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ON EXISTENCE OF SOLUTIONS FOR THE UNILATERAL PROBLEM ASSOCIATED TO THE DEGENERATE KIRCHHOFF EQUATIONS

By

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^{N}$ with smooth boundary Γ, T is a positive real number, $\rho:[0,T]\times\overline{\Omega}\rightarrow \mathbb{R}$ is a real function. In this paper, we consider the existence of solutions for the following nonlinear unilateral problem:

$$\begin{aligned}&\rho(t,x)u_{tt}(t,x)-\|\nabla u(t,x)\|_{2}^{2\gamma}\Delta u(t,x)\geq|u(t,x)|^{\alpha}u(t,x) \text{ on } [0,T]\times\Omega,\\ &u(t,x)=0 \text{ on } \sum=[0,T]\times\Gamma,\\ &u(0,x)=u_{0}(x),\ u_{t}(0,x)=u_{1}(x) \text{ on } \Omega,\end{aligned}$$

where Δ is the Laplacian in $\mathbb{R}^{N}$, $\alpha>0$ and $\gamma\geq 1$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^{N}$ with smooth boundary Γ, T is positive real number, $\rho:[0,T]\times\overline{\Omega}\rightarrow \mathbb{R}$ is a real function. In this paper, we consider the nonlinear unilateral problem:

$$(1.1)\quad \begin{aligned}&\rho(t,x)u_{tt}(t,x)-\|\nabla u(t,x)\|_{2}^{2\gamma}\Delta u(t,x)\geq|u(t,x)|^{\alpha}u(t,x) \text{ on } [0,T]\times\Omega,\\ &u(t,x)=0 \text{ on } \sum=[0,T]\times\Gamma,\\ &u(0,x)=u_{0}(x),\ u_{t}(0,x)=u_{1}(x) \text{ on } \Omega,\end{aligned}$$

where Δ is the Laplacian in $\mathbb{R}^{N}$, $\alpha>0$ and $\gamma\geq 1$.

Equation (1.1) has its origin in the mathematical description of small amplitude vibrations of an elastic string. Many authors have studied about the existence and uniqueness of solutions of (1.1) by using various methods.

When $\rho=1$, the unilateral problem (1.1) was studied in Ono [11], Brito [3], Ikehata [10], Yamada ([13], [14]) and the references therein.

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In this paper, we will study the existence and uniqueness of solutions of (1.1) in the degenerate case for bounded domains $\Omega \subset \mathbb{R}^N$ without geometrical restrictions and ρ is a positive function by using Galerkin method.

The contents of this paper are as follows; In section 2, we present the preliminaries and some lemmas and we give the statement of main theorem. In section 3, we deals with a priori estimates for solutions of (1.1) and prove our Theorem.

2. Preliminaries

We denote by Ω a bounded open set of $\mathbb{R}^N$ with a smooth boundary Γ, T is a positive real number, Q is the cylinder $[0,T] \times \Omega$ and $\sum = [0,T] \times \Gamma$ is the boundary of Q. The norm and inner product in the Hilbert space $L^2(\Omega)$ are denoted by

$$||u||_2 = \left(\int_\Omega |u(x)|^2 dx \right)^{\frac{1}{2}} \text{ and } (u,v) = \int_\Omega u(x)v(x)dx.$$

Let X be a Banach space. For a fixed p, $1 \leq p \leq \infty$, $L^p(0,T;X)$ denotes the space of L^p-integrable functions from $[0,T]$ into X, which is a Banach space with the norm

$$||u||_{L^p(0,T;X)} = \left(\int_0^T ||u(t)||_X^p dt \right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty,$$
$$||u||_{L^\infty(0,T;X)} = \text{esssup} \quad ||u(t)||_X.$$

By $W^{m,p}(\Omega)$, we represent the usual Sobolev space and $W_0^{m,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. The dual space of $W_0^{m,p}(\Omega)$ is indicated by $W^{-m,q}(\Omega)$, where $q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = 2$, we give a special notation, that is,

$$H^m(\Omega) = W^{m,2}(\Omega),\ H_0^m(\Omega) = W_0^{m,2}(\Omega), \text{ and } H^{-m}(\Omega) = (H_0^m(\Omega))'.$$

The norm and inner product in $H_0^1(\Omega)$ are denoted as follows:

$$||\nabla u||_2^2 = \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 dx \quad \text{and} \quad ((u,v)) = \sum_{i=1}^N \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx.$$

For a real function $h : \Omega \to \mathbb{R}$, we put $h^-(x) = \max\{-h(x), 0\}$ and $h^+(x) = \max\{h(x), 0\}$, $x \in \Omega$. Now, we introduce a closed and convex subset of $W_0^{2,4}(\Omega)$,

$$\mathcal{K} = \left\{ \psi \in W_0^{2,4}(\Omega) \mid |\Delta \psi| \leq 1, \psi \geq 0 \text{ a.e. on } \Omega \right\}$$

and $\beta : W_0^{2,4}(\Omega) \to W^{-2,\frac{4}{3}}(\Omega)$ denotes the penalty operator defined by $\beta(z) = \beta_1(z) + \beta_2(z)$, $z \in W_0^{2,4}(\Omega)$, where $\beta_1(z) : W^{2,4}(\Omega) \to \mathbb{R}$ defined by $v \mapsto \langle \beta_1(z), v\rangle = -\int_\Omega z^- v dx$ and $\beta_2(z) : W_0^{2,4}(\Omega) \to \mathbb{R}$ defined by $v \mapsto \langle \beta_2(z), v\rangle = \int_\Omega (1-|\Delta z|^2)^- \Delta z \Delta v dx$. We note that (see [5])

(*i*) β is a monotone operator;

$$\langle \beta(u) - \beta(v), u - v\rangle \geq 0 \text{ for all } u, v \in W_0^{2,4}(\Omega).$$

(*ii*) β is a hemicontinuous operator; the map $\lambda \to \langle \beta(u+\lambda v), w\rangle$ is continuous in $\mathbb{R}$.

(*iii*) $\beta(S)$ is bounded in $W^{-2,\frac{4}{3}}(\Omega)$ for all bounded set S in $W_0^{2,4}(\Omega)$.

(*iv*) $\beta(u) = 0$ if and only if $u \in \mathcal{K}$.

We first prepare the following well known lemmas which will be needed later.

