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IGUSA LOCAL ZETA FUNCTION OF THE QUADRATIC FORM $x_1^2+\cdots+x_n^2$

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Abstract. The purpose of this paper is to give the explicit form of Igusa local zeta function associated to the quadratic form $x_1^2+\cdots+x_n^2$.

0. Introduction

Let K be a finite algebraic extension of the p-adic field $\mathbf{Q}_p$, where p is a rational prime number. Throughout this paper, we assume $p\neq 2$. We denote by O_K the maximal compact subring of K. We fix a prime element π_K once and for all. We denote by $P_K=\pi_K O_K$ the unique maximal ideal of O_K and $O_K^{\times}=O_K-P_K$ the group of all units of O_K. The residue field O_K/P_K is a finite field and its cardinarity is denoted by q :

$$O_K/P_K\cong \mathbf{F}_q.$$

Every $\alpha\in K^{\times}$ can be uniquely expressed as

$$\alpha=\pi_K^{ord_K(\alpha)}ac_K(\alpha)$$

with $ord_K(\alpha)\in\mathbf{Z}$ and $ac_K(\alpha)\in O_K^{\times}$, and its absolute value is normalized by

$$|\alpha|_K=q^{-ord_K(\alpha)}.$$

Let $\Omega(K^{\times})$ be the group of all quasi-characters of $K^{\times}$ and $\widehat{O_K^{\times}}$ the group of all characters of $O_K^{\times}$. For every $\omega\in\Omega(K^{\times})$, there exist uniquely $s\in\mathbf{C}/\left(\frac{2\pi\sqrt{-1}}{\log q}\right)\mathbf{Z}$ and $\chi\in\widehat{O_K^{\times}}$ such that

$$\omega(\alpha)=|\alpha|_K^s\chi(ac_K(\alpha)),$$

then we write $\omega=(s;\chi)$ and put $\mathrm{Re}(\omega)=\mathrm{Re(s)}$. We call an $\omega=(s;\chi)\in\Omega(K^{\times})$ *ramified* or *unramified* quasi-character of $K^{\times}$, according to $\chi\neq 1$ or $\chi=1$, respectively.

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For a polynomial $f(x) = f(x_1, \cdots, x_n)$ in n-variables $x_1, \cdots, x_n$ with coefficients in K, we consider the following integral

$$Z_K(\omega) = Z_K(s;\chi) = \int_{O_K^n} |f(x)|_K^s \chi(ac_K(f(x)))dx \qquad (\omega = (s;\chi) \in \Omega(K^\times)),$$

in which we denote by dx the Haar measure on K^n normalized by $\mathrm{vol}(O_K^n) = \int_{O_K^n} dx = 1$. It is clear that $Z_K(\omega)$ is absolutely convergent for $\mathrm{Re}(\omega) > 0$. Morerover, J.-I. Igusa proved that $Z_K(\omega)$ has an analytic continuation to a rational function of $t = q^{-s}$ on the whole $\Omega(K^\times)$ ([1], [3]), which is called *Igusa local zeta function* associated to $f(x)$ after J.-P. Serre, and denoted by the same notation $Z_K(\omega)$.

Our purpose of this paper is to give the explicit form of Igusa local zeta function associated to a quadratic form

$$q(x) = x_1^2 + \cdots + x_n^2.$$

Theorem 1. *Let $Z_K(s;\chi)$ be Igusa local zeta function associated to the quadratic form $q(x)$:*

$$Z_K(s;\chi) = \int_{O_K^n} |q(x)|_K^s \chi(ac_K(q(x)))dx,$$

then the explicit form of $Z_K(\omega) = Z_K(s;\chi)$ is given as follows
(1) n : *odd*

$$Z_K(s;\chi) = \begin{cases} \dfrac{(1-q^{-1})(1-q^{-n}t)}{(1-q^{-1}t)(1-q^{-n}t^2)} & (\chi = 1) \\[2ex] \dfrac{(1-q^{-1})\chi_2(-1)^{[\frac{n}{2}]}q^{-[\frac{n}{2}]}}{1-q^{-n}t^2} & (\chi = \chi_2) \\[2ex] 0 & (\chi \neq 1,\ \chi_2), \end{cases}$$

