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AN EXAMPLE OF A p-QUASIHYPONORMAL OPERATOR

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Abstract. We give a p-quasihyponormal operator $T=U|T|$ such that (i) T is not q-quasihyponormal for all $q\in(0,p)$ and (ii) $|T|^{s}U|T|^{t}$ for $s,t>0$ is not q-quasihyponormal for all $q\in(0,\infty)$, which is a counter example for the main results in [1], [2].

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is called p-quasihyponormal if $T^{*}\{(T^{*}T)^{p}-(TT^{*})^{p}\}T\geq 0$ for $p>0$.

Let $\{\varepsilon_{n}; n\in \mathbb{Z}\}$ be the canonical orthonormal basis of $\ell^{2}(\mathbb{Z})$ and p_{n} the projection of $\ell^{2}(\mathbb{Z})$ to $\mathbb{C}\varepsilon_{n}$. Using the shift operator S on $\ell^{2}(\mathbb{Z})$ with $S\varepsilon_{n}=\varepsilon_{n+1}$ and positive 2×2 Hermitian matrices A and B, we define operators H and T on $\mathbb{C}^{2}\otimes\ell^{2}(\mathbb{Z})$ by

$$H=\sum_{n<0}A\otimes p_{n}+\sum_{n\geq 0}B\otimes p_{n}$$

and

$$T=(1\otimes S)H.$$

Denote the polar decomposition of T by $U|T|$. Then $U=1\otimes S$ and $|T|=H$. Since $|T^{*}|=U|T|U^{*}=\sum_{n\leq 0}A\otimes p_{n}+\sum_{n>0}B\otimes p_{n}$, it is easy to see that

$$T^{*}\left(|T|^{2p}-|T^{*}|^{2p}\right)T=A\left(B^{2p}-A^{2p}\right)A\otimes p_{-1}$$

for $p>0$. Hence T is p-quasihyponormal if and only if $A(B^{2p}-A^{2p})A\geq 0$.

In what follows we assume that A and B are of the form

$$\begin{pmatrix}\alpha & 0\\ 0 & 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix},$$

respectively. Let f be a function on the half interval $(0,\infty)$ defined by

$$f(p)=\left(\frac{9^{p}+1}{2}\right)^{\frac{1}{2p}}$$

Then it is strictly increasing.

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Theorem 1. *(i) T is p-quasihyponormal if and only if $\alpha \leq f(p)$.*

(ii) If $\alpha = f(p)$, then T is not q-quasihyponormal for $q \in (0,p)$, but q-quasihyponormal for $q \in [p,\infty)$.

Proof. (i) Since

$$B^{2p} = \frac{1}{2}\begin{pmatrix} 9^p+1 & 9^p-1 \\ 9^p-1 & 9^p+1 \end{pmatrix},$$

it is easy to see that T is p-quasihyponormal if and only if $(9^p+1)/2 - \alpha^{2p} \geq 0$.

(ii) It is immediate from (i). Q.E.D.

Theorem 2. *Let $T(s,t) = |T|^s U |T|^t$ for $s,t > 0$.*

(i) If $T(s,t)$ is p-quasihyponormal, then $\alpha \leq f(s)$.

(ii) If $\alpha = f(p)$ and $s \in (0,p)$, then $T(s,t)$ is not q-quasihyponormal.

Proof. (i) Since

$$\begin{aligned} &T(s,t)^*\left(|T(s,t)|^{2p} - |T(s,t)^*|^{2p}\right)T(s,t) \\ &= A^{s+t}\left\{(A^t B^{2s} A^t)^p - A^{2(s+t)p}\right\}A^{s+t} \otimes p_{-2} \\ &\quad + A^t B^s \left\{B^{2(s+t)p} - (B^s A^{2t} B^s)^p\right\} B^s A^t \otimes p_{-1}. \end{aligned}$$

Hence $T(s,t)$ is p-quasihyponormal if and only if

$$(A^t B^{2s} A^t)^p - A^{2(s+t)p} \geq 0, \quad \text{and} \quad A^t B^s \{B^{2(s+t)p} - (B^s A^{2t} B^s)^p\} B^s A^t \geq 0.$$

The former inequality implies that $\alpha \leq f(s)$.

(ii) It is immediate from (i). Q.E.D.

Remark. *The range of operator T is closed.*

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