Lemma 2.1. (Sovolev-Poincaré [2]) *If either* $1 \leq \alpha < \infty (N = 1, 2)$ *or* $1 \leq \alpha \leq \frac{4}{N-2} (N \geq 3)$, *then there is a constant* C_* *such that*

$$||u||_{\alpha+2} \leq C_* ||\nabla u||_2 \text{ for } u \in H_0^1(\Omega).$$

In other words, $C_* = \sup\left\{\frac{||u||_{\alpha+2}}{||\nabla||_2} \mid u \in H_0^1(\Omega),\ u \neq 0\right\}$ *is positive finite.*

Lemma 2.2. (Gagliardo-Nirenberg [2]) *Let* $1 \leq r < q \leq \infty$ *and* $p \leq q$. *Then the inequality*

$$||u||_{W^{k,q}} \leq C||u||_{W^{m,p}}^{\theta} ||u||_r^{1-\theta} \text{ for } W^{m,p}(\Omega) \cap L^r(\Omega)$$

holds with some $C > \theta$ *and* $\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$ *provided that* $0 < \theta \leq 1$ *(we assume* $0 < \theta < 1$ *if* $q = \infty$.).

For further a priori estimates, we need the generalized Gronwall's inequality which is due to Bihari and Langenhop.

Lemma 2.3. ([1], [6]) *If* $k > 0$ *and* $c \geq 0$ *are constants and* $g(s)$ *is positive, nondecreasing for* $s \geq 0$, *then the inequality*

$$\phi(t) \leq k + c\int_0^t \psi(s) g(\phi(s)) ds \tag{2.1}$$

implies that $\phi(t) \leq G^{-1}(c\int_0^t \psi(s)ds)$, *where* $G(\eta) = \int_k^\eta \frac{1}{g(s)} ds$, $\eta > k > 0$.

If $g(s) = s$, *then the inequality (2.1) is just the usual Gronwall's inequality and Lemma 2.3 reads as follows:*

$\phi(t) \leq k + c\int_0^t \psi(s)\phi(s)ds$ *implies that* $\phi(t) \leq k \exp\left(c\int_0^t \psi(s)ds\right)$, $t \geq 0$.

Now, we indicate the following proposition which is needed to obtain convergence results.

Lemma 2.4. (Teman [12]) *Let X and Y be two Banach spaces such that $X \subset Y$ with a continuous injection. If a function ϕ belongs to $L^\infty(0,T;X)$ and is weakly continuous with values in Y, then ϕ is weakly continuous with values in X.*

Lastly, the following useful imbedding result is needed.

Lemma 2.5. ([2]) *Suppose $m(\Omega) = \int_\Omega dx < \infty$ and $1 \le p \le q \le \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and $\|u\|_p \le m(\Omega)^{\frac{1}{p}-\frac{1}{q}}\|u\|_q$. Hence $L^q(\Omega) \hookrightarrow L^p(\Omega)$. Finally, if $u \in L^p(\Omega)$ for $1 \le p < \infty$ and if there is a constant k such that for all such p, $\|u\|_p \le k$, then $u \in L^\infty(\Omega)$ and $\|u\|_\infty \le k$.*

Now, we consider the following initial value problem:

$$\begin{aligned} &\rho(t)u_{tt} - \|\nabla u(t)\|_2^{2\gamma}\nabla u(t) \ge |u(t)|^\alpha u(t), \ t \in [0,T], \\ &u(0) = u_0, \ u_t(0) = u_1, \end{aligned} \tag{2.2}$$

where $T > 0$ and $\gamma \ge 1$.

Firstly, let us define the potential and energy associated with the problem (1.1) by

$$J(u) = \frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} - \frac{1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2},$$

$$\text{and } E(u) = \frac{\rho}{2}\|u'\|_2^2 + J(u).$$

We introduce the I-positive set

$$\mathcal{W} = \{u \in H_0^1(\Omega) \cap H^2(\Omega) \mid I(u) > 0\} \cup \{0\},$$

$$\text{where } I(u) = \|\nabla u\|_2^{2(\gamma+1)} - \|u\|_{\alpha+2}^{\alpha+2}$$

then we can state main Theorem.

Lemma 2.6. *Let $\rho \in C^1$ be a real nonincreasing continuous function such that $0 < \rho_0 \le \rho(t)$ for all $t \in [0,T]$. If $u_0 \in \mathcal{W}$, $u_1 \in int\ (\mathcal{K})$ and $2\gamma < \alpha \le \frac{4}{N-2}$ $(0 \le \alpha < \infty$ if $N = 1,2)$, then there exists a unique function u such that*

$$\begin{aligned} &u \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)), \ u' \in L^4(0,T;W_0^{2,4}(\Omega)), \\ &u'' \in L^\infty(0,T;L^2(\Omega)), \ u'(t) \in \mathcal{K} \ a.e. \ on \ [0,T] \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &(\rho u_{tt} - \|\nabla u\|_2^{2\gamma} - |u|^\alpha u, v - u') \ge 0 \ for \ all \ v \in \mathcal{K} \ a.e. \ on \ [0,T], \\ &u(0) = u_0, \ u_t(0) = u_1. \end{aligned} \tag{2.4}$$

3. Penalized problem and proof of the existence theorem

The result on the penalized problem is given by the following theorem.

Theorem 3.1. *Assume that the assumptions of Theorm 2.6 are hold. Then for each $\epsilon \in (0,1)$, there exists a function u_ϵ such that*

$$(3.1)\qquad \begin{aligned} &u_\epsilon \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)),\ u'_\epsilon \in L^4(0,T;W_0^{2,4}(\Omega)),\\ &u''_\epsilon \in L^\infty(0,T;L^2(\Omega)) \end{aligned}$$

and

$$(3.2)\qquad \begin{aligned} &(\rho u''_\epsilon - \|\nabla u_\epsilon\|_2^{2\gamma}\Delta u_\epsilon - |u_\epsilon|^\alpha u_\epsilon, v) + \tfrac{1}{\epsilon}\langle \beta(u'_\epsilon), v\rangle = 0 \text{ for all } u \in W_0^{2,4}(\Omega),\\ &u_\epsilon(0) = u_0,\ u'_\epsilon(0) = u_1. \end{aligned}$$

Note that if $u_\epsilon(t)$ is solution of (3.2), then $u_\epsilon(t)$ satisfies

$$(3.3)\qquad E(u_\epsilon(t)) \le E(u_0) + \frac{1}{2}\int_0^t \rho'(s)\|u'_\epsilon(s)\|_2^2\,ds.$$

In fact, taking $v = u'_\epsilon(t)$ in (3.2), then we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\rho(t)\|u'_\epsilon(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u_\epsilon(t)\|_2^{2(\gamma+1)} - \frac{1}{\alpha+2}\|u_\epsilon(t)\|_{\alpha+2}^{\alpha+2}\right)\\ &\quad + \frac{1}{\epsilon}\langle \beta(u'_\epsilon(t)), u'_\epsilon(t)\rangle = \frac{1}{2}\rho'(t)\|u'_\epsilon(t)\|_2^2. \end{aligned}$$

Integrating it from 0 to t, from the definition of $E(u_\epsilon(t))$ and the fact that

$$\begin{aligned} \langle \beta(u'_\epsilon), u'_\epsilon\rangle &= \int_\Omega |(u'_\epsilon)^-|^2\,dx\\ &\quad + \int_\Omega (1-|\Delta u'_\epsilon|^2)^-\,|\Delta u'_\epsilon|^2\,dx\\ &\le 0, \end{aligned}$$

we easily obtain (3.3).

