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A CHARACTERIZATION OF REAL HYPERSURFACES OF COMPLEX PROJECTIVE SPACE III

By

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Abstract. In [5] and [6] we showed a characterization of real hypersurfaces of type A_1 and A_2 (see Introduction) among all real hypersurfaces of complex projective space. In the present paper we will consider them under a weaker condition.

1. Introduction

Let CP^m, $m \geq 2$ be an m-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4, and let M be a real hypersurface CP^m. Let ν be a unit local normal vector field on M and $\xi = -J\nu$, where J denotes the complex structure of CP^m. M has an almost contact metric structure (ϕ, ξ, η, g) induced from J. We denote A and R the shape operator and the curvature tensor of M, respectively. Many differential geometeres have studied M (cf. [1], [3], [7] and [8]) by using the structure (ϕ, ξ, η, g).

Typical examples of real hypersurfaces in CP^m are homogeneous ones. Takagi [8] showed that all homogeneous real hypersurfaces in CP^m are realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or rank 2. Namely, he showed the following: Let M be a homogeneous real hypersurface of CP^m. Then M is a tube of radius r over one of the following Kaehler submanifolds:

(A_1) hyperplane CP^m, where $0 < r < \frac{\pi}{2}$,
(A_2) totally geodesic CP^k $(1 \leq k \leq m-2)$,
(B) complex quadric Q_{m-1}, where $0 < r < \frac{\pi}{4}$,
(C) $CP^1 \times CP^{\frac{m-1}{2}}$, where $0 < r < \frac{\pi}{4}$ and $m(\geq 5)$ is odd,
(D) complex Grassmann $CG_{2,5}$, where $0 < r < \frac{\pi}{4}$ and $m = 9$,
(E) Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $m = 15$.

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Due to his classification, we find that the number of distinct constant principal curvatures of a homogeneous real hypersurface is 2, 3, or 5. Here note that the vector ξ of any homogeneous real hypersurface M (which is a tube of radius r) is a principal curvature vector with principal curvature $\alpha = 2\cot 2r$ with multiplicity 1 (See [1]) and that in the case of type A_1 M has two distinct principal curvatures and in the case of type A_2 (resp. B) M has three distinct principal curvatures t, $-\frac{1}{t}$ and $\alpha = t - \frac{1}{t}$ (resp. $\frac{1+t}{1-t}$, $\frac{t-1}{t+1}$ and $\alpha = t - \frac{1}{t}$).

Contrary to homogeneous real hypersurfaces of CP^m, it is known that any ruled real hypersurface of CP^m is not complete and its structure vector field ξ is not principal ([4]).

In [2] Gotoh proved that if $m \geq 3$ and the shape operator A of a real hypersurface M satisfies $(R(Y,Z)A)X = 0$ for all tangent vectors X, Y, Z in $\xi^{\perp}$, then M is locally congruent to a geodesic hypersphere, where $\xi^{\perp}$ denotes the orthogonal complement of ξ in TM. The author in [5] showed that if $m \geq 2$ and

(1) $$R(AX,Y)(Z) - AR(X,Y)Z = 0$$

for any X, Y, Z tangent to TM, then M is congruent to an open part of a homogeneous real hypersurfaces of type A_1 and A_2. Also in [6] he proved that it remains true in the case where M satisfies (1) for any X, Y, Z in $\xi^{\perp}$. We say that M is *ruled* ([4]) if there is a foliation of M by complex hypersurfaces CP^{m-1}

The purpose of the present paper is to prove that if $m \geq 3$, then, it remains true except some case where M satisfies

(2) $$g\big(R(AX,Y)(Z) - AR(X,Y)Z, W\big) = 0$$

for any X, Y, Z and W in $\xi^{\perp}$, i.e.,

Theorem. *Let M be a real hypersurface of CP^m, $m \geq 3$. Then M satisfies (2) for any X, Y, Z and W in $\xi^{\perp}$ if and only if it is congruent to an open part of a homogeneous real hypersurfaces of type A_1 and A_2 or a ruled real hypersurface.*

2. Preliminaries

Let X be a tangent vector field to M. We write $JX = \phi X + \eta(X)\nu$, where ϕX is the tangent component of JX and $\eta(X) = g(X,\xi)$. As $J^2 = -Id$, where Id denotes the identity endomorphism on TCP^m, we get

(3) $$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\phi X) =, \quad \phi\xi = 0$$

for any X tangent to M. It is also easy to see that for any X, Y tangent to M

(4) $$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi$$

$$\nabla_X \xi = \phi A X. \tag{5}$$

Finally from the expression of the curvature tensor of CP^m, we see that the curvature tensor of M is given by

$$\begin{aligned} R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ - 2g(\phi X,Y)\phi Z + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi. \tag{7}$$

Now, we recall without proof the following results in order to prove our theorem:

Theorem 1. (Kimura [3]) *Let M be a real hypersurface of CP^m. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

Theorem 2. Okumura [7] *Let M be a real hypersurface of CP^m. Then the following are equivalent:*
(i) $\phi A = A\phi$.
(ii) M is locally congruent to one of homogeneous real hypersurfaces of type A_1 and A_2.

