YOKOHAMA MATHEMATICAL
JOURNAL VOL. 46, 1999

ON THE TAIL BEHAVIOUR OF THE SUPREMUM OF A RANDOM WALK DEFINED ON A MARKOV CHAIN

By

GEROLD ALSMEYER AND MIKHAIL SGIBNEV

(Received April 13, 1998)

Abstract. We study the exact tail behaviour of the supremum of a random walk defined on a finite Markov chain. The setting is that of subexponential distributions and the related $\mathcal{S}(\gamma)$-classes of convolution equivalent distributions with exponential tails ($\gamma > 0$).

1. Introduction

Let $\{\kappa_n\}_{n=0}^{\infty}$ be a Markov chain with state space $\mathcal{N} = \{1, \ldots, N\}$ and transition matrix $P = (p_{ij})$, where $p_{ij} = P(\kappa_n = j \mid \kappa_{n-1} = i)$, $n = 1, 2, \ldots$. We suppose that the chain $\{\kappa_n\}$ is ergodic (irreducible, aperiodic and positive recurrent) with stationary distribution $\pi = (\pi_1, \ldots, \pi_N)$ with $\pi_i > 0$, $i \in \mathcal{N}$.

For each pair $(i, j) \in \mathcal{N} \times \mathcal{N}$, let $\{X_m(i, j)\}_{m=1}^{\infty}$ be a sequence of independent identically distributed random variables with distribution F_{ij}. We assume that the sequences of random variables $\{X_m(i, j)\}_{m=1}^{\infty}$, $(i, j) \in \mathcal{N} \times \mathcal{N}$, and $\{\kappa_n\}_{n=0}^{\infty}$ are mutually independent. Write $S_0 = 0$ and $S_n = S_{n-1} + X_n(\kappa_{n-1}, \kappa_n)$ for $n \geq 1$. Denote $M_\infty = \sup_{n \geq 0} S_n$.

The object of this paper is to study the exact tail behaviour of the supremum M_∞ of the random walk $\{S_n\}$ defined on the Markov chain $\{\kappa_n\}$. The ideal setting for such a study will be provided by the class $\mathcal{S}$ of subexponential distributions and the related classes $\mathcal{S}(\gamma)$ (see Definition in Section 2). More precisely, we shall consider the asymptotic behaviour of the functions

$$\overline{W}_{ij}(x) = P(M_\infty > x, \kappa_\eta(x) = j \mid \kappa_0 = i)$$

as $x \to \infty$, where $\eta(x) \stackrel{\text{def}}{=} \min\{n \geq 1 : S_n > x\}$ and $\eta(x) \stackrel{\text{def}}{=} \infty$ on the event $\{M_\infty \leq x\}$.

This problem has already been considered by K. Arndt [1] under more stringent conditions than those of the present paper (for a detailed comparison of

1991 Mathematics Subject Classification: 60J15, 60J10

Key words and phrases: supremum, random walk, finite Markov chain, tail behaviour, subexponential distributions, convolution equivalent distributions with exponential tails, Banach algebras

results see Section 5).

The tail behaviour of the supremum of an ordinary random walk has been considered at various levels of generality by several authors. We mention the works of Borovkov [4, Chapter 4], Varaverbeke [26], Embrechts and Goldie [11], and Bertoin and Doney [2].

2. Preliminaries

Definition. The distribution G of a non-negative random variable Y is said to *belong to the class* $\mathcal{S}(\gamma)$, $\gamma \geq 0$, if the following conditions are satisfied:

$$\lim_{x\to\infty}(1-G(x+y))/(1-G(x)) = \exp(-\gamma y) \quad \text{for all } y \in \mathbf{R}; \tag{1}$$

$$\widehat{G}(\gamma) \stackrel{\text{def}}{=} E\exp(\gamma Y) < \infty;$$

$$\lim_{x\to\infty}(1-G^{*2}(x))/(1-G(x)) = 2\widehat{G}(\gamma).$$

The class $\mathcal{S} = \mathcal{S}(0)$ (later called the class of *subexponential* distributions) was introduced in [5], while the classes $\mathcal{S}(\gamma)$ for positive γ were first considered in [6] and [7]. There is a rather extensive literature concerning both the properties of $\mathcal{S}(\gamma)$-distributions themselves and numerous applications in various areas of probability theory (branching processes, queueing theory, infinite divisibility, etc.); see, e.g. [8], [9], [10], [12], [14], [17], [20], [21], [23] and [25]. The importance of such distributions has widely been illustrated by the fact that in many cases the exact asymptotic behaviour of probabilistic quantities of interest can be expressed in terms of the distributions of $\mathcal{S}(\gamma)$.

Denote by $\mathfrak{S}_\gamma$ the Banach algebra of all complex-valued measures ν on $\mathcal{B}$ such that

$$\|\nu\| = |\nu|(-\infty,0) + \int_0^\infty e^{\gamma x}|\nu|(dx) < \infty;$$

here $|\nu|(A)$ stands for the total variation of the measure ν on a set $A \in \mathcal{B}$. The addition of measures in $\mathfrak{S}_\gamma$ and multiplication of a measure by a scalar are defined in the usual way; the product of any two elements of $\mathfrak{S}_\gamma$ is their convolution, and the measure δ_0 of unit mass concentrated at the origin is the unit element of $\mathfrak{S}_\gamma$. Denote by $\widehat{\nu}(s)$ the Laplace transform of a measure ν : $\widehat{\nu}(s) = \int_{\mathbf{R}} \exp(sx)\nu(dx)$. The integral converges absolutely — with respect to $|\nu|$ — in the strip $\Pi(\gamma) \stackrel{\text{def}}{=} \{s \in \mathbf{C} : 0 \leq \Re s \leq \gamma\}$. Obviously, $\widehat{\delta}_0(s) \equiv 1$. We shall denote by $\widehat{\mathfrak{S}}_\gamma$ the isometric Banach algebra of Laplace transforms of elements of $\mathfrak{S}_\gamma$, i.e. $\widehat{\mathfrak{S}}_\gamma \stackrel{\text{def}}{=} \left\{\widehat{f}(s), s \in \Pi(\gamma) : f \in \mathfrak{S}_\gamma\right\}$ and $\|\widehat{f}\| \stackrel{\text{def}}{=} \|f\|$ for $f \in \widehat{\mathfrak{S}}_\gamma$.

Let ν be a finite complex-valued measure defined on $\mathcal{B}$. Define the measure

$\nu_{(1)}$ by

$$\nu_{(1)}(A) = \int_A n_1(x)dx, \qquad A \in \mathcal{B},$$

where $n_1(x) = -\nu(-\infty, x]$ for $x < 0$ and $n_1(x) = \nu(x,\infty)$ for $x \geq 0$. If $\int_{\mathbf{R}} |x|\,|\nu|(dx) < \infty$, then $\nu_{(1)}$ is a finite complex-valued measure and $\widehat{\nu}_{(1)}(s) = (\widehat{\nu}(s) - \widehat{\nu}(0))/s$, $\Re s = 0$.

Now choose an arbitrary distribution $G \in \mathcal{S}(\gamma)$. Put $\tau(x) = 1 - G(x)$. Define a functional $Q(\nu)$ on $\mathfrak{S}_\gamma$ by the formula

$$Q(\nu) = \sup_{x \geq 0} |\nu|(x,\infty)/\tau(x), \qquad \nu \in \mathfrak{S}_\gamma.$$

Consider the collections of measures $\mathfrak{S}(\tau) = \{\nu \in \mathfrak{S}_\gamma : Q(\nu) < \infty\}$ and

$$\mathfrak{S}l(\tau) = \left\{\nu \in \mathfrak{S}(\tau) : l(\nu) \overset{\text{def}}{=} \lim_{x\to\infty} \frac{\nu(x,\infty)}{\tau(x)} \in \mathbf{C} \text{ exists}\right\}$$

As shown in [20, Propositions 1 and 2] $\mathfrak{S}(\tau)$ is a Banach algebra with respect to some norm $\|\nu\|'$ equivalent to the norm $\|\nu\| + Q(\nu)$, and $\mathfrak{S}l(\tau)$ is a Banach subalgebra of $\mathfrak{S}(\tau)$. Moreover, for any two elements $\nu, \kappa \in \mathfrak{S}l(\tau)$, the following equality holds:

$$l(\nu * \kappa) = l(\nu)\widehat{\kappa}(\gamma) + l(\kappa)\widehat{\nu}(\gamma). \tag{2}$$

Let $\mathcal{A}$ be an arbitrary commutative complex Banach algebra with unit element e. The spectrum $\sigma(a)$ of an element $a \in \mathcal{A}$ is defined to be the set of all complex numbers λ such that the element $a - \lambda e$ does not have an inverse. If $f(z)$ is an analytic function in a domain containing the spectrum of an element $a \in \mathcal{A}$, then there exists an element $f(a) \in \mathcal{A}$ such that for each homomorphism $m : \mathcal{A} \to \mathbf{C}$ the following relation holds: $m(f(a)) = f(m(a))$ [27, Section 3]. The element $f(a)$ is called the *value of the analytic function $f(z)$ at the element $a \in \mathcal{A}$.*

We shall need the following result on the values of an analytic function at elements of $\mathfrak{S}l(\tau)$ [20, Theorem 3].

. **Theorem 1.** *Let $f(z)$ be an analytic function in a domain containing the spectrum $\sigma(\nu)$ of an element $\nu \in \mathfrak{S}_\gamma$, and let $f(\nu)$ be the value of $f(z)$ at $\nu \in \mathfrak{S}_\gamma$.*

(i) *If $\nu \in \mathfrak{S}(\tau)$, then $f(\nu) \in \mathfrak{S}(\tau)$.*

(ii) *If $\nu \in \mathfrak{S}l(\tau)$, then $f(\nu) \in \mathfrak{S}l(\tau)$ and the following equality holds:*

$$l[f(\nu)] = f'[\widehat{\nu}(\gamma)] \cdot l(\nu).$$

We also mention a result on the invertibility of elements in $\mathfrak{S}l(\tau)$. Theorem 2 of [20] says that each maximal ideal of the Banach algebra $\mathfrak{S}l(\tau)$ can be represented in the form $M = M_1 \cap \mathfrak{S}l(\tau)$, where M_1 is a maximal ideal of $\mathfrak{S}_\gamma$; and vice versa, if M_1 is an arbitrary maximal ideal of $\mathfrak{S}_\gamma$, then $M = M_1 \cap \mathfrak{S}l(\tau)$ is a maximal ideal of $\mathfrak{S}l(\tau)$. It follows [19, Section 11.5] that if an element $\nu \in \mathfrak{S}l(\tau)$ has an inverse ν^{-1} in $\mathfrak{S}_\gamma$, then $\nu^{-1} \in \mathfrak{S}l(\tau)$. The same is also true for the Banach algebra $\mathfrak{S}(\tau)$.

