times \mathbf{R}]\Big),$$

where $(F \cap \partial H_{\nu_0}(z_0)) \times \mathbf{R} = \{\xi \in F \cap \partial H_{\nu_0}(z_0),\ t \in \mathbf{R}\}$.

Proof. We denote $H = H_{\nu_0}(z_0)$ for notational simplicity. Let v be the function defined as in Lemma 1. Then, owing to the assertion v is an admissible function, and so by the minimality of u, we have the inequality $J(u) \leq J(v)$. Since F is a compact subset of Ω whereas $u = v$ in $\Omega\backslash F$, it holds that $u = v$ on $\partial\Omega$ in the sense of L^1-trace. Consequently, we obtain

$$\int_F \sqrt{1+|Du|^2} + \int_F Q^2\chi_{u>0}d\mathcal{L}^n \leq \int_F \sqrt{1+|Dv|^2}, \tag{26}$$

where we use the fact that $v = 0$ on F. Since $\mathcal{A}_0 = \partial H \cap F$, $\mathcal{A}_0$ is a closed set. Therefore $\mathcal{A}_0 \times \mathbf{R}$ is closed and so $\sqrt{1+|Du|^2}$-measurable. Thus, by the additivity of measure

$$\begin{aligned}\int_F \sqrt{1+|Du|^2} &= \int_{F\cap[\mathcal{A}_+\times\mathbf{R}]} \sqrt{1+|Du|^2} + \int_{F\cap[\mathcal{A}_0\times\mathbf{R}]} \sqrt{1+|Du|^2} \\ &\geq \mathcal{L}^n(F\cap[\mathcal{A}_+\times\mathbf{R}]) + \int_{F\cap[\mathcal{A}_0\times\mathbf{R}]} |D_1 u|.\end{aligned}$$

We here adopt Lemma 2 to derive

$$\int_F \sqrt{1+|Du|^2} \geq \mathcal{L}^n(F\cap[\mathcal{A}_+\times\mathbf{R}]) + \int_{\mathcal{A}_0} \sqrt{1+|Dv|^2}. \tag{27}$$

Coupling (26) and (27), we obtain

$$\mathcal{L}^n(F\cap[\mathcal{A}_+\times\mathbf{R}]) + \int_{\mathcal{A}_0}\sqrt{1+|Dv|^2} + \int_F Q^2\chi_{u>0}d\mathcal{L}^n \le \int_F \sqrt{1+|Dv|^2}\,. \tag{28}$$

On the other hand,

$$\begin{aligned}\int_F \sqrt{1+|Dv|^2} &= \int_{F\cap\partial H}\sqrt{1+|Dv|^2} + \int_{H^\circ\cap\partial F}\sqrt{1+|Dv|^2} + \int_{F^\circ}\sqrt{1+|Dv|^2}\\ &\le \int_{\mathcal{A}_0}\sqrt{1+|Dv|^2} + \int_{H^\circ\cap\partial F}|Dv| + \mathcal{L}^n(\partial F) + \mathcal{L}^n(F^\circ) \qquad (29)\\ &= \int_{\mathcal{A}_0}\sqrt{1+|Dv|^2} + \int_{H^\circ\cap\partial F}|Dv| + \mathcal{L}^n(F)\,.\end{aligned}$$

Since F is closed, $\partial F\subset F$, and therefore we infer from Lemma 1 that

$$\int_{H^\circ\cap\partial F}|Dv| \le \int_{H^\circ\cap F}|Dv| = 0\,.$$

This brings us from (29) to the estimation

$$\int_F\sqrt{1+|Dv|^2} \le \int_{\mathcal{A}_0}\sqrt{1+|Dv|^2} + \mathcal{L}^n(F)\,.$$

Combining this with (28), we obtain

$$\int_F Q^2\chi_{u>0}\,d\mathcal{L}^n \le \mathcal{L}^n(F) - \mathcal{L}^n(F\cap[\mathcal{A}_+\times\mathbf{R}])\,,$$

and by noticing that $F\backslash\{F\cap[\mathcal{A}_+\times\mathbf{R}]\} = F\cap[\mathcal{A}_0\times\mathbf{R}] = F\cap[(F\cap\partial H)\times\mathbf{R}]$ we arrive at our goal. □

4. A convex property of the set E_u

Our aim is to lead a property on the shape of $E_u = [\Omega(u=0)]^o$ for a minimizer u of J. To begin with we proceed our estimate adding the condition n-dimensional Legesgue measure of $\partial\,\Omega(u>0)$ is zero.