From (3.3) and the fact that ρ is nonincreasing, we get

$$(3.4)\qquad E(u_\epsilon(t)) \le E(u_0),\ t \ge 0.$$

Next, to obtain a priori bound, we need the following result.

Lemma 3.2. *Assume that either $1 \le \alpha < \infty (N = 1,2)$ or $1 \le \alpha \le \frac{4}{N-2} (N \ge 3)$ is satisfied. Let $u_\epsilon(t)$ be the solution of (3.2) with $u_0 \in \mathcal{W}$ and $u_1 \in H_0^1(\Omega)$. If u_0 and u_1 are sufficiently small in the sense that*

$$(3.5)\qquad C_*^{\alpha+2}\left(\frac{2(\alpha+2)(\gamma+1)}{\alpha-2\gamma}E(u_0)\right)^{\frac{\alpha-2\gamma}{2(\gamma+1)}} < 1,$$

then $u_\epsilon(t) \in \mathcal{W}$ *on* $[0,T]$*, that is,*

$$\|\nabla u_\epsilon\|_2^{2(\gamma+1)} - \|u_\epsilon\|_{\alpha+2}^{\alpha+2} > 0 \text{ on } [0,T].$$

Here, $E(u_0) = \frac{1}{2}\rho(0)\|u_1\|^2 + J(u_0)$.

Proof. Since $I(u_0) > 0$ and $u_\epsilon(t)$ is continuous in t,

$$I(u_\epsilon(t)) \geq 0 \text{ for some interval near } t = 0. \tag{3.6}$$

Let $t_{\max}$ be a maximal time (possibly $t_{\max} = T$) when (3.6) holds on $[0, t_{\max})$. Then $J(u_\epsilon(t))$ satisfies

$$\begin{aligned} J(u_\epsilon(t)) &= \frac{\alpha - 2\gamma}{2(\gamma+1)(\alpha+2)}\|\nabla u_\epsilon(t)\|_2^{2(\gamma+1)} + \frac{1}{\alpha+2} I(u_\epsilon(t)) \\ &\geq \frac{\alpha - 2\gamma}{2(\gamma+1)(\alpha+2)}\|\nabla u_\epsilon(t)\|_2^{2(\gamma+1)} \text{ on } [0, t_{\max}). \end{aligned} \tag{3.7}$$

Thus from (3.4) and (3.7), we have on $[0, t_{\max})$

$$\begin{aligned} \|\nabla u_\epsilon(t)\|_2^{2(\gamma+1)} &\leq \frac{2(\gamma+1)(\alpha+2)}{\alpha-2\gamma} J(u_\epsilon(t)) \\ &\leq \frac{2(\gamma+1)(\alpha+2)}{\alpha-2\gamma} E(u_\epsilon(t)) \\ &\leq \frac{2(\gamma+1)(\alpha+2)}{\alpha-2\gamma} E(u_0). \end{aligned} \tag{3.8}$$

It follows from the Sobolev-Poincaré's inequality, (3.5) and (3.8) that

$$\begin{aligned} \|u_\epsilon(t)\|_{\alpha+2}^{\alpha+2} &\leq C_*^{\alpha+2}\|\nabla u_\epsilon(t)\|_2^{\alpha+2} \\ &= C_*^{\alpha+2}\|\nabla u_\epsilon(t)\|_2^{\alpha-2\gamma}\|\nabla u_\epsilon(t)\|_2^{2(\gamma+1)} \\ &\leq C_*^{\alpha+2}\left(\frac{2(\gamma+1)(\alpha+2)}{\alpha-2\gamma}E(u_0)\right)^{\frac{\alpha-2\gamma}{2(\gamma+1)}}\|\nabla u_\epsilon(t)\|_2^{2(\gamma+1)} \\ &< \|\nabla u_\epsilon(t)\|_2^{2(\gamma+1)} \text{ on } [0, t_{\max}). \end{aligned} \tag{3.9}$$

Therefore, we get $I(u_\epsilon(t) > 0$ on $[0, t_{\max})$. This is implies that we can take $t_{\max} = T$. This ends the proof of Lemma 3.2. □

Proof of Theorem 3.1. We shall use the Galerkin's approximation. Let $(w_j)_{j\in N}$ be a basis of $W_0^{2,4}(\Omega)$ which is orthonormal in $L^2(\Omega)$. For each $m \in N$, we consider $u_{\epsilon m}(t) = \sum_{j=1}^m g_{\epsilon jm}(t)w_j$, the solution for the following system:

$$\begin{aligned} &(\rho(t)u''_{\epsilon m}(t), w) - \left(\|\nabla u_{\epsilon m}(t)\|_2^{2\gamma}\Delta u_{\epsilon m}(t), w\right) \\ &\quad + \frac{1}{\epsilon}\langle\beta(u'_{\epsilon m}(t)), w\rangle = |u_{\epsilon m}(t)|^{\alpha}(u_{\epsilon m}(t), w), \ w \in V_m \end{aligned} \tag{3.10}$$

($' = \frac{\partial}{\partial t}$ and $'' = \frac{\partial^2}{\partial t^2}$) with the initial conditions,

$$
\begin{aligned}
u_{\epsilon m}(0) &= u_{0\epsilon m} = \sum_{j=1}^{m}(u_0, w_j) \to u_0 \text{ strongly in } H_0^1(\Omega)\cap H^2(\Omega), \\
u'_{\epsilon m}(0) &= u_{1\epsilon m} = \sum_{j=1}^{m}(u_1, w_j) \to u_1 \text{ strongly in } H_0^1(\Omega),
\end{aligned} \tag{3.11}
$$

where V_m is the m-dimensional subspace of $W_0^{2,4}(\Omega)$ spanned by $\{w_1, w_2, \cdots, w_m\}$.