Lemma 1. *Let $G \in \mathcal{S}(\gamma)$ for some $\gamma \geq 0$ and let $\tau(x) = 1 - G(x)$. Suppose a finite positive measure ν is such that $\nu_{(1)} \in \mathfrak{S}l(\tau)$. Then also $\nu \in \mathfrak{S}l(\tau)$ and $l(\nu) = \gamma \cdot l(\nu_{(1)})$. Conversely, if $\nu \in \mathfrak{S}l(\tau)$ and $\gamma > 0$, then also $\nu_{(1)} \in \mathfrak{S}l(\tau)$.*

Proof. Suppose $\gamma > 0$ and choose an arbitrary $h > 0$. We have

$$\frac{\nu(t+h,\infty)\cdot h}{\tau(t)} \leq \int_t^{t+h} \frac{\nu(x,\infty)dx}{\tau(t)} \leq \frac{\nu(t,\infty)\cdot h}{\tau(t)}. \tag{3}$$

By (1), the middle term tends to $l(\nu_{(1)})(1-e^{-\gamma h})$ as $t \to \infty$, so that the inequalities (3) imply

$$\begin{aligned}\frac{(1-e^{-\gamma h})}{h} l(\nu_{(1)}) &\leq \liminf_{t\to\infty} \frac{\nu(t,\infty)}{\tau(t)} \\ &\leq \limsup_{t\to\infty} \frac{\nu(t,\infty)}{\tau(t)} \leq \frac{e^{\gamma h}(1-e^{-\gamma h})}{h} l(\nu_{(1)}).\end{aligned}$$

Letting $h \to 0$, we arrive at the conclusion of the lemma for $\gamma > 0$. The case $\gamma = 0$ is easily dealt with. We have

$$\limsup_{t\to\infty} \frac{\nu(t,\infty)}{\tau(t)} \leq \lim_{t\to\infty} \frac{\nu_{(1)}(t-1,\infty) - \nu_{(1)}(t,\infty)}{\tau(t)} = 0.$$

Suppose now that $\nu \in \mathfrak{S}l(\tau)$ and $\gamma > 0$. We have

$$\frac{\nu_{(1)}(t,\infty)}{\tau(t)} = \frac{\int_t^\infty \nu(x,\infty)dx}{\int_t^\infty \tau(x)dx} \cdot \frac{\int_t^\infty \tau(x)dx}{\tau(t)}.$$

The first factor on the right-hand side tends to $l(\nu)$ as $t \to \infty$. The desired assertion $\nu_{(1)} \in \mathfrak{S}l(\tau)$ will follow if we show that the second factor tends to $1/\gamma$, i.e.

$$\lim_{t\to\infty} \frac{\int_t^\infty \tau(x)dx}{\tau(t)} = \lim_{t\to\infty} \int_0^\infty \frac{\tau(t+x)}{\tau(x)} dx = \frac{1}{\gamma}. \tag{4}$$

By (1), there exists $T > 0$ such that $\tau(t+1)/\tau(t) < e^{-\gamma/2}$ for all $t \geq T$. The integrand $\tau(t+x)/\tau(t)$ tends to $\exp(-\gamma x)$ as $t \to \infty$ and, for $t \geq T$, it is majorized by the integrable function $f(x) \stackrel{\text{def}}{=} \exp(-n\gamma/2)$, where $x \in [n, n+1)$,

$n = 0, 1, \ldots$. Applying the dominated convergence theorem, we obtain (4). The lemma is proved.

Suppose a matrix $\mathbf{B} = \|B_{ij}\|$, say, is made up of elements of $\mathfrak{S}_\gamma$. Then we shall denote by $\widehat{\mathbf{B}}(s)$ the matrix whose entries are the Laplace transforms of the entries of $\mathbf{B}$, i.e. $\widehat{\mathbf{B}}(s) \stackrel{\text{def}}{=} \|\widehat{B}_{ij}(s)\|$. In this case we shall also write $\mathbf{B} \in \mathfrak{S}_\gamma$ and $\widehat{\mathbf{B}}(s) \in \widehat{\mathfrak{S}}_\gamma$. The same convention will be applied to other collections of measures, e.g. such as $\mathfrak{S}l(\tau)$. If $\mathbf{B} = \|B_{ij}\| \in \mathfrak{S}l(\tau)$, then $l(\mathbf{B}) \stackrel{\text{def}}{=} \|l(B_{ij})\|$. If $\mathbf{B} = \|B_{ij}\|$ is a matrix whose entries are finite complex-valued measures such that $\int_{-\infty}^{\infty} |x| B_{ij}(dx) < \infty$, $i, j \in \mathcal{N}$, then we shall denote by $\mathbf{B}_{(1)}$ the matrix $\|(B_{ij})_{(1)}\|$.

Denote by $\mathbf{E}$ the $N \times N$ matrix $\|E_{ij}\|$ whose entries are measures such that $E_{ij} = \delta_0$ for $i = j$ and $E_{ij} = 0$ for $i \neq j$ (recall that δ_0 is the unit element of the Banach algebra $\mathfrak{S}l(\tau)$).

Lemma 2. *Suppose that a matrix* $\mathbf{D} \in \mathfrak{S}l(\tau)$ *has an inverse* $\mathbf{D}^{-1} \in \mathfrak{S}l(\tau)$, *i.e.* $\mathbf{D} * \mathbf{D}^{-1} = \mathbf{E}$. *Then*

$$l(\mathbf{D}^{-1}) = -[\widehat{\mathbf{D}}(\gamma)]^{-1} \cdot l(\mathbf{D}) \cdot [\widehat{\mathbf{D}}(\gamma)]^{-1}. \tag{5}$$

Proof. The equalities $\mathbf{D} * \mathbf{D}^{-1} = \mathbf{E}$ and (2) imply

$$l(\mathbf{D}) \cdot [\widehat{\mathbf{D}}(\gamma)]^{-1} + \widehat{\mathbf{D}}(\gamma) \cdot l(\mathbf{D}^{-1}) = l(\mathbf{E}) = \mathbf{0}, \tag{6}$$

where $\mathbf{0}$ stands for the null matrix. Equality (5) follows from (6).

Denote by $\mathbf{A}$ the $N \times N$ matrix $\|p_{ij} F_{ij}\|$ and by $\mathbf{W}$ the $N \times N$ matrix $\|W_{ij}\|$, where W_{ij} is the measure defined on $\mathcal{B}$ by the relations

$$W_{ij}(x, \infty) \stackrel{\text{def}}{=} P(M_\infty > x, \kappa_{\eta(x)} = j \mid \kappa_0 = i), \qquad x > 0,$$

$W_{ij}(-\infty, 0) \stackrel{\text{def}}{=} 0$, $i, j \in \mathcal{N}$, and

$$W_{ij}(\{0\}) \stackrel{\text{def}}{=} \delta_{ij} - P(M_\infty > 0, \kappa_{\eta(0)} = j \mid \kappa_0 = i),$$

δ_{ij} is the Kronecker delta (the reason for this latter definition will become clear later).

Finally, we shall denote by $\lambda(\xi)$ the maximal positive eigenvalue of the matrix $\widehat{\mathbf{A}}(\xi)$; here ξ is a real number.

We shall need a matrix factorization involving the underlying matrix $\mathbf{A}$. Put

$$\mathbf{U}(s) = \|U_{ij}(s)\| \stackrel{\text{def}}{=} \left\| \frac{s+1}{s} \delta_{1i} + \delta_{ij}(1 - \delta_{1j}) \right\|,$$

$\mathbf{J} \stackrel{\text{def}}{=} \operatorname{diag}(\pi_1, \ldots, \pi_N)$ and $\mathbf{B}(s) \stackrel{\text{def}}{=} \mathbf{U}(s)\mathbf{J}$. We have

$$\mathbf{B}(s) = \begin{Vmatrix} \frac{s+1}{s}\pi_1 & \frac{s+1}{s}\pi_2 & \ldots & \frac{s+1}{s}\pi_N \\ 0 & \pi_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \pi_N \end{Vmatrix}$$

Let $\mathbf{I}$ be the identity matrix of order $N \times N$.

The following theorem about right canonical factorization is an easy consequence of already known results [16], [18] and [3]. Recall $\eta(x) = \inf\{n \geq 1 : S_n > x\}$ and define $\bar{s}_n = \max_{1 \leq m \leq n} S_m$ and $\chi(x) = S_{\eta(x)} - x$. Denote also $P_i(\cdot) = P(\cdot \mid \kappa_0 = i)$, $i \in \mathcal{N}$, and $E_\pi S_1 \stackrel{\text{def}}{=} \sum_{i,j=1}^N \pi_i p_{ij} EX_1(i,j)$, i.e. $E_\pi S_1$ is the expectation of a one-step increment of the random walk $\{S_n\}$ under the stationary distribution π of the chain $\{\kappa_n\}$.