(2) n : *even*

$$Z_K(s;\chi) = \begin{cases} \dfrac{(1-q^{-1})(1-\chi_2(-1)^{\frac{n}{2}}q^{-\frac{n}{2}})}{(1-q^{-1}t)(1-\chi_2(-1)^{\frac{n}{2}}q^{-\frac{n}{2}}t)} & (\chi = 1) \\[2ex] 0 & (\chi \neq 1), \end{cases}$$

in which we denote by $[x]$ the largest integer $\leq x$ and χ_2 the non-trivial character of $O_K^\times$ satisfying $\chi_2^2 = 1$.

In the unramified case, our result is already well-known ([2]), but in the ramified case, the above theorem gives new result.

In § 1, we give the proof for the case of $\chi \neq 1, \chi_2$. In § 2, we recall results concering with Mellin transformation and Fourier transformation over K and, by using these results, we prove the case of $\chi = \chi_2$, in § 3.

1. The case of $\chi \neq 1, \chi_2$

In this section, we shall give the proof for the case of $\chi \neq 1, \chi_2$.

Lemma 1. *Let $f(x) = f(x_1, \cdots, x_n)$ be a polynomial with coefficients in K and $Z_K(\omega)$ Igusa local zeta function associated to $f(x)$. If the units group $O_K^\times$ acts on O_K^n and this action gives a measure-preserving analytic homeomorphism from O_K^n onto itself, satisfying*

$$f(u \cdot x) = u^m f(x) \qquad (u \in O_K^\times)$$

with some positive intger m, then, for any $\chi \in \widehat{O_K^\times}$ satisfying $\chi^m \neq 1$, we have

$$Z_K(s; \chi) = 0.$$

Proof. Since the action of $O_K^\times$ on O_K^n gives a measure-preserving homeomorphism from O_K^n onto itself, we have , for $u \in O_K^\times$,

$$\int_{O_K^n} \omega(f(u \cdot x))dx = \int_{O_K^n} \omega(f(x))dx = Z_K(\omega).$$

On the other hand, the polynomial $f(x)$ satisfies $f(u \cdot x) = u^m f(x)$, hence we have, for $u \in O_K^\times$

$$\int_{O_K^n} \omega(f(u \cdot x))dx = \chi(u)^m \int_{O_K^n} \omega(f(x))dx = \chi(u)^m Z_K(\omega),$$

with $\omega = (s; \chi) \in \Omega(K^\times)$. Therefore we have

$$Z_K(\omega) = \chi(u)^m Z_K(\omega) \qquad (u \in O_K^\times,\ \omega = (s; \chi) \in \Omega(K^\times)).$$

Hence, if $\chi^m \neq 1$, we have $Z_K(\omega) = Z_K(s; \chi) = 0$. ■

Now we shall be back to our situation. If we consider the usual action of $O_K^\times$ on O_K^n:

$$u \cdot x = (ux_1, \cdots, ux_n) \qquad (u \in O_K^\times,\ x = (x_1, \cdots, x_n) \in O_K^n),$$

then we can see easily that, this action gives a measure-preserving homeomorphism and

$$q(u \cdot x) = u^2 q(x).$$

Therefore, by the above lemma, we have

$$Z_K(s;\chi) = 0 \qquad (\chi \neq 1, \chi_2).$$

2. Mellin transformation and Fourier transformation

Our task is only to give the proof for the case of $\chi = \chi_2$. For the completion of our task, in this section, we shall recall some results concering with Mellin transformation and Fourier trasnsformation over K. For these results, we refer to [1], [3].