3. Proof of the theorem

Let $\{E_1,\dots,E_{2m-2}\}$ be an orthonormal basis of $\xi^\perp$ at any point of M. If in (2) we take $X = E_j$, $Z = \phi E_j$, from (6) and applying the formulas (3) we have any $Y, W \in \xi^\perp$

$$\begin{aligned} -g(AE_j,\phi E_j)g(Y,X) + g(\phi Y,\phi E_j)g(\phi(AE_j,W) \\ -g(\phi AE_j,\phi E_j)g(\phi Y,W) - 2g(\phi AE_j,Y)g(\phi^2 E_j,W) \\ -g(A^2E_j,\phi E_j)g(AY,W) \\ +g(E_j,\phi E_j)g(AY,W) - g(\phi Y,\phi E_j)g(A\phi E_j,W) \\ +g(\phi E_j,\phi E_j)g(A\phi Y,W) + 2g(\phi E_j,Y)g(A\phi^2 E_j,W) \\ +g(AE_j,\phi E_j)g(A^2Y,W) = 0 \end{aligned} \tag{8}$$

Taking summation of (8) on j, we obtain

$$g((2m-3)A\phi Y + \phi AY - (\mathrm{trace}A - g(A\xi,\xi)\phi Y, W) = 0.$$

Hence we have

$$(2m-3)A\phi Y + \phi AY - (\mathrm{trace}A - g(A\xi,\xi))\phi Y = (2m-3)g(A\phi Y,\xi)\xi \tag{9}$$

(10) $$-(2m-3)\phi AY - A\phi Y + (\mathrm{trace}A - g(A\xi,\xi))\phi Y = -g(A\phi Y,\xi)\xi.$$

Taking Y-component of (9) to get

(11) $$2(m-2)g(A\phi Y, Y) = 0$$

If in (2) we take $Y = \phi E_j$, $Z = E_j$ and take summation on j; we obtain for $Y \in \xi^{\perp}$

(12) $$\begin{aligned} &2mA\phi Y - 2m\phi AY - A\phi A^2 Y + A^2\phi AY \\ &\quad = \{2mg(A\phi Y,\xi) - g(A\phi A^2 Y,\xi) + g(A^2\phi AY,\xi)\}\xi. \end{aligned}$$

Taking Y-component of (12) to get

(13) $$2mg(A\phi Y, Y) + g(A^2\phi AY, Y) = 0.$$

Combining (11) with (13), we have

(14) $$g(A^2\phi AY, Y) = 0,$$

since $m \geq 3$. Therefore putting $Y = E_j$ in (14) and taking summation on j,

(15) $$g(A^2\phi A\xi, \xi) = 0.$$

Now, define a cross section U of $\xi^{\perp}$ and a smooth function α on M by

$$A\xi = U + \alpha\xi.$$

Then $\phi A\xi = \phi U$ and $A^2\xi = AU + \alpha U + \alpha^2\xi$, so (15) implies

(16) $$g(\phi U, AU) = 0.$$

We also note

(17) $$g(\phi U, A\xi) = 0.$$

Thus from (16) and (17), we get the following by putting $X = U$ and $Z = \phi U$ in (2):

(18) $$\begin{aligned} &g(Y,U)g(\phi AU,W) - g(AU,U)g(\phi Y,W) \\ &2g(\phi AU,Y)g(U,W) - g(A^2U,\phi U)g(AY,W) \\ &-g(Y,U)g(A\phi U,W) + \|U\|^2 g(A\phi Y,W) \\ &-2g(\phi U,Y)g(AU,W) = 0, \end{aligned}$$

where $||U||^2 = g(U,U)$. From (18) we have

$$
\begin{aligned}
(19) \qquad & g(Y,U)\phi AU - g(AU,U)\phi Y + 2g(\phi AU,Y)U \\
& -g(A^2U,\phi U)AY - g(Y,U)A\phi U + ||U||^2 A\phi Y \\
& -2g(\phi U,Y)AU = \{-2g(A^2U,\phi U)g(Y,U) \\
& +||U||^2 g(A\phi Y,\xi) - 2g(\phi U,Y)g(AU,\xi)\}\xi.
\end{aligned}
$$