Therefore we can solve the system (3.10) by a Picard's iteration method. Hence the system (3.10) has a unique solution on some interval $[0, T_{\epsilon m})$ with $0 < T_{\epsilon m} \le T$. Note that $u_{\epsilon m}(t)$ is in the C^2-class. We shall see that $u_{\epsilon m}(t)$ can be extended to $[0, T]$. We can utilize a standard compactness argument for the limiting procedure, which allows us to employ the energy method for smooth solution $u_\epsilon(t)$ of the problem (3.2).

A Priori Estimates I

Taking $w = u'_{\epsilon m}(t)$ in (3.10) and then integrating it from 0 to t, $t \le T_{\epsilon m}$, then we have

$$
\begin{aligned}
&\frac{1}{2}\rho(t)\|u'_{\epsilon m}(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u_{\epsilon m}(t)\|_2^{2(\gamma+1)} - \frac{1}{\alpha+2}\|u_{\epsilon m}(t)\|_{\alpha+2}^{\alpha+2} \\
&\quad + \frac{1}{\epsilon}\int_0^t \langle \beta(u'_{\epsilon m}(s)), u'_{\epsilon m}(s)\rangle ds \\
&= E(u_0) + \frac{1}{2}\int_0^t \rho'(t)\|u'_{\epsilon m}(s)\|_2^2\, ds
\end{aligned} \tag{3.12}
$$

Now, applying the similar method as the one for Lemma 3.2, $I(u_{\epsilon m}(t)) \ge 0$ on $[0, T_{\epsilon m})$. Thus we have

$$
\begin{aligned}
J(u_{\epsilon m}(t)) &= \frac{1}{\alpha+2} I(u_{\epsilon m}(t)) + \frac{\alpha - 2\gamma}{2(\alpha+2)(\gamma+1)}\|\nabla u_{\epsilon m}(t)\|_2^{2(\gamma+1)} \\
&\ge \frac{\alpha - 2\gamma}{2(\alpha+2)(\gamma+1)}\|\nabla u_{\epsilon m}(t)\|_2^{2(\gamma+1)}.
\end{aligned} \tag{3.13}
$$

From the assumptions on ρ, we have

$$
\rho_0\|u'_{\epsilon m}(t)\|_2^2 \le \rho(t)\|u'_{\epsilon m}(t)\|_2^2 \text{ and } \frac{1}{2}\int_0^t \rho'(t)\|u'_{\epsilon m}(s)\|_2^2\, ds \le 0. \tag{3.14}
$$

Thus (3.13) and (3.14) imply that

$$\tag{3.15} \begin{aligned} &\frac{1}{2}\rho_0\|u'_{\epsilon m}(t)\|_2^2 + \frac{\alpha-2\gamma}{2(\alpha+2)(\gamma+1)}\|\nabla u_{\epsilon m}(t)\|_2^{2(\gamma+1)} \\ &+\frac{1}{\epsilon}\int_0^t \langle \beta(u'_{\epsilon m}(s)), u'_{\epsilon m}(s)\rangle ds \le E(u_0). \end{aligned}$$

Now, we have

$$\tag{3.16} |\Delta u'_{\epsilon m}|^4 - 1 \le 2\Big(1-|\Delta u'_{\epsilon m}|^2\Big)^- |\Delta u'_{\epsilon m}|^2.$$

Thus from (3.16), we get

$$\tag{3.17} \begin{aligned} \frac{1}{\epsilon}\int_0^t \langle \beta(u'_{\epsilon m}(s)), u'_{\epsilon m}(s)\rangle ds &= \frac{1}{\epsilon}\int_0^t\int_\Omega |(u'_{\epsilon m}(s)|^2)^- |\Delta u'_{\epsilon m}(s)|^2\, dxds \\ &\quad + \frac{1}{\epsilon}\int_0^t\int_\Omega (1-|\Delta u'_{\epsilon m}(s)|^2)^- |\Delta u'_{\epsilon m}(s)|^2\, dxds \\ &\ge \frac{1}{2\epsilon}\int_0^t\int_\Omega (|\Delta u'_{\epsilon u}(s)|^4 - 1) dxds. \end{aligned}$$

The inequality (3.15) and (3.17) give

$$\begin{aligned} \|\Delta u'_{\epsilon m}\|^4_{L^4(Q)} &= \int_0^t\int_\Omega |\Delta u'_{\epsilon m}(s)|^4\, dxds \\ &\le 2E(u_0)\epsilon + \frac{m(\Omega)T}{2\epsilon}, \end{aligned}$$

we choose constants $C_1 > 0$ and $C_2 > 0$ such that $E(u_0)\epsilon < C_1$ and $\frac{1}{2}m(\Omega)T < C_2\epsilon$. Then we have

$$\tag{3.18} \|\Delta u'_{\epsilon m}\|^4_{L^4(Q)} \le C_3(T),$$

where $C_3 = C_1 + C_2$.

Also, (3.15) implies

$$\tag{3.19} \|\beta(u_{\epsilon m})\|_{L^{\frac{4}{3}}(0,T;W^{-2,\frac{4}{3}})} \le C_4(T).$$

On the other hand, from (3.18) and Lemma 2.5, we get

$$\begin{aligned} \|\Delta u_{\epsilon m}(t)\|_2^2 &= \|\Delta u_{0\epsilon m}\|_2^2 + \int_0^t \frac{d}{ds}\|\Delta u_{\epsilon m}(s)\|_2^2\, ds \\ &= \|\Delta u_{0\epsilon m}\|_2^2 + 2\int_0^t \frac{d}{ds}(\Delta u_{\epsilon m}(s), \Delta u'_{\epsilon m}(s)) ds \\ &\le \|\Delta u_{0\epsilon m}\|_2^2 + 2\int_0^t \frac{d}{ds}\|\Delta u_{\epsilon m}(s)\|_{\frac{4}{3}}\|\Delta u'_{\epsilon m}(s)\|_4\, ds \end{aligned}$$

$$\leq \|\Delta u_{0\epsilon m}\|_2^2 + \frac{3}{2}\int_0^t \frac{d}{ds}\|\Delta u_{\epsilon m}(s)\|_{\frac{4}{3}}^{\frac{4}{3}} ds + \frac{1}{2}\int_0^t \|\Delta u'_{\epsilon m}(s)\|_4^4 ds$$

$$\leq \|\Delta u_{0\epsilon m}\|_2^2 + \frac{C_3}{2} + \frac{3}{2}m(\Omega)^{\frac{1}{4}} \int_0^t \|\Delta u_{\epsilon m}(s)\|_2^{\frac{4}{3}} ds,$$

where we have used Hölder's inequality and Young's inequality.