Theorem 2. *Let $\{S_n\}_{n=0}^\infty$ be a random walk defined on a Markov chain $\{\kappa_n\}_{n=0}^\infty$. Suppose the expectations $EX_1(i,j)$ $i,j \in \mathcal{N}$, are finite and $E_\pi S_1 \in (-\infty, 0)$. Assume also that the underlying distributions F_{ij} are absolutely continuous with respect to Lebesgue measure and $E\exp(\gamma X_1(i,j)) < \infty$ for some $\gamma \geq 0$ and all $i,j \in \mathcal{N}$, i.e. $F_{ij} \in \mathfrak{S}_\gamma$, $i,j \in \mathcal{N}$. If $\gamma > 0$, suppose additionally that $\lambda(\gamma) < 1$. Then, for $s \in \Pi(\gamma)$,*

$$\mathbf{B}(s)[\mathbf{I} - \widehat{\mathbf{A}}(s)] = [\mathbf{B}(s)\widehat{\mathbf{A}}_-(s)]\widehat{\mathbf{A}}_+(s), \tag{7}$$

where

$$\widehat{\mathbf{A}}_-(s) = \mathbf{I} - \left\| \sum_{n=1}^\infty \int_{-\infty}^0 e^{sx} P_i(\bar{s}_{n-1} < S_n \in dx, \kappa_n = j) \right\|$$

and

$$\widehat{\mathbf{A}}_+(s) = \mathbf{I} - \left\| \int_0^\infty e^{sx} P_i(\chi(0) \in dx, \kappa_{\eta(0)} = j) \right\|$$

Moreover, the matrices $[\mathbf{B}(s)\widehat{\mathbf{A}}_-(s)]$ and $\widehat{\mathbf{A}}_+(s)$ have inverses with entries being elements of the Banach algebra $\widehat{\mathfrak{S}}_\gamma$. The matrix $[\widehat{\mathbf{A}}_+(s)]^{-1}$ is given by

$$[\widehat{\mathbf{A}}_+(s)]^{-1} = \mathbf{I} + \left\| \sum_{n=1}^\infty \int_0^\infty e^{sx} P_i(\bar{s}_{n-1} < S_n \in dx, \kappa_n = j) \right\|.$$

Proof. If $\gamma = 0$, the assertion of the theorem is an immediate consequence of [3, Theorem 2(c)]. Suppose γ is positive. If $\Re s = 0$, then, as before, the factorization (7) is valid. Fix an arbitrarily small $\varepsilon > 0$. As pointed out in

[1, Proposition 1] with reference to [16], the matrix $\mathbf{I} - \widehat{\mathbf{A}}(s)$ admits the right canonical factorization $\mathbf{I} - \widehat{\mathbf{A}}(s) = \widehat{\mathbf{A}}_-(s)\widehat{\mathbf{A}}_+(s)$ for all $\varepsilon \le \Re s \le \gamma$, where the matrices $\widehat{\mathbf{A}}_-(s)$ and $\widehat{\mathbf{A}}_+(s)$ have the same meaning as before. Moreover, not only $\widehat{\mathbf{A}}_+(s)$, but also $\widehat{\mathbf{A}}_-(s)$ is invertible in the strip $\varepsilon \le \Re s \le \gamma$. Since the matrix $\mathbf{B}(s)$ is automatically invertible in the same strip and since $\varepsilon > 0$ is arbitrary, the desired factorization follows for $0 \le \Re s \le \gamma$.

3. Main result

Let $\mathbf{1}$ denote $N \times 1$ column vector whose entries equal 1 and $\boldsymbol{\pi}$ the $1 \times N$ row vector $\|\pi_1, \ldots, \pi_N\|$.

Theorem 3. *Let $\{S_n\}_{n=0}^{\infty}$ be a random walk defined on a Markov chain $\{\kappa_n\}_{n=0}^{\infty}$. Let G belong to the class $\mathcal{S}(\gamma)$ for some $\gamma \ge 0$. Suppose $\lambda(\gamma) < 1$ for $\gamma > 0$. In the case $\gamma = 0$, assume additionally that the expectations $EX_1(i,j)$, $i,j \in \mathcal{N}$, are finite and $E_\pi S_1 \in (-\infty, 0)$. If $\mathbf{A}_{(1)} \in \mathfrak{S}l(\tau)$ for $\gamma > 0$ or if $\boldsymbol{\pi}\cdot\mathbf{A}_{(1)} \in \mathfrak{S}l(\tau)$ for $\gamma = 0$, then $\mathbf{W} \in \mathfrak{S}l(\tau)$ and*

$$l(\mathbf{W}) = \begin{cases} \gamma[\mathbf{I} - \widehat{\mathbf{A}}(\gamma)]^{-1} l(\mathbf{A}_{(1)})[\widehat{\mathbf{A}}_+(\gamma)]^{-1}\widehat{\mathbf{A}}_+(0) & \text{if } \gamma > 0, \\ \dfrac{1}{-E_\pi S_1}\, l(\mathbf{1}\boldsymbol{\pi}\cdot\mathbf{A}_{(1)}) & \text{if } \gamma = 0. \end{cases} \tag{8}$$

Proof. In order to make the proof more readable, we split it into several stages, according to various technical assumptions.

I. Basic relation. As in Arndt [1], we start with the relation

$$\begin{aligned} &\mathbf{I} - \|P_i(M_\infty > 0, \kappa_{\eta(0)} = j)\| \\ &+ \left\| \int_{0+}^{\infty} e^{sx} P_i(M_\infty \in dx, \kappa_{\eta(x)} = j) \right\| \stackrel{\text{def}}{=} \widehat{\mathbf{W}}(s) = \left[\widehat{\mathbf{A}}_+(s)\right]^{-1}\widehat{\mathbf{A}}_+(0), \end{aligned} \tag{9}$$

which follows from Presman [18, Theorem 2.2]. If we prove that $\mathbf{A}_+^{-1} \in \mathfrak{S}l(\tau)$, then relation (9) will imply $\mathbf{W} \in \mathfrak{S}l(\tau)$ and

$$l(\mathbf{W}) = l(\mathbf{A}_+^{-1})\widehat{\mathbf{A}}_+(0). \tag{10}$$

Denote $\widehat{\mathbf{Q}}(s) = \|\widehat{Q}_{ij}(s)\| = \mathbf{B}(s)\left[\mathbf{I} - \widehat{\mathbf{A}}(s)\right]$.

Before we proceed further, we will prove the following two lemmas.

Lemma 3. *Let $\{S_n\}_{n=0}^{\infty}$ be a random walk defined on a Markov chain $\{\kappa_n\}_{n=0}^{\infty}$. Let G belong to the class $\mathcal{S}(\gamma)$ for some $\gamma \ge 0$. Suppose $\lambda(\gamma) < 1$ for $\gamma > 0$. Assume that all the expectations $EX_1(i,j)$ are finite and $E_\pi S_1 \in (-\infty, 0)$.*

If $\mathbf{A}_{(1)} \in \mathfrak{S}l(\tau)$ *for* $\gamma > 0$ *or if* $\boldsymbol{\pi}\cdot\mathbf{A}_{(1)} \in \mathfrak{S}l(\tau)$ *for* $\gamma = 0$, *then* $\mathbf{Q} \in \mathfrak{S}l(\tau)$ *and*

$$(11)\qquad l(\mathbf{Q}) = \begin{cases} -\gamma\mathbf{B}(\gamma)\cdot l(\mathbf{A}_{(1)}) & \text{if } \gamma > 0, \\ \left\| \begin{matrix} -l\big((\boldsymbol{\pi}\cdot\mathbf{A}_{(1)})_1\big) & \dots & -l\big((\boldsymbol{\pi}\cdot\mathbf{A}_{(1)})_N\big) \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{matrix} \right\| & \text{if } \gamma = 0. \end{cases}$$

Lemma 4. *Suppose that the hypotheses of Lemma 3 are satisfied. Assume additionally that all the distributions* $F_{i,j}$ *are absolutely continuous. Then* $[\widehat{\mathbf{Q}}(s)]^{-1} \in \widehat{\mathfrak{S}l}(\tau)$.

Proof of Lemma 3. Consider the $(1,1)$-entry of the matrix $\widehat{\mathbf{Q}}(s)$ (the argument will also apply to the remaining $(1,j)$-entries of $\widehat{\mathbf{Q}}(s)$). The entry under consideration is equal to

$$\frac{s+1}{s}\left[\pi_1 - \sum_{i=1}^{N} \pi_i p_{i1} \widehat{F}_{i1}(s)\right] = \frac{s+1}{s}\sum_{i=1}^{N} \pi_i p_{i1}\big[1 - \widehat{F}_{i1}(s)\big]$$

$$(12)\qquad = \sum_{i=1}^{N} \pi_i p_{i1}\big[1 - \widehat{F}_{i1}(s) - (\widehat{F_{i1}})_{(1)}(s)\big],$$

Row i of the matrix $\widehat{\mathbf{Q}}(s)$ for $i > 1$ is equal to row i of $\mathbf{I} - \widehat{\mathbf{A}}(s)$ multiplied by π_i, $i = 2, \dots, N$.

Let us compute the value of $l(\widehat{\mathbf{Q}}(s))$. Suppose $\gamma > 0$. By the hypotheses of the lemma, $(F_{ij})_{(1)} \in \mathfrak{S}l(\tau)$, $i,j \in \mathcal{N}$. By Lemma 1, $F_{ij} \in \mathfrak{S}l(\tau)$ and $l(F_{ij}) = \gamma \cdot l\big((F_{ij})_{(1)}\big)$. Hence $l\widehat{\mathbf{Q}}(s))$ is equal to the matrix

$$\left\| \begin{matrix} -(\gamma+1)\sum_{i=1}^{N} \pi_i p_{i1} l\big((F_{i1})_{(1)}\big) & \dots & -(\gamma+1)\sum_{i=1}^{N} \pi_i p_{iN} l\big((F_{iN})_{(1)}\big) \\ -\gamma\pi_2 p_{21} l\big((F_{21})_{(1)}\big) & \dots & -\gamma\pi_2 p_{2N} l\big((F_{2N})_{(1)}\big) \\ \dots & \dots & \dots \\ -\gamma\pi_N p_{N1} l\big((F_{N1})_{(1)}\big) & \dots & -\gamma\pi_N p_{NN} l\big((F_{NN})_{(1)}\big) \end{matrix} \right\|$$

which, in turn, is equal to $-\gamma\mathbf{B}(\gamma)\cdot l(\mathbf{A}_{(1)})$.

Let $\gamma = 0$. By Lemma 1, $\boldsymbol{\pi}\cdot\mathbf{A} \in \mathfrak{S}l(\tau)$ and $l(\boldsymbol{\pi}\cdot\mathbf{A}) = \mathbf{0}$ (the null row vector). It follows that $l(p_{ij}F_{ij}) = 0$ for all $i,j \in \mathcal{N}$ since $p_{ij}F_{ij} \le (\boldsymbol{\pi}\cdot\mathbf{A})_j(x,\infty)/\pi_j$. Hence $\mathbf{Q} \in \mathfrak{S}l(\tau)$ and the corresponding equality for $l(\mathbf{Q})$ holds.

