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ON A TYPE OF CONTACT METRIC 3-MANIFOLDS

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Abstract. We prove that every 3-dimensional contact metric manifold satisfying $\nabla_{\xi}\tau = 0$ ($\tau = L_{\xi}g$) and $R(X,\xi)\cdot C = 0$, is flat or locally isometric to $S^3(1)$.

1. Introduction

Let M be a contact Riemannian manifold and (η, g, φ, ξ) its contact metric structure. Takahashi in [9] proved that a Sasakian manifold satisfying $R(X,Y)\cdot R = 0$, for any X, Y (where $R(X,Y)$ acts on R as a derivation), is of constant curvature 1.

D. Perrone in [8] studied the contact Riemannian manifolds M satisfying $R(X,\xi)\cdot R = 0$ and he proved: (i) If $\dim M > 3$ and $\ell =: R(\cdot,\xi)\xi = -k\varphi^2$ for some function k defined on M, then either $\ell = 0$, or M is Sasakian of constant curvature 1. (ii) If $\dim M = 3$ and $\nabla_{\xi}\tau = 0$, (where $\tau = L_{\xi}g$), then M is flat or of constant curvature 1. Furthermore, he proved: Let M be a $(2n+1)$-dimensional contact Riemannian manifold satisfying $R(X,Y)\cdot S = 0$, and $R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y]$ where S is the Ricci tensor and k is a function on M. Then M is an Einstein-Sasakian manifold or it is locally isometric to a Riemannian product $E^{n+1}\times S^n(4)$.

B.J. Papantoniou in [7] has given a classification of the contact metric manifolds satisfying $R(X,\xi)\cdot R = 0$ or $R(X,\xi)\cdot S = 0$ under the condition that ξ belongs to the (k,μ)-nullity distribution.

Denoting by C the Weyl conformal curvature tensor, Chaki and Tarafdar in [5] proved that every Sasakian manifold of dimension > 3 satisfying $R(X,Y)\cdot C = 0$ is locally isometric to a unit sphere.

Chr. Baikoussis and Th. Koufogiorgos in [1] extended this result to the contact metric manifolds of dimension ≥ 3 with ξ belonging to the k-nullity distribution.

In [6] we introduced the notion of a 3-τ-manifold, which is a 3-dimensional

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contact metric manifold on which $\nabla_\xi \tau = 0$, $(\tau = L_\xi g)$. In [4] it is proved that $Q\varphi = \varphi Q$ implies $\nabla_\xi \tau = 0$. Recently, D.E. Blair in [3] found out an example of a 3-τ-manifold which does not satisfy the condition $Q\varphi = \varphi Q$. Therefore, it is meaningful to undertake the study of the 3-τ-manifolds.

In the present paper we generalize the above mentioned results of [1] and [5]. We consider a 3-dimensional contact metric manifold M satisfying $\nabla_\xi \tau = 0$ and $R(X,\xi)\cdot C = 0$. We prove that M is flat or locally isometric to $S^3(1)$. The standard contact form on S^3 and the standard Liouville contact form on $S^1(T^*E^2)$ are examples of the theorem.

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2. Some remarks on contact manifolds

Let M be a $(2n+1)$-dimensional C^∞ manifold which admits a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions: $\varphi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$. In this case we say that M has an almost contact structure. On a C^∞ manifold with an almost contact structure (φ,ξ,η) there always exists a Riemannian metric g satisfying $g(\varphi Y, \varphi Z) = g(Y,Z) - \eta(Y)\eta(Z)$, for any vector fields Y and Z on M. The structure (φ,ξ,η,g) is called almost contact metric structure.

A $(2n+1)$-dimensional manifold M^{2n+1} is called a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. A manifold with a contact structure η admits an almost contact metric structure (φ,ξ,η,g) satisfying the relation $d\eta(Y,Z) = g(Y,\varphi Z)$ for all vector fields Y and Z on M^{2n+1}. In this case the structure is called contact metric structure and the manifold M^{2n+1} equipped with such a structure is said to be a contact metric manifold.

We denote by R the curvature tensor and by L the Lie differentiation. We define the operators ℓ and h by the relations: $\ell = R(\cdot\,,\xi)\xi$, $h = L_\xi\varphi/2$. The tensors ℓ and h are symmetric of type $(1,1)$ and satisfy: $h\xi = \ell\xi = 0$, $\operatorname{Tr} h = \operatorname{Tr} h\varphi = 0$, $h\varphi = -\varphi h$. Moreover, on every $(2n+1)$-dimensional contact metric manifold the following formulas hold

$$\nabla_Y \xi = -\varphi Y - \varphi h Y, \qquad \varphi\ell\varphi - \ell = 2(\varphi^2 + h^2), \qquad \nabla_\xi \varphi = 0,$$

$$\nabla_\xi h = \varphi - \varphi\ell - \varphi h^2, \qquad \operatorname{Tr}\ell = g(Q\xi,\xi) = 2n - \operatorname{Tr} h^2,$$

where ∇ is the Riemannian connection of g and Q is the Ricci operator.