For a polynomial $f(x) = f(x_1, \cdots, x_n)$ with coefficients in K, we denote by C_f the critical set of $f(x)$, namely,

$$x \in C_f \iff \mathrm{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x)\right) = 0.$$

In this section, the following condition is assumed;

$$C_f \subset f^{-1}(0),$$

this condition is satisfied if $f(x)$ is homogeneous.

We put $U(i) = f^{-1} - C_f$ for $i \in K$. Let a be an element of $U(i)$, that is $f(a) = i$ and further at least one partial derevative of $f(x)$, say $\frac{\partial f}{\partial x_k}(x)$, does not vanish at a, then

$$\theta_i(x) = (-1)^k \left(\frac{\partial f}{\partial x_k}(x)\right)^{-1} \cdot dx_1 \wedge \cdots \wedge \check{dx_k} \wedge \cdots \wedge dx_n|_{U(i)}$$

is a well-defined non-vanishing regular $(n-1)$-form around $a \in U(i)$ and thereby gives rise to a global regular and non-vanishing $(n-1)$-form on $U(i)$. Moreover, $\theta_i(x)$ induces on $U(i)$ a Borel measure $|\theta_i|_K$ (cf. Chapter III-§1.3 in [1]).

For any Φ in the Schwartz-Bruhat space of K^n, the integral

$$F_\Phi(i) = \int_{U(i)} \Phi(x)|\theta_i(x)|_K$$

defines a continuous function F_Φ on $K^\times$. When Φ is the chracteristic function of O_K^n, the funnction F_Φ is simply denoted by F. We define the another function F^* on K, associated to $f(x)$ by

$$F^*(i) = \int_{O_K^n} \psi(i^* f(x))dx \qquad (i^* \in K),$$

in which we denote by ψ a fixed additive character of K such that ψ is non-trivial on $\pi_K^{-1}O_K$ and trivial on O_K and dx the Haar measure on K^n, introduced in §1.

From Theorem 1.6 of Chapter III in [1], we have the following results. The function F can be uniquely extended to a continuous function on K and its Mellin transform is nothing but Igusa local zeta function Z_K assocated to $f(x)$:

$$Z_K(\omega) = \int_K F(i)\omega(i)di \qquad (\omega \in \Omega(K^\times)),$$

in which di is the Haar measure on K normalized by $\mathrm{vol}(O_K) = \int_{O_K} di = 1$. Moreover, F^* coincides with the Fourier transform of F:

$$F^*(i^*) = \int_K F(i)\psi(ii^*)di \qquad (i^* \in K),$$

and hence, from the Fourier-inversion formula, we have

$$F(i) = \int_K F^*(i^*)\psi(-ii^*)di \qquad (i \in K).$$

3. The case of $\chi = \chi_2$

In this section, we shall give the proof for the case of $\chi = \chi_2$, by using the relation of three functions F, F^* and Z_K, introduced in § 2.

For $e \in \mathbf{Z}$ and $\chi \in \widehat{O_K^\times}$, we put

$$g_{e,\chi} = \int_{O_K}^{\times} \chi(u)\psi(\pi_K^{-e}u)du.$$

We denote by e_χ the smallest natural number e such that χ is trivial on the set $1 + \pi_K^e O_K$, and put

$$g_\chi = g_{e_\chi,\chi},$$

then we have

$$\overline{g_\chi} = \chi(-1)g_{\chi^{-1}}, \qquad |g_\chi|^2 = q^{-e_\chi}$$

([1]). Since we assume $p \neq 2$, we have $e_{\chi_2} = 1$ from Lemma 2.5 of Chapter III in [1]. Hence, by the above formulae, we have

$$g_{\chi_2}^2 = \chi_2(-1)q^{-1} \tag{1}$$

Let F^* be the F^*−function associated to the quadratic form $q(x)$:

$$F^*(i^*) = \int_{O_K^n} \psi(i^*q(x))dx \qquad (i^* \in K).$$

The following lemma gives the explicit form of the function F^*.