Proof of Lemma 4. Remembering the structure of an inverse matrix, we arrive at the desired conclusion as follows. The adjoint matrix of $\widehat{\mathbf{Q}}(s)$ is obviously an element of $\widehat{\mathfrak{S}l}(\tau)$, and so is $\det\{\widehat{\mathbf{Q}}(s)\} \overset{\text{def}}{=} d(s)$ as well. The determinant

$d(s)$, $s \in \Pi(\gamma)$, is the Laplace transform of a measure of the form $a\delta_0 + \beta$, where $a = \prod_{i=1}^N \pi_i \neq 0$ and $\beta \in \mathfrak{S}l(\tau) \subset \mathfrak{S}_\gamma$ is absolutely continuous (recall δ_0 is the measure of unit mass concentrated at the origin). We need to check only $a = \prod_{i=1}^N \pi_i$. Applying the Riemann-Lebesgue lemma, we see that the $(1,1)$-entry of $\widehat{\mathbf{Q}}(s)$ tends to π_1 as $s \to \infty$, $\Re s = 0$. Similarly, the (i,j)-enrty of $\widehat{\mathbf{Q}}(s)$ tends to $\pi_i\delta_{ij}$ as $s \to \infty$, $\Re s = 0$; here $i = 2, \ldots, N$, $j = 1, \ldots, N$. Since $\widehat{\beta}(s) \to 0$ as $s \to \infty$, $\Re s = 0$, and since the determinant of a matrix is a continuous function of its entries, the equality $a = \prod_{i=1}^N \pi_i$ is proved.

Next we note that the function $\varphi(x) \stackrel{\text{def}}{=} \exp(\gamma x) \vee 1$ is clearly *submultiplicative*, i.e.

$$\varphi(0) = 1, \qquad \varphi(x+y) \leq \varphi(x)\varphi(y) \quad \text{for all} \quad x, y \in \mathbf{R}.$$

Moreover,

$$0 = \lim_{x \to -\infty} \frac{\log \varphi(x)}{x} \leq \lim_{x \to \infty} \frac{\log \varphi(x)}{x} = \gamma.$$

Consider the subalgebra $\mathcal{V}_\gamma$, say, of $\mathfrak{S}_\gamma$ consisting of all elements $a\delta_0 + \beta$, where $a \in \mathbf{C}$ and β is absolutely continuous. By the general theory of such algebras [13, Section 18], $1/d(s)$ is the Laplace transform of a measure $\nu \in \mathcal{V}_\gamma \subset \mathfrak{S}_\gamma$ since $d(s) = a + \widehat{\beta}(s) \neq 0$, $s \in \Pi(\gamma)$, and $a \neq 0$. By Theorem 1, $\nu \in \mathfrak{S}l(\tau)$. Summing up, we conclude that $[\widehat{\mathbf{Q}}(s)]^{-1} \in \widehat{\mathfrak{S}}l(\tau)$.

We now return to the proof of Theorem 3.

II. Absolutely continuous case. Here, we assume that all the distributions F_{ij} are absolutely continuous with finite means $EX_1(i,j)$ such that $E_\pi S_1 < 0$, even in the case $\gamma > 0$.

It follows from Theorem 2 that

$$[\widehat{\mathbf{A}}_+(s)]^{-1} = \{\widehat{\mathbf{Q}}(s)\}^{-1}[\mathbf{B}(s)\widehat{\mathbf{A}}_-(s)].$$

By Theorem 2, $\mathbf{B}(s)\widehat{\mathbf{A}}_-(s)$ is a matrix of the Laplace transforms of finite (not necessarily positive) measures concentrated on $(-\infty, 0)$. Hence, automatically, $\mathbf{B}(s)\widehat{\mathbf{A}}_-(s) \in \widehat{\mathfrak{S}}l(\tau)$ with $l(\mathbf{B}(s)\widehat{\mathbf{A}}_-(s)) = \mathbf{0}$ (the null matrix). By (7) and Lemma 4,

$$[\widehat{\mathbf{A}}_+(s)]^{-1} = [\widehat{\mathbf{Q}}(s)]^{-1}[\mathbf{B}(s)\widehat{\mathbf{A}}_-(s)] \in \widehat{\mathfrak{S}}l(\tau)$$

and, by (2),

$$\begin{aligned} l([\widehat{\mathbf{A}}_+(s)]^{-1}) &= l([\widehat{\mathbf{Q}}(s)]^{-1}) \cdot \mathbf{B}(\gamma)\widehat{\mathbf{A}}_-(\gamma) + [\widehat{\mathbf{Q}}(\gamma)]^{-1} \cdot l(\mathbf{B}(s)\widehat{\mathbf{A}}_-(s)) \\ &= l([\widehat{\mathbf{Q}}(s)]^{-1}) \cdot \mathbf{B}(\gamma)\widehat{\mathbf{A}}_-(\gamma). \end{aligned} \tag{13}$$

By Lemma 2 with $\mathbf{D} = \mathbf{Q}$, we obtain from (13) that

$$l([\widehat{\mathbf{A}}_+(s)]^{-1}) = -[\widehat{\mathbf{Q}}(\gamma)]^{-1} \cdot l(\widehat{\mathbf{Q}}(s)) \cdot [\widehat{\mathbf{A}}_+(\gamma)]^{-1}. \tag{14}$$

Let $\gamma > 0$. By Lemma 3, it follows from (14) that

$$l([\widehat{\mathbf{A}}_+(s)]^{-1}) = \gamma[\mathbf{I} - \widehat{\mathbf{A}}(\gamma)]^{-1} \cdot l(\mathbf{A}_{(1)}) \cdot [\widehat{\mathbf{A}}_+(\gamma)]^{-1}. \tag{15}$$

Relations (10) and (13) now imply (8), provided that all the distributions F_{ij} are absolutely continuous and have finite means $EX_1(i,j)$ such that $E_\pi S_1 \in (-\infty, 0)$.

Suppose $\gamma = 0$. In order to obtain an explicit expression for $l([\widehat{\mathbf{A}}_+(s)]^{-1})$ (see (14)), let us compute $[\widehat{\mathbf{Q}}(0)]^{-1} \cdot l(\widehat{\mathbf{Q}}(s))$. Since all the rows of $l(\widehat{\mathbf{Q}}(s))$, except for the first one, vanish, we need to compute only the first column of the matrix $[\widehat{\mathbf{Q}}(0)]^{-1}$ in order to obtain the desired product of matrices. Notice that the sum of the entries in row $i, i = 2, \dots, N$, vanish. Denote by $\mathbf{Q}(i,j)$ the matrix obtained from $\widehat{\mathbf{Q}}(0)$ by deleting row i and column j. We have

$$\det \widehat{\mathbf{Q}}(0) = -\sum_{i,j=1}^{N} \pi_i p_{ij} \mu_{ij} \cdot \det \mathbf{Q}(1,1) = -E_\pi S_1 \cdot \det \mathbf{Q}(1,1). \tag{16}$$

This may be seen as follows. We add to the first column of $\widehat{\mathbf{Q}}(0)$ the remaining columns. The first column of the matrix thus obtained will have the first entry equal to $-\sum_{i,j=1}^{N} \pi_i p_{ij} \mu_{ij}$, and the remaining ones will vanish, whence the desired conclusion (16) will follow. We assert that column 1 of $[\widehat{\mathbf{Q}}(0)]^{-1}$ consists of identical entries equal to $-1/E_\pi S_1$. In fact,

$$\left([\widehat{\mathbf{Q}}(0)]^{-1}\right)_{j1} \stackrel{\text{def}}{=} \frac{\text{cofactor of } \widehat{Q}_{1j}(0)}{\det \widehat{\mathbf{Q}}(0)}. \tag{17}$$

Denote by $\mathbf{Q}(1)$ the $(N-1) \times N$ matrix obtained from $\widehat{\mathbf{Q}}(0)$ by deleting row 1. The cofactor of $\widehat{Q}_{1j}(0)$ is equal to $(-1)^{j+1}$ times the determinant of the matrix $\mathbf{Q}(1,j)$. If we replace column 1 of $\mathbf{Q}(1,j)$ by the sum of all columns of $\mathbf{Q}(1,j)$, then the resulting matrix $\mathbf{S}$, say, can also be obtained from the matrix $\mathbf{Q}(1,1)$ by moving column j to first place and then multiplying this column by -1. Hence

$$\det \mathbf{Q}(1,j) = \det \mathbf{S} = (-1) \cdot (-1)^{j-2} \cdot \det \mathbf{Q}(1,1) = (-1)^{j-1} \det \mathbf{Q}(1,1). \tag{18}$$

Relations (16)–(18) imply the desired assertion about column 1 of $[\widehat{\mathbf{Q}}(0)]^{-1}$. It now follows that the matrix $[\widehat{\mathbf{Q}}(0)]^{-1} \cdot l(\widehat{\mathbf{Q}}(0))$ consists of identical rows each equal to

$$\left\| \frac{1}{E_\pi S_1} l((\boldsymbol{\pi} \cdot \mathbf{A}_{(1)})_1) \;\; \frac{1}{E_\pi S_1} l((\boldsymbol{\pi} \cdot \mathbf{A}_{(1)})_2) \quad \cdots \quad \frac{1}{E_\pi S_1} l((\boldsymbol{\pi} \cdot \mathbf{A}_{(1)})_N) \right\|,$$

that it

$$[\widehat{\mathbf{Q}}(0)]^{-1} \cdot l(\widehat{\mathbf{Q}}(s)) = \frac{1}{E_\pi S_1} l(\mathbf{1}\boldsymbol{\pi} \cdot \mathbf{A}_{(1)}). \tag{19}$$

Taking relations (10), (14) and (19) into account, we arrive at the conclusion of the theorem in the case $\gamma = 0$:

$$l(\mathbf{W}) = l([\widehat{\mathbf{A}}_+(s)]^{-1})\widehat{\mathbf{A}}_+(0) = \frac{1}{-E_\pi S_1} l(\mathbf{1}\boldsymbol{\pi}\cdot\mathbf{A}_{(1)}),$$

provided the distributions F_{ij}, $i, j \in \mathcal{N}$, are absolutely continuous.