Here, we set $g(s) = s^{\frac{2}{3}}$ on $s \geq 0$. Then we have

$$\|\Delta u_{\epsilon m}(t)\|_2^2 \leq C_5 + C_6 \int_0^t g\Big(\|\Delta u_{\epsilon m}(s)\|_2^2\Big) ds,$$

where $C_5 = \|\Delta u_{0\epsilon m}\|_2^2 + \frac{C_3}{2}$ and $C_6 = \frac{3}{2}m(\Omega)^{\frac{1}{4}}$.

Note that $g(s)$ is continuous and nondecreasing on $s \geq 0$. By applying Bihari-Langenhop's inequality, we obtain

$$\|\Delta u_{\epsilon m}(t)\|_2^2 \leq G^{-1}(C_6 t), \quad \text{where} \quad G(y) = \int_{C_5}^y \frac{1}{g(s)} ds, \ y \geq C_5 > 0$$

and so

$$\|\Delta u_{\epsilon m}(t)\|_2^2 \leq \frac{1}{3}\left(3C_5^{\frac{1}{3}} + C_6 T\right)^3 \equiv C_7(T). \tag{3.20}$$

Moreover, it follows from Lemma 2.1 and 2.5 that

$$\begin{aligned} \|\nabla u'_{\epsilon m}(t)\|_2 &\leq C\|\Delta u'_{\epsilon m}(t)\|_2 \\ &\leq \|\Delta u_{\epsilon m}(t)\|_4 \\ &\leq C_8(T). \end{aligned} \tag{3.21}$$

A Priori Estimates II

Firstly, we note that from assumption on ρ,

$$\rho_0\|u''_{\epsilon m}(0)\|_2^2 \leq \rho(0)\|u''_{\epsilon m}(0)\|_2^2$$

and since $u_{1\epsilon m} \to u_1 \in \text{int}(\mathcal{K})$, we can assert that $u_{1\epsilon m} \in \mathcal{K}$ for m large enough. Then $\langle \beta(u_{1\epsilon m}), u''_{\epsilon m}(0)\rangle = 0$. Hence, taking $t = 0$ and $w = u''_{\epsilon m}(0)$ in (3.10), we have

$$\begin{aligned} \rho_0\|u''_{\epsilon m}(0)\|_2^2 \leq\ & \|\nabla u_{\epsilon m}(0)\|_2^{2\gamma}\|\Delta u_{\epsilon m}(0)\|_2\|u''_{\epsilon m}(0)\|_2 \\ &+ \|u_{\epsilon m}(0)\|_{2(\alpha+1)}^{\alpha+1}\|u''_{\epsilon m}(0)\|_2. \end{aligned}$$

Thus

$$\rho_0\|u''_{\epsilon m}(0)\|_2 \leq \|\nabla u_{\epsilon m}(0)\|_2^{2\gamma}\|\Delta u_{\epsilon m}(0)\|_2 + \|u_{\epsilon m}(0)\|_{2(\alpha+1)}^{\alpha+1}. \tag{3.22}$$

Since $\alpha \le \frac{4}{N-2} < \frac{4}{N-4}$, it follows from Gagliardo-Nirenberg inequality that

$$\|u_{\epsilon m}(0)\|_{2(\alpha+1)}^{\alpha+1} \le C\|\Delta u_{\epsilon m}(0)\|_2^{\theta}\|u_{\epsilon m}(0)\|_2^{1-\theta} \le C_9(T) \tag{3.23}$$

for some constant $C_9(T)$. Hence (3.15), (3.20), (3.22) and (3.23) imply that

$$\|u''_{\epsilon m}(0)\|_2 \le C_{10}(T), \tag{3.24}$$

where C_{10} does not depend on m and ϵ.
Let us define

$$\begin{aligned}
\psi_h(t) &= \frac{1}{h}(u_{\epsilon m}(t+h) - u_{\epsilon m}(t))\\
f_h(t) &= \frac{1}{h}(|u_{\epsilon m}(t+h)|^{\alpha}\, u_{\epsilon m}(t+h) - |u_{\epsilon m}(t)|^{\alpha}\, u_{\epsilon m}(t))\\
\rho_h(t) &= \frac{1}{h}(\rho(t+h) - \rho(t))\\
M_h(t) &= \frac{1}{h}(\|\nabla u_{\epsilon m}(t+h)\|^{2\gamma} - \|\nabla u_{\epsilon m}(t)\|^{2\gamma}).
\end{aligned} \tag{3.25}$$

Therefore, from (3.10), we have

(3.26)
$$\begin{aligned}
&(\rho(t+h)\psi''_h(t), \psi'_h(t)) + (\rho_h(t)u''_{\epsilon m}(t), \psi'_h(t)) + \frac{1}{2}\|\nabla u_{\epsilon m}(t+h)\|_2^{2\gamma}\frac{d}{dt}\|\nabla\psi_h(t)\|_2^2\\
&\quad + \frac{1}{\epsilon h^2}\langle\beta(u'_{\epsilon m}(t+h)) - \beta(u'_{\epsilon m}(t)), u'_{\epsilon m}(t+h) - u'_{\epsilon m}(t)\rangle\\
&= -(M_h(t)\Delta u_{\epsilon m}(t), \psi'_h(t)) + (f_h(t), \psi'_h(t)).
\end{aligned}$$

Hence using monotonicity of β, we have

$$\begin{aligned}
&\frac{1}{2}\rho(t+h)\frac{d}{dt}\|\psi'_h(t)\|_2^2 + (\rho_h(t)u''_{\epsilon m}(t), \psi'_h(t)) + \frac{1}{2}\|\nabla u_{\epsilon m}(t+h)\|_2^{2\gamma}\frac{d}{dt}\|\nabla\psi_h(t)\|_2^2\\
&\le \left|(M_h(t)\Delta u_{\epsilon m}(t), \psi'_h(t))\right| + (f_h(t), \psi'_h(t)).
\end{aligned}$$