A K-contact manifold is a contact metric manifold for which ξ is Killing. If the almost complex structure J on $M^{2n+1} \times \mathbf{R}$, defined by $J\left(Y, f\frac{d}{dt}\right) =$

$(\varphi Y - f\xi, \eta(Y)\frac{d}{dt})$ where f is a real valued function, is integrable, then the structure is said to be normal and the manifold is called Sasakian. A Sasakian manifold is K-contact. The inverse holds only for dimension 3. A Sasakian manifold may be characterized by $R(Y,Z)\xi = \eta(Z)Y - \eta(Y)Z$, for all vector fields Y and Z on the manifold. For details we refer to [2].

The k-nullity distribution of a Riemannian manifold (M,g) for a real number k is given by $N(k) : p \to N_p(k) = \{W \in T_p(M) \mid R(Y,Z)W = k[g(Z,W)Y - g(Y,W)Z]\}$, for all vectors $Y, Z \in T_p(M)$ and for every $p \in M$. In what follows writting ξ belongs to the k-nullity distribution, we mean that k is a real fixed constant.

On every 3-dimensional Riemannian manifold the curvature tensor $R(Y,Z)W$ is given by

$$\begin{aligned} R(Y,Z)W &= g(Z,W)QY - g(Y,W)QZ + g(QZ,W)Y \\ &\quad - g(QY,W)Z - \frac{S}{2}[g(Z,W)Y - g(Y,W)Z], \end{aligned} \tag{2.1}$$

and the conformal curvature tensor $C(Y,Z)$ (like the Weyl conformal tensor $C(X,Y)Z$ defined on every Riemannian manifold of dimension $n > 3$) is given by

$$C(Y,Z) = (\nabla_Y Q)Z - (\nabla_Z Q)Y + \frac{1}{4}[(Z \cdot S)Y - (Y \cdot S)Z], \tag{2.2}$$

where Q is the Ricci operator, $S = \operatorname{Tr} Q$ is the scalar curvature and Y, Z and W are arbitrary vector fields. In [6] the authors studied the 3-τ-manifolds. We shall give some of its results, which we shall use in the present paper.

Proposition A. *Let M^3 be a non-Sasakian 3-τ-manifold. If X is a unit eigenvector of h with eigenvalue λ and orthogonal to ξ then*

$$\operatorname{Tr} \ell = 2(1 - \lambda^2) < 2, \tag{2.3}$$

$$\nabla_\xi X = \nabla_\xi(\varphi X) = 0, \quad \nabla_X \xi = -(\lambda + 1)\varphi X, \quad \nabla_{\varphi X}\xi = (1 - \lambda)X, \tag{2.4}$$

$$\nabla_X X = \frac{1}{2\lambda}[\varphi X \cdot \lambda + \eta(QX)]\varphi X, \quad \nabla_{\varphi X}(\varphi X) = \frac{1}{2\lambda}[X \cdot \lambda + \eta(Q\varphi X)]X, \tag{2.5}$$

$$\begin{aligned} \nabla_X(\varphi X) &= -\frac{1}{2\lambda}[\varphi X \cdot \lambda + \eta(QX)]X + (\lambda + 1)\xi, \\ \nabla_{\varphi X} X &= -\frac{1}{2\lambda}[X \cdot \lambda + \eta(Q\varphi X)]\varphi X + (\lambda - 1)\xi, \end{aligned} \tag{2.6}$$

$$\begin{aligned} X \cdot \lambda &= \frac{\lambda - 1}{2\lambda}\eta(Q\varphi X) - \frac{1}{2\lambda}\eta(\nabla_\xi QX), \\ \varphi X \cdot \lambda &= \frac{\lambda + 1}{2\lambda}\eta(QX) - \frac{1}{2\lambda}\eta(\nabla_\xi Q\varphi X) \end{aligned} \tag{2.7}$$

Corollary A. *On a 3-τ-manifold holds $\xi \cdot \operatorname{Tr} \ell = 0$.*

Proposition B. *Let M^3 be a 3-dimensional contact metric manifold. If for every Y of the contact distribution B holds $QY \in B$, then the conditions $Q\varphi = \varphi Q$ and $\nabla_\xi \tau = 0$ are equivalent.*

Furthermore, for every vector field Y on a 3-τ-manifold we have ([6])

$$QY = aY + b\eta(Y)\xi + \eta(Y)Q\xi + \eta(QY)\xi, \tag{2.8}$$

where $a = \frac{1}{2}(S - \operatorname{Tr} \ell)$ and $b = -\frac{1}{2}(S + \operatorname{Tr} \ell)$.