III. General case. We now drop the absolute continuity assumption, but retain for the time being the requirements that the expectations $EX_1(i,j)$, $i, j \in \mathcal{N}$, be finite and that $E_\pi S_1$ be negative. (Later, these restrictions will be removed in the case $\gamma > 0$.) Since a direct attack to evaluate $l(\mathbf{W})$ does not seem to be viable, we shall make a detour to compute first the matrix $l(\mathbf{A}_+^{-1})$, whence, by (9), the desired conclusion (8) will follow. To this end, let us consider a family $\{Y_m(i,j)\}_{m=1}^{\infty}$, $(i,j) \in \mathcal{N} \times \mathcal{N}$, of sequences of independent indentically distributed random variables with uniform distribution U_h on $[0,h]$, where $h > 0$ will be taken sufficiently small. Moreover, we assume that all the sequences $\{X_m(i,j)\}_{m=1}^{\infty}$, $\{Y_m(i,j)\}_{m=1}^{\infty}$, $(i,j) \in \mathcal{N} \times \mathcal{N}$, and $\{\kappa_n\}_{n=0}^{\infty}$ are mutually independent. Along with the random walk $\{S_n\}_{n=0}^{\infty}$, let us consider a similar random walk $\{S_n^*\}_{n=0}^{\infty}$ defined on the same chain: $S_0^* = 0$ and $S_n^* = S_{n-1}^* + X_n(\kappa_{n-1}, \kappa_n) + Y_n(\kappa_{n-1}, \kappa_n)$, $n \geq 1$. Its underlying matrix is given by $\mathbf{A}_h = \|p_{ij} F_{ij} * U_h\|$. Obviously, $\widehat{\mathbf{A}}_h(s) = \frac{e^{hs}-1}{hs}\widehat{\mathbf{A}}(s)$ and $\lambda_h(\xi) = \frac{e^{h\xi}-1}{h\xi}\lambda(\xi)$, where $\lambda_h(\xi)$ stands for the maximal positive eigenvalue of $\widehat{\mathbf{A}}_h(\xi)$, ξ real. Choosing $h > 0$ sufficiently small, we can achieve that both $\lambda_h(\gamma) < 1$ and $E_\pi S_1^* = E_\pi S_1 + h/2 < 0$. Denote $M_\infty^* = \sup_{n \geq 0} S_n^*$ and $\eta^*(x) = \min\{n \geq 1 : S_n^* > x\}$. Clearly,

$$P_i(M_\infty > x) \leq P_i(M_\infty^* > x) = \sum_{j=1}^{N} P_i(M_\infty^* > x, \kappa_{\eta^*(x)} = j).$$

Applying the theorem already proved for absolutely continuous F_{ij}, we infer that the ratios $P_i(M_\infty^* > x)/\tau(x)$ tend to finite limits as $x \to \infty$, whence the boundedness of all $W_{ij}(x,\infty)/\tau(x)$ follows, i.e. $\mathbf{W} \in \mathfrak{S}(\tau)$. Then, by (9), we have

$$[\widehat{\mathbf{A}}_+(s)]^{-1} = \widehat{\mathbf{W}}(s)[\widehat{\mathbf{A}}_+(0)]^{-1} \in \widehat{\mathfrak{S}}(\tau).$$

Our goal is to establish (14), whence the desired assertion for $l(\mathbf{W})$ will follow. Denote, for short, the tails of the entries of $\mathbf{A}_+^{-1}$ by $a_{ij}(x)$, i.e.

$$a_{ij}(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} P_i\left(\{\bar{s}_{n-1} < S_n, \kappa_n = j\} \backslash \{\bar{s}_{n-1} < S_n \leq x, \kappa_n = j\}\right).$$

Fix an arbitrary pair $(k,l) \in \mathcal{N} \times \mathcal{N}$. Put $h_{kl} \stackrel{\text{def}}{=} \limsup_{x\to\infty} a_{kl}(x)/\tau(x)$. By the above, $h_{kl} < \infty$. Choose a sequence $\{x_n\}$ such that $x_n \to \infty$ and

$a_{kl}(x_n)/\tau(x_n) \to h_{kl}$ as $n \to \infty$. Let D be a denumerable dense set in $\mathbf{R}$ such that $0 \in D$. Passing to a subsequence of $\{x_n\}$, which we will again denote by $\{x_n\}$, we can achieve that all the limits

$$h_{ij}(y) \stackrel{\text{def}}{=} \lim_{n\to\infty} a_{ij}(x_n - y)/\tau(x_n), \qquad y \in D, \tag{20}$$

exist (they are automatically finite by the above). The functions $h_{ij}(y)$, $y \in D$, are monotone non-decreasing. For each $(i,j) \in \mathcal{N} \times \mathcal{N}$, define $h_{ij}(y)$ on $\mathbf{R}\backslash D$ by continuity from the right. By the monotonicity of $h_{ij}(y)$ relation (20) will hold for each y belonging to the continuity set of $h_{ij}(y)$. Set $K_{ij}(y) \stackrel{\text{def}}{=} h_{ij}(y)/e^{\gamma y}$. Then $K_{kl}(y) \le K_{kl}(0) = h_{kl}$ for all $y \in \mathbf{R}$. Put $\mathbf{K}(y) \stackrel{\text{def}}{=} \|K_{ij}(y)\|$ and

$$\widehat{\mathbf{B}}_1(s) \stackrel{\text{def}}{=} \operatorname{diag}(e^{-s^2/2}, \dots, e^{-s^2/2}) \cdot \widehat{\mathbf{Q}}(s).$$

It follows from Theorem 2 that

$$\widehat{\mathbf{B}}_1(s)\big[\widehat{\mathbf{A}}_+(s)\big]^{-1} = \operatorname{diag}(e^{-s^2/2}, \dots, e^{-s^2/2}) \cdot \mathbf{B}(s)\widehat{\mathbf{A}}_-(s).$$

The entries of $\mathbf{B}(s)\widehat{\mathbf{A}}_-(s)$ represent the Laplace transforms of measures concentrated on $(-\infty, 0]$. Trivially, $l(\mathbf{B}(s)\widehat{\mathbf{A}}_-(s)) = \mathbf{0}$. The matrix $\operatorname{diag}(e^{-s^2/2}, \dots, e^{-s^2/2})$ is the Laplace transform of $\operatorname{diag}(N(0,1), \dots, N(0,1))$. Clearly, $l(N(0,1)) = 0$. Hence, by (2), $l(\widehat{\mathbf{B}}_1(s)[\widehat{\mathbf{A}}_+(s)]^{-1}) = \mathbf{0}$, i.e.

$$\lim_{x\to\infty} (\mathbf{B}_1 * \mathbf{A}_+^{-1})(x,\infty)/\tau(x) = \mathbf{0}. \tag{21}$$

On the other hand,

$$\begin{aligned}(\mathbf{B}_1 * \mathbf{A}_+^{-1})(x,\infty) &= \int_{-\infty}^{x/2} \mathbf{B}_1(dy)\mathbf{A}_+^{-1}(x-y,\infty) \\ &\quad + \int_0^{x/2} \mathbf{B}_1(x-y,\infty)\mathbf{A}_+^{-1}(dy) \\ &\quad + \mathbf{B}_1(x/2,\infty) \cdot \mathbf{A}_+^{-1}(x/2,\infty) \\ &\stackrel{\text{def}}{=} I_1(x) + I_2(x) + I_3(x).\end{aligned} \tag{22}$$

Lemma 3 says that $\widehat{\mathbf{Q}}(s) \in \widehat{\mathfrak{S}}l(\tau)$. By (2), we also have

$$l(\mathbf{B}_1) = \operatorname{diag}(e^{-\gamma^2/2}, \dots, e^{-\gamma^2/2}) \cdot l(\widehat{\mathbf{Q}}(s)). \tag{23}$$

By (1) and the dominated convergence theorem,

$$\lim_{x\to\infty} \frac{I_2(x)}{\tau(x)} = l(\mathbf{B}_1)\big[\widehat{\mathbf{A}}_+(\gamma)\big]^{-1}. \tag{24}$$

Since the measures $(\mathbf{B}_1)_{ij}$ have no atoms (actually, they are absolutely continuous), the dominated convergence theorem may again be applied to yield

(25) $$\lim_{n\to\infty} \frac{I_1(x_n)}{\tau(x_n)} = \int_{\mathbf{R}} \mathbf{B}_1(dy)e^{\gamma y}\mathbf{K}(y)$$

although the convergence in (20) may be violated at points belonging to a denumerable set. (This is one of the reasons of why we have introduced the matrix $\mathbf{B}_1$ instead of working directly with $\widehat{\mathbf{Q}}(s)$.) We also have [20, (28)] that $\lim_{x\to\infty}[\tau(x/2)]^2/\tau(x) = 0$. Since, obviously, $|I_3(x)| \le Q(\mathbf{B}_1)\cdot Q(\mathbf{A}_+^{-1})[\tau(x/2)]^2$ (here $Q(\mathbf{B}_1) \stackrel{\text{def}}{=} \|Q((\mathbf{B}_1)_{ij})\|$ and, similarly, for $Q(\mathbf{A}_+^{-1})$), we obtain

(26) $$\lim_{x\to\infty} \frac{I_3(x)}{\tau(x)} = \mathbf{0}.$$

Collecting relations (21), (22), and (24)–(26), we arrive at the following matrix equality: $\int_{\mathbf{R}} \mathbf{B}_1(dy)e^{\gamma y}\mathbf{K}(y) + l(\mathbf{B}_1)[\widehat{\mathbf{A}}_+(\gamma)]^{-1} = \mathbf{0}$. It should be clear that if we replace x by $x - z$ in (21), (22), $I_2(x)$ and $I_3(x)$ and x_n by $x_n - z$ in $I_1(x_n)$ and repeat that above reasoning, the last equality will become, by (1),

(27) $$\int_{\mathbf{R}} \mathbf{B}_1(dy)e^{\gamma y}\mathbf{K}(y+z) \equiv -l(\mathbf{B}_1)[\widehat{\mathbf{A}}_+(\gamma)]^{-1}.$$

It is from this identity that we shall extricate the needed information about the exact value of h_{kl}. Assume first that the matrix function $\mathbf{K}(y)$ is equal to a constant matrix almost everywhere (a.e.) with respect ot Lebesgue measure, i.e. $\mathbf{K}(y) = \mathbf{K}^* = \|K^*_{ij}\|$ a.e. Then (27) becomes $\widehat{\mathbf{B}}_1(\gamma)\mathbf{K}^* = -l(\mathbf{B}_1)[\widehat{\mathbf{A}}_+(\gamma)]^{-1}$, or, recalling the definition of $\widehat{\mathbf{B}}_1(s)$ and (23),

$$\widehat{\mathbf{Q}}(\gamma)\mathbf{K}^* = -l(\widehat{\mathbf{Q}}(s))[\widehat{\mathbf{A}}_+(\gamma)]^{-1}.$$