Putting $Y = U$ in (9) and (10) and summing up those equations, we have

$$(20) \qquad A\phi U - \phi AU = 0,$$

since $m \geq 3$ and (17). Putting $Y = U$ in (19), we obtain

$$
\begin{aligned}
(21) \qquad & -g(AU,U)\phi U + ||U||^2 A\phi U \\
& = g(A^2U,\phi U)AU - g(A^2U,\phi U)||U||^2\xi.
\end{aligned}
$$

From (20) we know that

$$(22) \qquad g(A^2U,\phi U) = g(AU,A\phi U) = g(AU,\phi AU) = 0.$$

Combining (21) with (22) we have

$$(23) \qquad ||U||^2 A\phi U = g(AU,U)\phi U.$$

Assume that $||U||^2 \neq 0$ at a point, say x. By (23) there exists a certain real number λ such that

$$(24) \qquad A\phi U = \lambda\phi U.$$

If Y is perpendicular to all of U, ϕU and ξ, from (19)

$$(25) \qquad A\phi Y = \lambda\phi Y.$$

Let $T_xM = V \oplus \mathrm{span}\{U,\xi\}$ be the orthogonal decomposition. Then the above argument implies

$$(26) \qquad A|_V = \lambda I_V,$$

where I_V stands for the identity transformation of V. Further we decompose V orthogonally as $V = V' \oplus \mathrm{span}\{\phi U\}$. Note that $\dim V' \geq 1$ by the assumption $m \geq 3$. Since V' is invariant by ϕ, (9) reduces to

$$-(\mathrm{trace}A - \alpha)\phi Y + (2m-2)\lambda\phi Y = 0,$$

for each $Y \in V'$. So we have

$$\mathrm{trace}A - (2m-2)\lambda - \alpha = 0.$$

On the other hand, (26) implies

$$\mathrm{trace} A = (2m-3)\lambda + g(AU, U) + \alpha.$$

Thus $g(AU, U) = \lambda$, which implies

$$AU = \lambda U + \|U\|^2 \xi. \tag{27}$$

Putting $Y = W = U$ and $Z = X$ in (2) and substituting (24), (26) and (27), we obtain

$$\lambda g(X, X)\|U\|^4 = 0.$$

Thus we know $\lambda = 0$. Then

$$A\xi = U + \alpha\xi, \quad AU = \|U\|^2 \xi.$$

Now, it is proved by the same argument with the proof in [4] that a real hypersurface M of CP^m whose the second fundamental form A satisfies

$$A\xi = \alpha\xi + U, \quad AU = \|U\|^2\xi, \quad AX = 0 \tag{28}$$

for any vector X orthogonal to ξ and U, is a ruled hypersurface of CP^m, where α is a smooth function on M. Here we show that there is a foliation of M by complex hyperplane CP^{m-1}. We have only to see that the distribution T_0 defined by $T_0(x) = \{X \in T_x(M) : \eta(X) = 0\}$ is integrable and totally geodesic in M. We consider the open set M_0 of M defined by $\|U\|$, say $\nu \neq 0$. Let T_1 be a distribution defined by $T_1(x) = \{X \in T_x(M) : \eta(X) = g(X, U) = g(X, \phi U) = 0\}$. Let $X \in T_1$. Then we have $AX = 0$ and $\nabla_X \xi = 0$ by (5) and (28). Using (7) and (28), we have $(\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X$, and

$$\begin{aligned}(\nabla_X A)\xi - (\nabla_\xi A)X &= \nabla_X(A\xi) - A\nabla_X\xi - \nabla_\xi(AX) + A\nabla_\xi X \\ &= \nabla_X(\nu V + \alpha\xi) + A\nabla_\xi X \\ &= (X\nu)V + \nu\nabla_X V + (X\alpha)\xi + A\nabla_\xi X,\end{aligned}$$

where $V = U/\|U\|$. Hence we have $\phi X + (X\nu)V + \nu\nabla_X V + (X\alpha)\xi + A\nabla_\xi X = 0$. This equation yields $\phi X + \nu\nabla_X V = 0$, since all other terms are linear combinations of ξ and V. Thus we get

$$\nabla_X V = -(1/\nu)\phi X \tag{29}$$

on M_0. This shows that $M = M_0$, because if $\{x_j\}$ is a sequence of points in M_0 such that it converges to some boundary point of M_0 [hence $\nu(x_j) \to 0$ as $j \to \infty$], then the sequence $\{\|\nabla_X V\|(x_j)\}$ diverges. Moreover, we have

$$X\nu = 0, \tag{30}$$

because $g(\nabla_\xi X, AV) = \nu g(\nabla_\xi X, \xi) = -\nu g(X, \nabla_\xi \xi) = -\nu g(X, \phi A\xi) = 0$, using (28). Next, (7) also implies that $(\nabla_X A)V - (\nabla_V A)X = (\nabla_X A)\phi V - (\nabla_{\phi V} A)X = 0$. By using (4), (28)–(30), we have