Letting h tend to zero, then we have

(3.27)
$$\begin{aligned}
&\frac{1}{2}\rho(t)\frac{d}{dt}\|u''_{\epsilon m}(t)\|_2^2 + (\rho'(t)u''_{\epsilon m}(t)) + \frac{1}{2}\|\nabla u_{\epsilon m}(t)\|^{2\gamma}\frac{d}{dt}\|\nabla u'_{\epsilon m}(t)\|_2^2\\
&\le \left|\left(\frac{d}{dt}\|\nabla u_{\epsilon m}(t)\|_2^{2\gamma}\Delta u_{\epsilon m}(t), u''_{\epsilon m}(t)\right)\right| + \left(\frac{d}{dt}(|u_{\epsilon m}(t)|^{\alpha}\, u_{\epsilon m}(t)), u''_{\epsilon m}(t)\right).
\end{aligned}$$

Integrating (3.27) from 0 to t, then we have

$$
\begin{aligned}
&\frac{1}{2}\rho(t)\|u''_{\epsilon m}(t)\|_2^2 + \frac{1}{2}\int_0^t \|\nabla u_{\epsilon m}(s)\|_2^{2\gamma}\frac{d}{ds}\|\nabla u'_{\epsilon m}(s)\|_2^2\,ds \\
&\le \frac{1}{2}\rho(0)\|u''_{\epsilon m}(0)\|_2^2 + \frac{3}{2}\max_{t\in[0,T]}\{|\rho'(t)|\}\int_0^t \|u''_{\epsilon m}(s)\|_2^2\,ds \\
&\quad + \int_0^t \left|\left(\frac{d}{ds}(|u_{\epsilon m}(s)|^\alpha u_{\epsilon m}(s), u''_{\epsilon m}(s)\right)\right|\,ds \\
&\quad + \int_0^t \left|\left(\frac{d}{ds}(\|\nabla u_{\epsilon m}(s)\|_2^{2\gamma})\Delta u_{\epsilon m}(s), u''_{\epsilon m}(s)\right)\right|\,ds.
\end{aligned}
\tag{3.28}
$$

On the other hand, taking into account (3.15), (3.20) and (3.21), we can verify that

$$
\begin{aligned}
&\int_0^t \left|\left(\frac{d}{ds}(\|\nabla u_{\epsilon m}(s)\|_2^{2\gamma})\Delta u_{\epsilon m}(s), u''_{\epsilon m}(s)\right)\right|\,ds \\
&= 2\gamma\int_0^t \|\nabla u_{\epsilon m}(s)\|_2^{2\gamma-2}(\nabla u_{\epsilon m}(s), \nabla u'_{\epsilon m}(s))(\Delta u_{\epsilon m}(s), u''_{\epsilon m}(s))ds \\
&\le 2\gamma\int_0^t \|\nabla u_{\epsilon m}(s)\|_2^{2\gamma-1}\|\nabla u_{\epsilon m}(s)\|_2\|\nabla u_{\epsilon m}(s)\|_2\|u''_{\epsilon m}(s)\|_2\,ds \\
&\le C_{11}(T)\int_0^t \|u''_{\epsilon m}(s)\|_2\,ds.
\end{aligned}
\tag{3.29}
$$

Now, Schwarz's inequality implies that

$$
\begin{aligned}
\left(\frac{d}{ds}(|u_{\epsilon m}(s)|^\alpha\, u_{\epsilon m}(s)), u''_{\epsilon m}(s)\right) &= (\alpha+1)(|u_{\epsilon m}(s)|^\alpha\, u'_{\epsilon m}(s), u''_{\epsilon m}(s)) \\
&\le (\alpha+1)\||u_{\epsilon m}(s)|^\alpha\, u'_{\epsilon m}(s)\|_2\,\|u''_{\epsilon m}(s)\|_2.
\end{aligned}
$$

If $\alpha \le \frac{2}{N-2}$, then we have

$$
\begin{aligned}
\||u_{\epsilon m}(s)|^\alpha\, u'_{\epsilon m}(s)\|_2 &= \|u_{\epsilon m}(s)\|_{N\alpha}^\alpha, u'_{\epsilon m}(s)\|_{\frac{2N}{N-2}} \\
&\le C\|\nabla u_{\epsilon m}(s)\|_2^\alpha\,\|\nabla u'_{\epsilon m}(s)\|_2 \\
&\le C_{12}(T).
\end{aligned}
$$

Thus

$$
\int_0^t \left(\frac{d}{ds}(|u_{\epsilon m}(s)|^\alpha\, u_{\epsilon m}(s)), u''_{\epsilon m}(s)\right) ds \le C_{13}(T)\int_0^t \|u''_{\epsilon m}(s)\|_2\,ds.
\tag{3.30}
$$

On the other hand, we have after integration by parts

$$(3.31)\quad \begin{aligned}&\int_0^t \|\nabla u_{\epsilon m}(s)\|_2^{2\gamma}\frac{d}{ds}\|\nabla u'_{\epsilon m}(s)\|_2^2\,ds\\ &= \|\nabla u_{\epsilon m}(t)\|_2^{2\gamma}\|\nabla u'_{\epsilon m}(t)\|_2^2 - \|\nabla u_{\epsilon m}(0)\|_2^{2\gamma}\|\nabla u'_{\epsilon m}(0)\|_2^2\\ &\quad - 2\gamma\int_0^t \|\nabla u_{\epsilon m}(s)\|_2^{2\gamma-2}(\nabla u_{\epsilon m}(s),\nabla u'_{\epsilon m}(s))\|\nabla u'_{\epsilon m}(s)\|_2^2\,ds.\end{aligned}$$

However, we have

$$(3.32)\quad \begin{aligned}&2\gamma\int_0^t \|\nabla u_{\epsilon m}(s)\|_2^{2\gamma-2}(\nabla u_{\epsilon m}(s),\nabla u'_{\epsilon m}(s))\|\nabla u'_{\epsilon m}s\|_2^2\,ds\\ &\le 2\gamma\int_0^t \|\nabla u_{\epsilon m}(s)\|_2^{2\gamma-1}\|\nabla u'_{\epsilon m}(s)\|_2^3\,ds\\ &\le C_{14}(T)\end{aligned}$$

and

$$(3.33)\quad 0\le \|\nabla u_{\epsilon m}(0)\|_2^{2\gamma}\|\nabla u'_{\epsilon m}(0)\|_2^2 \le C_{15}(T).$$