Hence

(28) $$\mathbf{K}^* = -[\widehat{\mathbf{Q}}(\gamma)]^{-1}l(\widehat{\mathbf{Q}}(s))[\widehat{\mathbf{A}}_+(\gamma)]^{-1}.$$

The function $h_{kl}(y)$ is monotone non-decreasing and, by definition, $h_{kl}(y) = e^{\gamma y}K_{kl}(y)$. Hence, letting $y \to 0+$ over the set $\{y \in \mathbf{R} : \mathbf{K}(y) = \mathbf{K}^*\}$, we obtain $h_{kl} = h_{kl}(0) \le \mathbf{K}^*_{kl}$. On the other hand, $h_{kl} \ge \mathbf{K}_{kl}(y)$ *for all* $y \in \mathbf{R}$, and hence $h_{kl} \ge \mathbf{K}^*_{kl}$. Thus, $h_{kl} = \mathbf{K}^*_{kl}$. In other words, we have proved that

$$h_{kl} \stackrel{\text{def}}{=} \limsup_{x\to\infty} \frac{(\mathbf{A}_+^{-1})_{kl}(x,\infty)}{\tau(x)} = \mathbf{K}^*_{kl}.$$

In a similar way, we can demonstrate that

$$h_{kl} \stackrel{\text{def}}{=} \liminf_{x\to\infty} \frac{(\mathbf{A}_+^{-1})_{kl}(x,\infty)}{\tau(x)} = \mathbf{K}^*_{kl}.$$

It follows that the limit $\lim_{x\to\infty}(\mathbf{A}_+^{-1})_{kl}(x,\infty)/\tau(x)$ exists and is equal to the corresponding entry on the right-hand side of (28). Since the pair $(k,l)\in\mathcal{N}\times\mathcal{N}$ was chosen arbitrarily, the proof of (14) and, at the same time, that of (8) will be complete as soon as we prove the following lemma.

Lemma 5. *Let $\mathbf{H}=\|H_{ij}\|$ be an absolutely continuous matrix-valued measure such that $\widehat{\mathbf{H}}(s)$ is invertible for all s with $\Re s=0$. Suppose that the densities $p_{ij}(x)$, $x\in\mathbf{R}$, of the entries H_{ij} belong to the space $\mathcal{S}_1$ of rapidly decreasing functions. Let $\mathbf{M}(x)$, $x\in\mathbf{R}$, be a bounded Lebesgue-measurable matrix function such that $\mathbf{M}*\mathbf{H}(x)\stackrel{\text{def}}{=}\int_{\mathbf{R}}\mathbf{H}(dy)\mathbf{M}(x-y)\equiv\mathbf{0}$. Then $\mathbf{M}(x)\equiv 0$. a.e.*

Proof of Lemma 5. We shall adapt ot the matrix case the proof due to V.M. Kruglov of a similar assertion [22, Lemma 3]. First recall some basic facts from the theory of tempered distributions (see, e.g. [19, Chapter 7]). A function $f\in C^\infty(\mathbf{R})$ is called *rapidly decreasing* if

$$\sup_{0\le k\le n}\sup_{x\in\mathbf{R}}(1+x^2)|f^{(k)}(x)|<\infty,\qquad n=0,1,\dots. \tag{29}$$

The space $\mathcal{S}_1$ of rapidly decreasing functions is a locally convex space defined by the countable family of norms (29). A continuous linear functional u on the space $\mathcal{S}_1$ of rapidly decreasing functions is called a *tempered distribution*, and we write $u\in\mathcal{S}_1'$. The *Fourier transform* $\mathcal{F}(u)\in\mathcal{S}_1'$ of a tempered distribution u is defined as follows: $\mathcal{F}(u)(\psi)\stackrel{\text{def}}{=}u(\mathcal{F}(\psi))$, $\psi\in\mathcal{S}_1$, where

$$\mathcal{F}(\psi)(t)\stackrel{\text{def}}{=}(2\pi)^{-1/2}\int_{\mathbf{R}}e^{-itx}\psi(x)dx,\qquad t\in\mathbf{R}.$$

A bounded Lebesgue-measurable function $g(x)$, $x\in\mathbf{R}$, defines an element of $\mathcal{S}_1'$ by the formula $g(\psi)\stackrel{\text{def}}{=}\int_{\mathbf{R}}g(x)\psi(x)dx$, $\psi\in\mathcal{S}_1$. Similarly, a finite complex-valued measure u with density $p(x)\in\mathcal{S}_1$ defines a continuous linear functional u on $\mathcal{S}_1$ by

$$u(\psi)\stackrel{\text{def}}{=}\int_{\mathbf{R}}\psi(x)u(dx)=\int_{\mathbf{R}}p(x)\psi(x)dx,\qquad \psi\in\mathcal{S}_1.$$

The Fourier transform $\mathcal{F}(u)\in\mathcal{S}_1'$ of such an element $u\in\mathcal{S}_1'$ may be identified with the function $\mathcal{F}(p)(t)$ (which equals $(2\pi)^{-1/2}\widehat{u}(-it)$, $t\in\mathbf{R}$), i.e.

$$\mathcal{F}(u)(\psi)=\int_{\mathbf{R}}\mathcal{F}(p)(t)\psi(t)dt,\qquad \psi\in\mathcal{S}_1.$$

Next, the function $u*g(x)\stackrel{\text{def}}{=}\int_{\mathbf{R}}g(x-y)u(dy)$ is clearly an element of $\mathcal{S}_1'$ and, by Theorem 7.19(c) of [19], $\mathcal{F}(u*g)=\mathcal{F}(p)\cdot\mathcal{F}(g)$.

Let us now turn to the matrix case of the lemma. Choose an arbitrary pair $(k,l) \in \mathcal{N} \times \mathcal{N}$ and consider the (k,l)-entry of $\mathbf{H} * \mathbf{M}(x) \overset{\text{def}}{=} \int_{\mathbf{R}} \mathbf{H}(dy)\mathbf{M}(x-y)$. We have

$$(\mathbf{H} * \mathbf{M})_{kl}(x) = \sum_{j=1}^{N} \int_{\mathbf{R}} H_{kj}(dy) M_{jl}(x-y) \equiv 0.$$

An action of the Fourier transform on both sides of this identity understood as elements of $\mathcal{S}_1'$ yields $\sum_{j=1}^{N} \mathcal{F}(p_{kj}) \cdot \mathcal{F}(M_{jl}) = 0$, or in the matrix form

$$(2\pi)^{-1/2}\widehat{\mathbf{H}}(-it) \cdot \mathcal{F}(\mathbf{M}) = \mathbf{0}. \tag{30}$$

Let $\mathcal{D}(\mathbf{R})$ be the space of functions $f \in C^\infty(\mathbf{R})$ with compact support [19, Chapter 6]. The space $\mathcal{D}(\mathbf{R}) \subset \mathcal{S}_1$ is dense in $\mathcal{S}_1$ [19, Theorem 7.10(a)]. Therefore, each tempered distribution u is uniquely determined by its restriction to $\mathcal{D}(\mathbf{R})$. Multiplying both sides of (30) (this time considered as linear functionals on $\mathcal{D}(\mathbf{R})$), by the matrix $(2\pi)^{1/2}\left[\widehat{\mathbf{H}}(-it)\right]^{-1} \in C^\infty(\mathbf{R})$, we obtain that $\mathcal{F}(\mathbf{M}) = \mathbf{0}$ (as a matrix of linear functionals on $\mathcal{D}(\mathbf{R})$). By the above, $\mathcal{F}(\mathbf{M}) = \mathbf{0}$ in $\mathcal{S}_1'$. Taking the inverse Fourier transform, we come to the conclusion that the linear functionals on $\mathcal{S}_1$ given by the functions $M_{ij}(x)$ vanish. Hence $M_{ij}(x) = 0$ a.e. The lemma is proved.

Par acquit de conscience, one should verify that the ingredients of relation (27) satisfy the hypotheses of Lemma 5. Let $\gamma > 0$. Put

$$\mathbf{M}(x) = \mathbf{K}(-x) + \left[\widehat{\mathbf{B}}_1(\gamma)\right]^{-1} l(\mathbf{B}_1)\left[\widehat{\mathbf{A}}_+(\gamma)\right]^{-1}$$

and $\mathbf{H}(dx) = e^{\gamma x}\mathbf{B}_1(dx) \overset{\text{def}}{=} e^{\gamma x}\|p_{ij}(x)\|dx$, where the densities

$$e^{\gamma x}p_{ij}(x) = (2\pi)^{-1/2}\int_{\mathbf{R}} e^{\gamma x}e^{-(x-y)^2/2}Q_{ij}(dy), \qquad x \in \mathbf{R},$$

obviously belong to $\mathcal{S}_1$; (we recall that $\widehat{\mathbf{Q}}(s) = \|\widehat{Q}_{ij}(s)\| \overset{\text{def}}{=} \mathbf{B}(s)[\mathbf{I} - \widehat{\mathbf{A}}(s)]$). The hypothesis $\lambda(\gamma) < 1$ of Theorem 2 ensures, by the Perron-Frobenius theory, the invertibility of $\widehat{\mathbf{H}}(s)$ for all s with $\Re s = 0$.

Let $\gamma = 0$. Put $\mathbf{M}(x) = \mathbf{K}(-x) + [\widehat{\mathbf{Q}}(0)]^{-1} l(\mathbf{Q})[\widehat{\mathbf{A}}_+(0)]^{-1}$ and $\mathbf{H}(dx) = \mathbf{B}_1(dx) = \|p_{ij}(x)\|dx$, where the densities $p_{ij}(x)$ are the same as in the case $\gamma > 0$. For $s = 0$, $\widehat{\mathbf{H}}(s) = \widehat{\mathbf{Q}}(0)$ is invertible since, by the Perron-Frobenius theory, $\det \mathbf{Q}(1,1) \neq 0$, and $\det \widehat{\mathbf{Q}}(0) = -E_\pi S_1 \cdot \det \mathbf{Q}(1.1)$. If $s \neq 0$ and $\Re s = 0$, then the matrix $\widehat{\mathbf{H}}(s) = \operatorname{diag}(e^{-s^2/2}, \ldots, e^{-s^2/2}) \cdot \widehat{\mathbf{Q}}(s)$ is invertible since each of the three factors on the right-hand side is invertible (the invertibility of $\mathbf{I} - \widehat{\mathbf{A}}(s)$ again follows from the Perron-Frobenius theory). Thus, we have proved that relation (27) satisfies the hypotheses of Lemma 5. The proof of the theorem for the case $\gamma = 0$ is complete.