Lemma 2.
(1) n : *odd*

$$F^*(i^*) = \begin{cases} |i^*|_K^{-\frac{n}{2}} & (-ord_K(i^*) : \textit{positive even}) \\ q^{\frac{1}{2}} g_{\chi_2} \chi_2(-1)^{[\frac{n}{2}]} \chi_2(ac_K(i^*)) |i^*|_K^{-\frac{n}{2}} & (-ord_K(i^*) : \textit{positive odd}) \\ 1 & (-ord_K(i^*) \leq 0), \end{cases}$$

(2) n : *even*

$$F^*(i^*) = \begin{cases} |i^*|_K^{-\frac{n}{2}} & (-ord_K(i^*) : \textit{positive even}) \\ \chi_2(-1)^{[\frac{n}{2}]} |i^*|_K^{-\frac{n}{2}} & (-ord_K(i^*) : \textit{positive odd}) \\ 1 & (-ord_K(i^*) \leq 0). \end{cases}$$

Proof. Since ψ is trivial on O_K, we have

$$F^*(i^*) = 1 \qquad (-ord_k(i^*) \leq 0),$$

hence we may consider the case of $-ord_K(i^*) > 0$.

We denote by F_0^* the F^*−function associated to a monomial x^2:

$$F_0^*(i^*) = \int_{O_K} \psi(i^* x^2) dx,$$

then we have

$$F^*(i^*) = F_0^*(i^*)^n. \tag{2}$$

On the other hand, J.-I. Igusa gives the explicit form of $F_0^*(i^*)$:

$$F_0^*(i^*) = \begin{cases} |i^*|_K^{-\frac{1}{2}} & (-ord_K(i^*) : \text{positive even}) \\ q^{\frac{1}{2}} g_{\chi_2} \chi_2^{-1}(ac_K(i^*)) |i^*|_K^{-\frac{1}{2}} & (-ord_K(i^*) : \text{positive odd}) \end{cases}$$

((111) in [1]). We put these formulae into (2), then, by using (1), we have our result. ■

We shall consider the explicit form of the function $F(i)$ associated to the quadratic form $q(x)$. For $i \in K - O_K$, the set $U(i) \cap O_K^n$ is empty. Hence, by the definition of $F(i)$, we have $F(i)=0$ ($i \in K - O_K$) and so we may consider $F(i)$ for $i \in O_K$. Since F is the inverse Fourier transform of F^*, we have

$$\begin{aligned} F(i) &= \int_K F^*(i^*)\psi(-ii^*)di^* \\ &= \int_{O_K} F^*(i^*)\psi(-ii^*)di^* + \int_{K-O_K} F^*(i^*)\psi(-ii^*)di^*. \end{aligned}$$

By Lemma 2, $F^*(i^*)=1$ for $i^* \in O_K$ and ψ is trivial on O_K, hence we have

$$\int_{O_K} F^*(i^*)\psi(-ii^*)di^* = 1.$$

On the other hand, we have

$$\begin{aligned} \int_{K-O_K} F^*(i^*)\psi(-ii^*)di^* &= \sum_{k\geq 1} \int_{\pi_K^{-2k} O_K^\times} F^*(i^*)\psi(-ii^*)di^* \\ &+ \sum_{k\geq 1} \int_{\pi_K^{-(2k-1)} O_K^\times} F^*(i^*)\psi(-ii^*)di^*, \end{aligned}$$

and, by putting the explicit form of $F^*(i^*)$ given in Lemma 2 into each partial integral in the R.H.S. of the above formula, we obtain the following lemma.