Thus (3.31)-(3.33) imply that

$$\begin{aligned}&\|\nabla u_{\epsilon m}(t)\|_2^{2\gamma}\|\nabla u'_{\epsilon m}(t)\|_2^2 - C_{14}(T) - C_{15}(T)\\ &\quad\le \int_0^t \|\nabla u_{\epsilon m}(s)\|_2^{2\gamma}\frac{d}{ds}\|\nabla u'_{\epsilon m}(s)\|_2^2\,ds.\end{aligned}$$

Thus from (3.28), (3.29), (3.30) and (3.34), we get

(3.34)

$$\begin{aligned}&\frac{1}{2}\rho_0\|u''_{\epsilon m}(t)\|_2^2 + \frac{1}{2}\|\nabla u_{\epsilon m}(t)\|_2^{2\gamma}\|\nabla u'_{\epsilon m}(t)\|_2^2\\ &\quad\le C_{14}(T)+C_{15}(T)+\frac{1}{2}\rho(0)\|u''_{\epsilon m}(0)\|_2^2+\frac{3}{2}\max_{t\in[0,T]}\{|\rho'(t)|\}\int_0^t \|u''_{\epsilon m}(s)\|_2^2\,ds\\ &\qquad+(C_{11}(T)+C_{13}(T))\int_0^t \|u''_{\epsilon m}(s)\|_2\,ds\\ &\quad\le C_{16}(T)\left(1+\int_0^t (\|u''_{\epsilon m}(s)\|_2+\|u''_{\epsilon m}(s)\|_2^2)ds\right).\end{aligned}$$

Hence Bihari-Langenhop's inequality implies that

$$(3.35)\quad \|u''_{\epsilon m}(t)\|_2 \le C_{17}(T),$$

where C_{17} does not depend on m and ϵ.

Limiting process

It follows from immediately from estimates (3.15), (3.18), (3.20), (3.21) and (3.35) that there exist functions $\{u_\epsilon\}$, ϕ_ϵ and a subsequence of $u_{\epsilon m}$ which we will denote by $\{u_{\epsilon m}\}$ such that

$$u_{\epsilon m} \to u_\epsilon \text{ in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \text{ weak}^*, \tag{3.36}$$

$$u'_{\epsilon m} \to u'_\epsilon \text{ in } L^4(0,T;W_0^{2,4}(\Omega)) \text{ weakly}, \tag{3.37}$$

$$u'_{\epsilon m} \to u'_\epsilon \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak}^*, \tag{3.38}$$

$$u''_{\epsilon m} \to u''_\epsilon \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak}^*, \tag{3.39}$$

$$\beta(u'_{\epsilon m}) \to \phi_\epsilon \text{ in } L^{\frac{4}{3}}(0,T;W^{-2,\frac{4}{3}}(\Omega)) \text{ weakly}. \tag{3.40}$$

Using Aubin-Lions' compactness lemma, we can extract from $\{u_{\epsilon m}\}$ a subsequence still denoted by $\{u_{\epsilon m}\}$ such that

$$u_{\epsilon m} \to u_\epsilon \text{ strongly in } L^2(0,T;H_0^1(\Omega)), \tag{3.41}$$

$$u'_{\epsilon m} \to u'_\epsilon \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.42}$$

Now, letting $m \to \infty$ in (3.10), we can find that u satisfies the equation:

$$\begin{aligned}(\rho(t)u''_\epsilon(t),w) - (\|\nabla u_\epsilon(t)\|_2^{2\gamma}\Delta u_\epsilon(t),w) + \frac{1}{\epsilon}\langle\phi_\epsilon,w\rangle \\ = |u_\epsilon(t)|^\alpha(u_\epsilon(t),w) \text{ a.e. on } [0,T] \text{ for all } w\in W_0^{2,4}(\Omega).\end{aligned} \tag{3.43}$$

It follows from (3.41) that

$$u_{\epsilon m}(0) = u_{0\epsilon m} \to u_\epsilon(0) \text{ strongly in } H_0^1(\Omega).$$

From (3.11), we get $u_\epsilon(0) = u_0$. Also, from (3.42), we obtain

$$(u'_{\epsilon m}(0) - u'_\epsilon(0),w) \to 0 \text{ as } m\to\infty \text{ for each } w\in H_0^1(\Omega),$$

which together with (3.11) imply that

$$u'_\epsilon(0) = u_1.$$

Finally, by means of monotonicity and hemicontinuity, we get

$$\phi_\epsilon = \beta(u'_\epsilon) \text{ in } L^{\frac{4}{3}}(0,T;W^{-2,\frac{4}{3}}(\Omega)).$$

This completes the proof of Theorem 3.1. □

Now we can prove theorem 2.6.

Proof of Theorem 2.6. Let $(\epsilon_n)_{n\in N}$ be a sequence of real numbers such that

$$0 < \epsilon_n < 1 \text{ for all } n \in N \text{ and } \lim_{n\to\infty} \epsilon_n = 0.$$

For each $n \in N$, we get a function which satisfies Theorem 3.1. Since the estimates were uniform on ϵ and n, we can see that there exist a subsequence of u_{ϵ_n}, again called u_{ϵ_n}, and a function u such that

$$u \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)),\ u' \in L^4(0,T; W_0^{2,4}(\Omega)),$$
$$u'' \in L^\infty(0,T; L^2(\Omega))$$

and

$$\begin{aligned}
&u_{\epsilon_n} \to u \text{ in } L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \text{ weak}^*,\\
&u'_{\epsilon_n} \to u' \text{ in } L^4(0,T; W_0^{2,4}(\Omega)) \text{ weakly},\\
&u'_{\epsilon_n} \to u' \text{ in } L^\infty(0,T; L^2(\Omega)) \text{ weak}^*,\\
&u''_{\epsilon_n} \to u'' \text{ in } L^\infty(0,T; L^2(\Omega)) \text{ weak}^*,\\
&\beta(u'_{\epsilon_n}) \to \beta(u') \text{ in } L^{\frac{4}{3}}(0,T; W^{-2,\frac{4}{3}}(\Omega)) \text{ weakly}.
\end{aligned} \tag{3.44}$$