In order to complete our proof in the case $\gamma > 0$, it remains to remove the restriction that the expectations $EX_1(i,j)$, $i,j \in \mathcal{N}$, are finite. By truncating the random variables $X_m(i,j)$ at a sufficiently remote negative level and considering a new random walk: $S_0^* = 0$ and $S_n^* = S_{n-1}^* + X_n^*(\kappa_{n-1},\kappa_n)$, $n \geq 1$, we can achieve that the expectations $EX_1^*(i,j)$, $i,j \in \mathcal{N}$, are finite, $\lambda^*(\gamma) < 1$ and $E_\pi S_1^* \in (-\infty, 0)$. The latter relation follows from the fact that $[\lambda^*(\xi)]'_{\xi=0} = E_\pi S_1^*$ [15]. Actually, the function $\lambda^*(\xi)$, $\xi \leq \gamma$, is convex [15], whence $[\lambda^*(\xi)]'_{\xi=0} < 0$. Obviously, $P_i(M_\infty > x) \leq P_i(M_\infty^* > x)$. As before, the ratios $P_i(M_\infty^* > x)/\tau(x)$ are bounded. Now repeating the preceding reasoning based on Lemma 2, we come to the desired conclusion (8) for $\gamma > 0$. Theorem 3 is proved.

4. Further results

Summing over $j \in \mathcal{N}$ the probabilities $P_i(M_\infty > x, \kappa_{\eta(x)} = j)$ and taking the obvious equality $\widehat{\mathbf{A}}_+(0)\mathbf{1} = \|P_i(M_\infty = 0)\|$ into account, we obtain the following result about the exact asymptotic behaviour of $P_i(M_\infty > x)$.

Theorem 4. *Under the hypotheses of Theorem 3, we have*

$$\left\| \lim_{x\to\infty} \frac{P_i(M_\infty > x)}{\tau(x)} \right\| = \begin{cases} \gamma\big[\mathbf{I} - \widehat{\mathbf{A}}(\gamma)\big]^{-1} l(\mathbf{A}_{(1)})\big[\widehat{\mathbf{A}}_+(\gamma)\big]^{-1}\widehat{\mathbf{A}}_+(0)\mathbf{1} & \text{if } \gamma > 0, \\ \dfrac{1}{-E_\pi S_1}\, l(\mathbf{1}\boldsymbol{\pi}\cdot\mathbf{A}_{(1)})\mathbf{1} & \text{if } \gamma = 0. \end{cases}$$

As a byproduct of the preceding section, a corresponding result about the asymptotic behaviour of the joint distribution of the first positive sum and the state of the chain $\{\kappa_n\}$ when the random walk hits the positive half-axis is readily obtained.

Theorem 5. *Under the hypotheses of Theorem 3, we have*

$$\left\| \lim_{x\to\infty} \frac{P_i(\chi(0) > x, \kappa_{\eta(0)} = j)}{\tau(x)} \right\| = \begin{cases} -\gamma\big[\widehat{\mathbf{A}}_-(\gamma)\big]^{-1}\, l(\mathbf{A}_{(1)}) & \text{if } \gamma > 0, \\ \dfrac{1}{-E_\pi S_1}\widehat{\mathbf{A}}_+(0)\, l(\mathbf{1}\boldsymbol{\pi}\cdot\mathbf{A}_{(1)}) & \text{if } \gamma = 0. \end{cases}$$

Proof. Denote $\Xi(A) = \left\|\frac{P_i(\chi(0)\in A, \kappa_{\eta(0)}=j)}{\tau(x)}\right\|$, $A \in \mathcal{B}$. By Theorem 2, $\Xi = \mathbf{I} - \mathbf{A}_+$. Hence $l(\Xi) = -l(\mathbf{A}_+)$. Now relations (7), (6) with $\mathbf{D} = \mathbf{A}_+$ and (15) yield the assertion of the theorem.

In just the same manner as in the case of the supremum, Theorem 5 implies the following result on the exact asymptotic tail behaviour of the first positive sum $\chi(0)$.

Theorem 6. *Under the hypotheses of Theorem 3, we have*

$$\left\| \lim_{x\to\infty} \frac{P_i(\chi(0) > x)}{\tau(x)} \right\| = \begin{cases} -\gamma\left[\widehat{\mathbf{A}}_-(\gamma)\right]^{-1} l(\mathbf{A}_{(1)})\mathbf{1} & \text{if } \gamma > 0, \\ \dfrac{1}{-E_\pi S_1}\widehat{\mathbf{A}}_+(0)\, l\big(\mathbf{1}\boldsymbol{\pi}\cdot\mathbf{A}_{(1)}\big)\mathbf{1} & \text{if } \gamma = 0. \end{cases}$$

We complement the assertion of Theorem 3 with a result in the opposite direction.

Theorem 7. *Let $\{S_n\}_{n=0}^{\infty}$ be a random walk defined on a Markov chain $\{\kappa_n\}_{n=0}^{\infty}$. Let G belong to the class $\mathcal{S}(\gamma)$ for some $\gamma \geq 0$. Suppose $\lambda(\gamma) < 1$ for $\gamma > 0$. In the case $\gamma = 0$, assume additionally that the expectations $EX_1(i,j)$, $i, j \in \mathcal{N}$, are finite and $E_\pi S_1 \in (-\infty, 0)$. If $\gamma > 0$ and $\mathbf{W} \in \mathfrak{S}l(\tau)$, then $\mathbf{A}_{(1)} \in \mathfrak{S}l(\tau)$ (which is equivalent to $\mathbf{A} \in \mathfrak{S}l(\tau)$). If $\gamma = 0$ and $\mathbf{W} \in \mathfrak{S}l(\tau)$, then $\boldsymbol{\pi}\cdot\mathbf{A}_{(1)} \in \mathfrak{S}l(\tau)$. In both cases the corresponding parts of relation* (8) *hold.*

Proof. We readily obtain from $\mathbf{W} = \mathbf{A}_+^{-1}\widehat{\mathbf{A}}_+(0)$ that $\mathbf{A}_+ \in \mathfrak{S}l(\tau)$. Regardless of whether the matrix $\mathbf{A}$ has absolutely continuous entries or not, the factorization $\mathbf{I} - \widehat{\mathbf{A}}(s) = \widehat{\mathbf{A}}_-(s)\widehat{\mathbf{A}}_+(s)$ still holds, and we have

$$\widehat{\mathbf{Q}}(s) \stackrel{\text{def}}{=} \mathbf{B}(s)\left[\mathbf{I} - \widehat{\mathbf{A}}(s)\right] = \left[\mathbf{B}(s)\widehat{\mathbf{A}}_-(s)\right]\widehat{\mathbf{A}}_+(s)$$

[18, Theorem 2.1] (at the same time, one cannot assert anything about the invertibility of the factors involved).

Consider first the case $\gamma = 0$. By the above, $\widehat{\mathbf{Q}}(s) \in \widehat{\mathfrak{S}l}(\tau)$. The $(1,j)$-entries of $\widehat{\mathbf{Q}}(s)$ equal

$$\pi_j - (\boldsymbol{\pi}\cdot\widehat{\mathbf{A}}(s))_j - (\boldsymbol{\pi}\cdot\widehat{\mathbf{A}}_{(1)}(s))_j \in \widehat{\mathfrak{S}l}(\tau).$$

We now show that $\boldsymbol{\pi}\cdot\mathbf{A} \in \mathfrak{S}l(\tau)$, whence the desired assertion $\boldsymbol{\pi}\cdot\mathbf{A}_{(1)} \in \mathfrak{S}l(\tau)$ will follow.

Suppose a finite positive measure ν, say, is such that $\Upsilon \stackrel{\text{def}}{=} \nu + \nu_{(1)} \in \mathfrak{S}l(\tau)$. We have

$$\begin{aligned} \nu(t,\infty) &\leq \int_{t-1}^{t} \nu(x,\infty)dx \\ &= \Upsilon(t-1,\infty) - \Upsilon(t,\infty) - \nu(t-1,\infty) + \nu(t,\infty). \end{aligned}$$

Hence $\nu(t-1,\infty) \leq \Upsilon(t-1,\infty) - \Upsilon(t,\infty)$. Dividing both sides of this inequality by $\tau(t-1)$ and recalling (1) for $\gamma = 0$, we obtain

$$\begin{aligned} \limsup_{t\to\infty} \frac{\nu(t,\infty)}{\tau(t)} &\leq \lim_{t\to\infty} \left[\frac{\Upsilon(t,\infty)}{\tau(t)} - \frac{\Upsilon(t+1,\infty)}{\tau(t+1)} \cdot \frac{\tau(t+1)}{\tau(t)}\right] \\ &= l(\Upsilon) - l(\Upsilon)\cdot 1 = 0. \end{aligned}$$

Therefore, $\nu \in \mathfrak{S}l(\tau)$ and $l(\nu) = 0$, whence $\nu_{(1)} \in \mathfrak{S}l(\tau)$. Putting successively $\nu = (\boldsymbol{\pi}\cdot\mathbf{A})_j$, $j \in \mathcal{N}$, we come to the conclusion $\boldsymbol{\pi}\cdot\mathbf{A}_{(1)} \in \mathfrak{S}l(\tau)$.

Let $\gamma > 0$. The factorization $\mathbf{I} - \widehat{\mathbf{A}}(s) = \widehat{\mathbf{A}}_-(s)\widehat{\mathbf{A}}_+(s)$ and relation $\mathbf{A}_+ \in \mathfrak{S}l(\tau)$ imply $\mathbf{A} \in \mathfrak{S}l(\tau)$. However, in case of $\gamma > 0$, $\mathbf{A}_{(1)} \in \mathfrak{S}l(\tau) \Leftrightarrow \mathbf{A} \in \mathfrak{S}l(\tau)$, which follows from Lemma 1. The proof of Theorem 7 is complete.