Lemma 3. *For $i \in O_K$, we put $e = ord_K(i) > 0$ and $u = ac_K(i)$, then we have*
(1) n : *odd*

$$\begin{aligned} F(i) = 1 &+ \sum_{k\geq 1} q^{-(n-2)k} g_{2k-e,1} \\ &+ g_{\chi_2}\chi_2(-1)^{[\frac{n}{2}]}\chi_2(-u)q^{[\frac{n}{2}]} \sum_{k\geq 1} q^{-(n-2)k} g_{2k-e-1,\chi_2}, \end{aligned}$$

(2) n : *even*

$$\begin{aligned} F(i) = 1 &+ \sum_{k\geq 1} q^{-(n-2)k} g_{2k-e,1} \\ &+ \chi_2(-1)^{[\frac{n}{2}]} q^{\frac{n}{2}-1} \sum_{k\geq 1} q^{-(n-2)k} g_{2k-e-1,1}. \end{aligned}$$

The following formulae are given by J.-I. Igusa,

$$
(3) \qquad g_{e,\chi} = \begin{cases} 1 - q^{-1} & (e \le 0, \quad \chi = 1) \\ -q^{-1} & (e = 1, \quad \chi = 1) \\ 0 & (e > 1, \quad \chi = 1) \\ 0 & (e = e_\chi, \quad \chi \ne 1) \end{cases}
$$

((52) in [1]). We put (3) into the formulae in Lemma 3, then, by using (1), we have the explicit form of $F(i)$.

Lemma 4. *For $i \in O_K$, using same notations in as Lemma 3, we have*
(1) n : *odd*

$$
F(i) = \begin{cases} 1 + A_n + \chi_2(-1)^{[\frac{n}{2}]}\chi_2(u) q^{-\left(\frac{(n-2)e}{2} + [\frac{n}{2}]\right)} & (e : \textit{even}) \\ 1 + A_n - q^{-\left(\frac{(n-2)e}{2} + 1\right)} & (e : \textit{odd}), \end{cases}
$$

in which we put

$$
A_n = \frac{(1 - q^{-1})(1 - q^{-\frac{(n-2)(e-1)}{2}})}{1 - q^{-(n-2)}}.
$$

(2) n : *even*

$$
F(i) = \begin{cases} 1 + B_n + C_n q^{-(\frac{n}{2}-1)e} & (e : \textit{even}) \\ 1 + B_n - D_n q^{-(\frac{n}{2}-1)(e-1)} & (e : \textit{odd}), \end{cases}
$$

in which we put

$$
B_n = \frac{(1 - q^{-1})(1 + \chi_2(-1)^n q^{n-1})}{1 - q^{-(n-2)}} q^{-(n-2)},
$$

$$
C_n = A_n + \chi_2(-1)^{\frac{n}{2}} q^{-\frac{n}{2}}
$$

and

$$
D_n = A_n - q^{-(n-2)} \left(-q^{-1} + \chi_2(-1)^n q^{n-1}(1 - q^{-1})\right).
$$

As introduced in § 2, Igusa local zeta function Z_K is the Mellin transform of F. Hence we have

$$
Z_K(s; \chi_2) = \int_K F(i)|i|_K^s \chi_2(ac_K(i))di
$$

$$
(4) \qquad \begin{aligned} &= \sum_{\substack{\ell \geq 0 \\ \ell:\text{even}}} (q^{-1}t)^{\ell} \int_{O_K^n} F(\pi_K^{\ell} u)\chi_2(u)du \\ &+ \sum_{\substack{\ell \geq 0 \\ \ell:\text{odd}}} (q^{-1}t)^{\ell} \int_{O_K^n} F(\pi_K^{\ell} u)\chi_2(u)du \end{aligned}
$$

We put the formulae in Lemma 4 into (4), then, by using (3), we have

$$
Z_K(s;\chi_2) = \begin{cases} \dfrac{(1-q^{-1})\chi_2(-1)^{[\frac{n}{2}]}q^{-[\frac{n}{2}]}}{1-q^{-n}t^2} & (n : \text{odd}) \\ 0 & (n : \text{even}), \end{cases}
$$

and hence we complete the proof for Theorem 1.

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