Thus, using monotonicity of β,

$$\begin{aligned}
&\int_0^T (\rho(t)u''_{\epsilon_n}(t) - \|\nabla u_{\epsilon_n}(t)\|_2^{2\gamma}\Delta u_{\epsilon_n}(t) - |u_{\epsilon_n}(t)|^\alpha u_{\epsilon_n}(t),\ v - u'_{\epsilon_n}(t))dt\\
&= -\frac{1}{\epsilon_n}\int_0^T \langle\beta(u'_{\epsilon_n}(t)),\ v - u'_{\epsilon_n}(t)\rangle dt\\
&= \frac{1}{\epsilon_n}\int_0^T \langle\beta(v(t)) - \beta(u'_{\epsilon_n}(t)),\ v - u'_{\epsilon_n}(t)\rangle dt\\
&\quad - \frac{1}{\epsilon_n}\int_0^T \langle\beta(v(t)),\ v(t) - u'_{\epsilon_n}(t)\rangle dt\\
&\geq -\frac{1}{\epsilon_n}\int_0^T \langle\beta(v(t)),\ v(t) - u'_{\epsilon_n}(t)\rangle dt\\
&= \frac{1}{\epsilon_n}\int_0^T\int_\Omega (v(t))^-(v(t) - u'_{\epsilon_n}(t))dxdt\\
&\quad - \int_0^T\int_\Omega (1 - |\Delta v(t)|^2)^-\Delta v(t)\Delta(v(t) - u'_{\epsilon_n}(t))dxdt \geq 0
\end{aligned} \tag{3.45}$$

for all $v \in L^4(0,T; W_0^{2,4}(\Omega))$ with $v(t) \in \mathcal{K}$ a.e. on$[0,T]$.

Letting n tend to ∞, we have

$$\int_0^T (\rho(t)u''(t) - \|\nabla u(t)\|_2^{2\gamma}\Delta u(t) - |u(t)|^\alpha u(t),\ v - u'(t))dt \geq 0 \tag{3.46}$$

for all $v \in L^4(0,T; W_0^{2,4}(\Omega))$with $v(t) \in \mathcal{K}$a.e. on $[0,T]$.

Thus we get (2.4). Now, in order to prove $u'(t) \in \mathcal{K}$ a.e. on $[0,T]$, we observe that from (3.15),

$$\|\beta(u'_{\epsilon_n}(s))\|_{L^\infty(0,T;W^{-2,\frac{4}{3}}(\Omega))} \leq \epsilon_n E(u_0) \text{ for all } n \in N.$$

Since $\lim_{n\to\infty} \epsilon_n = 0$, letting n tend to ∞, we have

$$\beta(u'_{\epsilon_n}) \to 0 \text{ in } L^\infty(0,T;W^{-2,\frac{4}{3}}(\Omega)).$$

From (3.44) and the previous convergence for $\beta(u'_{\epsilon_n})$, we have

$$\beta(u) = 0 \text{ in } L^\infty(0,T;W^{-2,\frac{4}{3}}(\Omega))$$

which shows that $u'(t) \in \mathcal{K}$ a.e. on $[0,T]$.

We shall prove the uniqueness of the solutions of (2.2)-(2.3). Let $u(t)$ and $v(t)$ be two solutions of (2.2)-(2.3) and $w(t) = u(t) - v(t)$. Then $w(t)$ satisfies that

$$\begin{aligned} &(\rho(t)w''(t) - \|\nabla u(t)\|_2^{2\gamma}\Delta w(t) - \left(\|\nabla u(t)\|_2^{2\gamma} - \|\nabla v(t)\|_2^{2\gamma}\right)\Delta v(t) \\ &\qquad - (|u(t)|^\alpha u(t) - |v(t)|^\alpha v(t)),\ w'(t)) \\ &\leq 0 \end{aligned} \tag{3.47}$$

$$\text{and} \quad w(0) = w'(0) = 0.$$

Integrating the inequality (3.46) from 0 to t, then we have

$$\begin{aligned} &\frac{1}{2}\rho_0\|w'(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^{2\gamma}\|\nabla w(t)\|_2^2 \\ &\leq \int_0^t \left|\|\nabla u(s)\|_2^{2\gamma} - \|\nabla v(s)\|_2^{2\gamma}\right| \|\nabla v(s)\|_2\|w'(s)\|_2\, ds \\ &\quad + \int_0^t \|\, |u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\|_2\|w'(s)\|_2\, ds \\ &\quad + \int_0^t \left(\frac{d}{ds}\|\nabla u(s)\|_2^{2\gamma}\right) \|\nabla w(s)\|_2^2\, ds. \end{aligned}$$

Note that from mean value theorem,

$$\left|\|\nabla u(s)\|_2^{2\gamma} - \|\nabla v(s)\|_2^{2\gamma}\right| \leq C_{18}(T)(\|\nabla u(s)\|_2^{2\gamma-1} + \|\nabla v(s)\|_2^{2\gamma-1})\|\nabla w(s)\|_2$$

and

$$\frac{d}{ds}\|\nabla u(s)\|_2^{2\gamma} \leq C_{19}(T)\|\nabla u(s)\|_2^{2\gamma-1}\|\nabla u'(s)\|_2.$$

Also, for $1 \le \alpha \le \frac{4}{N-2}$ ($1 \le \alpha < \infty$ if $N = 1, 2$)

$$\| \, |u(s)|^{\alpha} \, u(s) - |v(s)|^{\alpha} \, v(s)\|_2 \le C_{20}(T)(\|\nabla u(s)\|_2^{\alpha} + \|\nabla v(s)\|_2^{\alpha})\|\nabla w(s)\|_2.$$

Thus

$$\begin{aligned}
&\frac{1}{2}\rho_0\|w'(t)\|_2^2 + \frac{1}{2}M(\|\nabla u(t)\|_2^2)\|\nabla w(t)\|_2^2 \\
&\quad \le C_{18}(T)\int_0^t (\|\nabla u(s)\|_2^{2\gamma-1} + \|\nabla v(s)\|_2^{2\gamma-1})\|\nabla w(s)\|_2\|\Delta v(s)\|_2\|w'(s)\|_2 \, ds \\
&\qquad + C_{19}(T)\int_0^t \|\nabla u(s)\|_2^{2\gamma-1}\|\nabla u'(s)\|_2\|\nabla w(s)\|_2^2 \, ds \\
&\qquad + C_{20}(T)\int_0^t (\|\nabla u(s)\|_2^{\alpha} + \|\nabla v(s)\|_2^{\alpha})\|\nabla w(s)\|_2\|w'(s)_2 \, ds.
\end{aligned}$$

By Hölder inequality and Young's inequality, we have

$$\|w'(t)\|_2^2 + \|\nabla w(t)\|_2^2 \le C_{21}(T)\int_0^t (\|w'(s)\|_2^2 + \|\nabla w(s)\|_2^2) ds.$$

Gronwall's inequality implies that $w \equiv 0$. This completes the proof of Theorem 2.6. □

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