5. Comparison of results

As was noted in Section 1, the problem of finding the exact tail behaviour for the distribution of the supremum M_∞ has been considered by Arndt [1, Theorems 1 and 2]. Let $\gamma = \sup\{\xi : p_{ij}E\exp\{\xi X_1(i,j)\} < \infty \text{ for all } i,j \in \mathcal{N}\}$. Theorem 1 of [1] deals with the case $\lambda(\gamma) \geq 1$ for $\gamma > 0$, while Theorem 2 of [1] is concerned with the cases $\gamma = 0$ and $\lambda(\gamma) < 1$ for $\gamma > 0$, as in the present paper. In Theorem 2 of [1], the role of the comparison function $\tau(x)$ is played by $e^{-\gamma x}h(x)$, where $h(x)$ is a *superpower function*, that is (i) $\lim_{x\to\infty} h(x+t)/h(x) = 1$ and (ii) for each $s \in (0,1]$, $0 < h(sx)/h(x) \leq c(s) < \infty$, $x > 0$, where $c(s)$ is bounded on every interval $[s_1, 1]$ with $s_1 > 0$. We will show that the functions of the form $e^{-\gamma x}h(x)$ are a particular case of the comparison functions provided by the tails of distribution in $\mathcal{S}(\gamma)$. More precisely, we will demonstrate that a distribution G with $\tau(x) = 1 - G(x) = e^{-\gamma x}h(x)$, where $h(x)$ is superpower, is an element of $\mathcal{S}(\gamma)$. First, notice that in [1, Theorem 2] it is tacitly assumed that in case of $\gamma > 0$ the superpower function is such that $\int_0^\infty h(x)dx < \infty$. In fact, the hypotheses of Theorem 2 of [1] include (a) $\widehat{\mathbf{A}}(\gamma) < \infty$ and (b) $(\mathbf{A}_{(1)})_{ij}(x,\infty) \sim ce^{-\gamma x}h(x)$, as $x \to \infty$ for some $c > 0$ and $i,j \in \mathcal{N}$ (we are using for the most part the notation of the present paper). Since $\widehat{\mathbf{A}}(\gamma) < \infty \Leftrightarrow \widehat{\mathbf{A}}_{(1)}(\gamma) < \infty$ for $\gamma > 0$, it follows that $\int_0^\infty h(x)dx < \infty$. This point settled, we proceed as follows. Consider the equalities

$$
\begin{aligned}
\frac{G^{*2}(x,\infty)}{\tau(x)} &= 2\int_0^{x/2} \frac{\tau(x-y)}{\tau(x)} G(dy) + \frac{[\tau(x/2)]^2}{\tau(x)} \\
&= 2\int_0^{x/2} \frac{h(x-y)}{h(x)} e^{\gamma y} G(dy) + \frac{[h(x/2)]^2}{h(x)} \\
&\stackrel{\text{def}}{=} 2I_1(x) + I_2(x).
\end{aligned} \tag{31}
$$

We have $\lim_{x\to\infty} I_2(x) = 0$ since $I_2(x) \leq c(1/2)h(x/2)$ and $\lim_{x\to\infty} h(x) = 0$. This follows from the fact that $\int_0^\infty (\mathbf{A}_{(1)})_{ij}(x,\infty)e^{\gamma x}dx < \infty$, the integrand being asymptotically $c \cdot h(x)$, which together with property (i) of superpower functions yields $\lim_{x\to\infty} h(x) = 0$. Furthermore, in case of $\gamma > 0$, $\int_0^\infty e^{\gamma x}G(dx) < \infty$ since $\int_0^\infty e^{\gamma x}[1 - G(x)]dx = \int_0^\infty h(x)dx < \infty$. Hence, by the dominated convergence theorem, $\lim_{x\to\infty} I_1(x) = \widehat{G}(\gamma)$. By (31), $\lim_{x\to\infty} G^{*2}(x,\infty)/\tau(x) = 2\widehat{G}(\gamma)$, that is $G \in \mathcal{S}(\gamma)$.

Although the class of superpower functions $h(x)$ is rather large (in particular, it contains the class of regularly varying functions at infinity), it does not, however, coincide with the class made up by the tails of subexponential distributions; e.g. the function $\tau(x) = \exp(-x^{\alpha})$ with $0 < \alpha < 1$ is the tail of a subexponential distribution $G \in \mathcal{S}(0)$, but obviously property (ii) of superpower functions is violated, so that $\tau(x)$ is not superpower. Next, it is assumed in [1, Theorem 2] that the matrix $\omega = \|p_{ij}F_{ij}(0,\infty)\|$ is irreducible, which is presumably due to the form in which the assertions of [1, Theorem 2] are presented (see relations (6) and (7) therein). Finally, in the case $\gamma = 0$ Theorem 2 of [1] stipulates that the distributions F_{ij} have no singular components and at least one of the F_{ij} with $p_{ij} > 0$ does possess an absolutely continuous component. Summing up, Theorem 3 of the present paper generalizes Theorem 2 of [1] essentially in two ways: first, it covers a broader range of asymptotic behaviour of the tails involved and, secondly, the requirement that in the case $\gamma = 0$ at least one of the F_{ij} possess an absolutely continuous component has been removed.

Let $\{S_n\}$ be an ordinary random walk generated by the partial sums of a sequence $\{X_n\}$ of independent identically distributed random variables with common distribution F. This corresponds to the case when the state space $\mathcal{N}$ of the chain $\{\kappa_n\}$ reduces to the one-element set $\{1\}$. In this setting, a result due to Veraverbeke [26, Theorem 2(B)] says that if $\mu \stackrel{\text{def}}{=} EX_1 \in (-\infty, 0)$, then $F_{(1)} \in \mathcal{S}(0)$ if and only if $W \in \mathcal{S}(0)$ and both relations imply

$$P(M_\infty > x) \sim \frac{1}{-\mu} F_{(1)}(x,\infty) \qquad \text{as } x \to \infty,$$

which coincides with the assertions of Theorems 3 and 7 of the present paper when $\gamma = 0$ and $\mathcal{N} = \{1\}$. In case of $\gamma > 0$, Theorem 1 of Bertoin and Doney [2] coincides with the corresponding statement of Theorem 3 of the present paper when $\mathcal{N} = \{1\}$. As far as the result due to Bertoin and Doney [2, Theorem 1] is concerned, an alternative proof relying on Banach algebra techniques was given in [24].

Acknowledgement.

A draft version of this paper was written while the second author was visiting the Institute of Mathematical Statistics at the Wilhelm-University of Münster. He wishes to express gratitude to the Institute for its hospitality and to the Deutsche Forschungsgemeinschaft for its financial support, which made this visit possible.

References

[1] K. Arndt, Asymptotic properties of the distribution of the supremum of a random walk on a Markov chain, *Theory Probab. Appl.* **25** (1980), 309–324.

[2] J. Bertoin and R.A. Doney, Some asymptotic results for transient random walks, *J. Appl. Probab.* **28** (1996), 206–226.

[3] K.A. Borovkov, Stability theorems and estimates of the rate of convergence of the components of factorizations for walks defined on Markov chains, *Theory Probab. Appl.* **25** (1980), 325–334.

[4] A.A. Borovkov, *Stochastic Processes in Queuing Theory*, Springer, New York, 1976.

[5] V.P. Chistyakov, A theorem on sums of independent positive random variables and its applications to branching processes, *Theory Probab. Appl.* **9** (1964), 640–648.

[6] J. Chover, P. Ney and S. Wainger, Degeneracy properties of subcritical branching processes, *Ann. Probab.* **1** (1973), 663–673.

[7] J. Chover, P. Ney and S. Wainger, Functions of probability measures, *J. Analyse Math.* **26** (1973), 255–302.

[8] D.B.H. Cline, Convolution tails, product tails and domains of attraction, *Probab. Theory Related Fields* **72** (1986), 529–557.

[9] D.B.H. Cline, Convolutions of distributions with exponential and subexponential tails, *J. Austral. Math. Soc.* (Series A) **43** (1987). 347–365.

[10] P. Embrechts and C.M. Goldie, On closure and factorization properties of subexponential and related distributions, *J. Austral. Math. Soc.* (Series A) **29** (1980), 243–256.

[11] P. Embrechts and C.M. Goldie, On convolution tails, *Stochastic Process. Appl.* **13** (1982), 263–278.

[12] P. Embrechts, C.M. Goldie and N. Veraverbeke, Subexponentiality and infinite divisibility, *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **49** (1979), 335–347.

[13] I.M. Gelfand, D.A. Raikov and E.M. Shilov, *Commutative Normed Rings*, Chelsea, New York, 1964.

[14] C. Klüppelberg, Subexponential distributions and characterizations of related classes, *Probab. Theory Related Fields* **82** (1989), 259–269.

[15] H.D. Miller, A convexity property in the theory of random variables defined on a Markov chain, *Ann. Math. Statist.* **32** (1961), 1260–1270.

[16] H.D. Miller, A matrix factorization problem in the theory of random variables defined on a finite Markov chain, *Proc. Cambr. Phil. Soc.* **58** (1962), 268–285.

[17] E.J.G. Pitman, Subexponential distribution functions, *J. Austral. Math. Soc.* (Series A) **29** (1980), 337–347.

[18] E.L. Presman, Factorization methods and boundary problems for sums of random variables given on a Markov chain, *Math. USSR Izv.* **3** (1969), 815–852.

[19] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[20] M.S. Sgibnev, Banach algebras of measures of class $\mathcal{S}(\gamma)$, *Siberian Math. J.* **29** (1988), 647–655.

[21] M.S. Sgibnev, Asymptotics of infinitely distributions on **R**, *Siberian Math. J.* **32** (1990), 115–119.

[22] M.S. Sgibnev, Asymptotics of infinitely distributions on $\mathbf{R}^n$, *Siberian Advances in Mathematics* **1** (1991), 122–141.

[23] M.S. Sgibnev, Asymptotic behaviour of higher renewal moments, *Theory Probab. Appl.* **36** (1991), 508–518.

[24] M.S. Sgibnev, On the asymptotic behaviour of the harmonic renewal measure, *J. Theo-*

retical Probab. (1996), Accepted for publication.

[25] J.L. Teugels, The class of subexponential distributions, *Ann. Probab.* **3** (1975), 1000–1011.

[26] N. Veraverbeke, Asymptotic behaviour of Wiener-Hopf factors of a random walk, *Stochastic Process. Appl.* **5** (1977), 27–37.

[27] J. Wermer, *Banach Algebras and Several Complex Variables*, Second Edition. Springer, New York, 1976.

Gerold Alsmeyer
Institute of Mathematical Statistics,
Department of Mathematics,
University of Münster,
Einsteinstraße 62,
D-48149 Münster,
GERMANY

Mikhail Sgibnev
Institute of Mathematics,
Siberian Branch of the
Russian Academy of Sciences,
Novosibirsk 90, 630090
RUSSIA