$$\begin{aligned}(\nabla_X A)V - (\nabla_V A)X &= \nabla_X(AV) - A\nabla_X V - \nabla_V(AX) + A\nabla_V X\\ &= \nabla_X(\nu\xi) + (1/\nu)A\phi X + A\nabla_V X\\ &= \nu\nabla_X\xi + A\nabla_V X,\\ &= A\nabla_V X,\end{aligned}$$

and

$$\begin{aligned}(\nabla_X A)\phi V - (\nabla_{\phi V} A)X &= \nabla_X(A\phi V) - A\nabla_X(\phi V) - \nabla_{\phi V}(AX) + A\nabla_{\phi V} X\\ &= -A\big((\nabla_X\phi)V + \phi\nabla_X V\big) + A\nabla_{\phi V} X\\ &= -A\big(\eta(V)AX - g(AX, V)\xi + (1/\nu)X\big) + A\nabla_{\phi V} X\\ &= A\nabla_{\phi V} X.\end{aligned}$$

Hence $\nabla_V X$ and $\nabla_{\phi V} X$ are orthogonal to ξ and V. On the other hand, the Codazzi equation (7) implies that $(\nabla_\xi A)V - (\nabla_V A)\xi = \phi V$, and we get

$$\begin{aligned}(\nabla_\xi A)V - (\nabla_V A)\xi &= \nabla_\xi(AV) - A\nabla_\xi V - \nabla_V(A\xi) + A\nabla_V\xi\\ &= \nabla_\xi(\nu\xi) - A\nabla_\xi V - \nabla_V(\nu V + \alpha\xi)\\ &= (\xi\nu)\xi + \nu\phi A\xi - A\nabla_\xi V - (V\nu)V - \nu\nabla_V - V(V\alpha)\xi\\ &= (\xi\nu - V\alpha)\xi + \nu^2\phi V - A\nabla_\xi V - (V\nu)V - \nu\nabla_V V,\end{aligned}$$

by using (28), (29) and $\nabla_V\xi = 0$. This implies that $g(\nabla_V V, \phi V) = \nu - 1/\nu$. Since $g(\nabla_V X, V) = 0$ for $X \in T_1$, we have

(31) $$\nabla_V V = (\nu - 1/\nu)\phi V \quad \text{and} \quad \nabla_V(\phi V) = -(\nu - 1/\nu)V.$$

using (4) and (28). Similarly, we obtain $(\nabla_\xi A)\phi V - (\nabla_{\phi V} A\xi) = -V$, and we have

$$\begin{aligned}(\nabla_\xi A)\phi V - (\nabla_{\phi V} A)\xi &= \nabla_\xi(A\phi V) - A\nabla_\xi(\phi V) - \nabla_{\phi V}(A\xi) + A\nabla_{\phi V}\xi\\ &= -A\big((\nabla_\xi\phi)V + \phi\nabla_\xi V\big) - \nabla_{\phi V}(\nu V + \alpha\xi)\\ &= -A\big(\eta(V)A\xi - g(A\xi, V)\xi + \phi\nabla_\xi V\big) - (\phi V\nu)V\\ &\quad - \nu\nabla_{\phi V} V - (\phi V\alpha)\xi\\ &= \nu A\xi - A\phi\nabla_\xi V - (\phi V\nu)V - \nu\nabla_{\phi V} V - (\phi V\alpha)\xi.\end{aligned}$$

These equations yield

(32) $$\nabla_{\phi V} V = 0,$$

because $\nabla_{\phi V} V$ is orthogonal to ξ and V, and

$$\phi V \nu = \nu^2 + 1. \tag{33}$$

Then we have

$$\nabla_{\phi V}(\phi V) = 0, \tag{34}$$

by using (4) and (32). From these equations, we can easily show that if X and Y are contained in T_0, then $\nabla_X Y \in T_0$ and $[X, Y] = \nabla_X Y - \nabla_Y X \in T_0$. Hence T_0 is integrable and totally geodesic in M. Moreover, (28) means that the integral manifold of T_0 is a totally geodesic in CP^m. Since T_0 is J-invariant, its integral manifold is a complex hypersurface CP^{m-1}. Conversely, assume that (28). Then we know M satisfies (2).

Next, assume that $\lambda \neq 0$. It asserts $\phi A \xi = 0$, i.e., ξ is a principal vector. Hence from (9) and (10) it holds $A\phi - \phi A = 0$ on M. Thus by virtue of Theorem 1 and 2, we can obtain Theorem.

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