Lemma 4. *Let u be a minimizer of J. Assume that $\mathcal{L}^n(\partial\,\Omega(u>0)) = 0$. Then for any $z_0\in\mathbf{R}^n$, $\nu_0\in\mathbf{S}^{n-1}$ and $F\in\mathcal{F}_{\nu_0}(z_0)$*

$$\int_F Q^2 d\mathcal{L}^n \le \mathcal{L}^n(F)\,.$$

Proof. From the hypothesis, $u > 0$ holds $\mathcal{L}^n$-almost everywhere in $\Omega \backslash E_u$. Since $F \subset \Omega \backslash E_u$, $u > 0$ holds $\mathcal{L}^n$-almost everywhere in F. Thus,

$$\int_F Q^2 \chi_{u>0} \, d\mathcal{L}^n = \int_F Q^2 d\mathcal{L}^n ,$$

and hence from Theorem 1 the desired estimate follows. □

Let us call E_u *is convex in* $\mathcal{L}^n$*-measure* when for any $z_0 \in \mathbf{R}^n$, $\nu_0 \in S^{n-1}$ and any $F \in \mathcal{F}_{\nu_0}(z_0)$ it holds that $\mathcal{L}^n(F) = 0$. Here $H_{\nu_0}(z_0) = \{z \in \mathbf{R}^n \mid \langle z - z_0, \nu_0 \rangle \geq 0\}$.

Theorem 2. (A convex property) *Let* $Q^2_{\min} = \operatorname{essinf}_\Omega Q^2 > 1$. *If* $\mathcal{L}^n(\partial \Omega (u > 0)) = 0$, *then* E_u *is convex in* $\mathcal{L}^n$*-measure.*

Proof. Let $z_0 \in \mathbf{R}^n$, $\nu_0 \in S^{n-1}$ and $F \in \mathcal{F}_{\nu_0}(z_0)$. Then from Lemma 4 we directly have $Q^2_{\min} \mathcal{L}^2(F) \leq \mathcal{L}^n(F)$. Therefore, remembering the assumption $Q^2_{\min} > 1$, we must have $\mathcal{L}^n(F) = 0$. □

We close this section by considering the sufficient condition for obtaining $F \cap (H_{\nu_0}(z_0))^o = \emptyset$ instead of $\mathcal{L}^n(F) = 0$ asserted in Theorem 2. We now in particular choose a representative u (refer to Section 2) such that

$$u(z_0) = \lim_{\rho \downarrow 0} \frac{1}{\mathcal{L}^n(B_\rho(z_0))} \int_{B_\rho(z_0)} u \, d\mathcal{L}^n \tag{30}$$

holds for any $z_0 \in \mathbf{R}^n$ at which the limit is finite-definite. The existence of such a representative is guaranteed by the well-known theorem due to Lebesgue.

Corollary. *Suppose that* $Q^2_{\min} = \operatorname{essinf}_\Omega Q^2 > 1$ *and* Ω *is a convex domain. Let* u *be as above and assume that* $\mathcal{L}^n(\partial \Omega (u > 0)) = 0$. *Let* $z_0 \in \mathbf{R}^n$ *and* $\nu_0 \in \mathbf{S}^{n-1}$. *If the cardinarity of elements of* $\mathcal{F}_{\nu_0}(z_0)$ *is at most countable, then* $F \cap (H_{\nu_0}(z_0))^o$ *is empty for any* $F \in \mathcal{F}_{\nu_0}(z_0)$.

Proof. Let $F \in \mathcal{F}_{\nu_0}(z_0)$ and denote $H = H_{\nu_0}(z_0)$. Assume that $F \cap H^o$ is not empty and let $x \in F \cap H^o$. Then we shall show that there exists a positive number ρ_x such that $\mathcal{L}^n(B_\rho(x)(u > 0)) = 0$ for any $\rho < \rho_x$, where $B_\rho(x)(u > 0) = \{z \in B_\rho(x) \mid u(z) > 0\}$. For this purpose, suppose on the contrary that there exists a sequence $\{\rho_j\}_{j=1}^\infty \subset \mathbf{R}^+$ with $\lim_{j \to \infty} \rho_j = 0$ such that $\mathcal{L}^n(B_{\rho_j}(x)(u > 0)) > 0$ for each $j \in \mathbf{N}$. Then

$$B_{\rho_j}(x)(u > 0) \cap \Delta \neq \emptyset \quad \text{for each } j \in \mathbf{N} . \tag{31}$$

Here Δ is the connected component of $H \backslash E_u$ containing a point of Ω^c, where we notice that Δ is uniquely determined because of the hypothesis of convexity of Ω. Suppose on the contrary that (31) does not hold. Then $B_{\rho_j}(x)(u > 0) \subset \bigcup_{F\in\mathcal{F}_{\nu_0}(z_0)} F$ holds for sufficiently large $j \in \mathbf{N}$, and hence from the countability asuumption we deduce for such j that

$$\mathcal{L}^n(B_{\rho_j}(x)(u > 0)) \le \mathcal{L}^n\Big(\bigcup_{F\in\mathcal{F}_{\nu_0}(z_0)} F\Big) \le \sum_{F\in\mathcal{F}_{\nu_0}(z_0)} \mathcal{L}^n(F).$$

The right side is equal to zero owing to Theorem 2, and thus we have a contradiction. Now, from (31), $x \in \overline{\Delta} = \Delta$, which contradicts to the fact $x \in F$. We thus have proved the existence of ρ_x as described above. In particular, $u = 0$ holds $\mathcal{L}^n$-almost everywhere in $B_{\frac{\rho_x}{2}}(x)$. Recalling here (30), we have $u = 0$ everywhere in $B_{\frac{\rho_x}{2}}(x)$. As a result, we finally reach the contradiction $x \in E_u$, and therefore $F \cap H^o = \emptyset$. $\square$

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