3. 3-τ-Manifolds with $R(Y, \xi) \cdot C = 0$

Let M^3 be a 3-τ-manifold satisfying

$$R(Y, \xi) \cdot C = 0, \tag{3.1}$$

for all vectors Y tangent to M^3. In (3.1) $R(Y, \xi)$ acts on C as a derivation. Hence, we have that (3.1) is equivalent to

$$R(W, \xi)C(Y, Z) - C(R(W, \xi)Y, Z) - C(Y, R(W, \xi)Z) = 0, \tag{3.2}$$

for all tangent vectors W, Y and Z of M^3.

In what follows, we can define $\{X, \varphi X, \xi\}$ as an orthonormal frame, such that X be a unit eigenvector of h with eigenvalue λ, and φX is also a unit eigenvector of h with eigenvalue $-\lambda$, ([2], p. 22). We denote

$$\begin{aligned}
A_1 &= \eta(\nabla_X QX) - \frac{1}{2\lambda}\left[\varphi X \cdot \lambda + \eta(QX)\right]\eta(Q\varphi X) - \frac{1}{4}\xi \cdot S, \\
A_2 &= \eta(\nabla_{\varphi X} Q\varphi X) - \frac{1}{2\lambda}\left[X \cdot \lambda + \eta(Q\varphi X)\right]\eta(QX) - \frac{1}{4}\xi \cdot S, \\
B_1 &= \eta(\nabla_X Q\varphi X) + \frac{1}{2\lambda}\left[\varphi X \cdot \lambda + \eta(QX)\right]\eta(QX) + 2(\lambda + 1)(\lambda^2 - 1), \\
B_2 &= \eta(\nabla_{\varphi X} QX) + \frac{1}{2\lambda}\left[X \cdot \lambda + \eta(Q\varphi X)\right]\eta(Q\varphi X) + 2(\lambda - 1)(\lambda^2 - 1), \\
C_1 &= (\lambda + 3)\eta(Q\varphi X) - 2\lambda(X \cdot \lambda) - \frac{1}{4}X \cdot S, \\
C_2 &= (\lambda - 3)\eta(QX) - 2\lambda(\varphi X \cdot \lambda) - \frac{1}{4}\varphi X \cdot S.
\end{aligned} \tag{3.3}$$

The relation (2.2) using (2.3), (2.4), (2.5), (2.6), (2.8) and (3.3) gives

$$\begin{aligned}
C(X, \xi) &= A_1 X + B_1 \varphi X + C_1 \xi \\
C(\varphi X, \xi) &= B_2 X + A_2 \varphi X + C_2 \xi, \\
C(X, \varphi X) &= C_2 X - C_1 \varphi X + (B_1 - B_2)\xi.
\end{aligned} \tag{3.4}$$

The relation (3.2) because of (2.1) and (2.5), for all vector fields Y and Z on M^3, gives

$$\begin{aligned} R(X,\xi)C(Y,Z) = \left[\frac{\operatorname{Tr}\ell}{2}\eta(C(Y,Z)) + \eta(Q\varphi X)g(\varphi X, C(Y,Z))\right] X \\ - g(X,C(Y,Z))\eta(Q\varphi X)\varphi X - \frac{\operatorname{Tr}\ell}{2} g(X,C(Y,Z))\xi \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} R(\varphi X,\xi)C(Y,Z) &= -g(\varphi X, C(Y,Z))\eta(QX)X \\ &+ \left[\frac{\operatorname{Tr}\ell}{2}\eta(C(Y,Z)) + \eta(QX)g(X,C(Y,Z))\right]\varphi X \\ &+ \frac{\operatorname{Tr}\ell}{2} g(\varphi X, C(Y,Z))\xi. \end{aligned} \tag{3.6}$$

Setting (i) $W = X$, $Y = \varphi X$, $Z = \xi$, (ii) $W = \varphi X$, $Y = X$, $Z = \xi$, (iii) $W = X$, $Y = X$, $Z = \xi$, (iv) $W = \varphi X$, $Y = \varphi X$, $Z = \xi$ in (3.2) and making use of (3.3), (3.4), (3.5), (3.6) we get respectively

$$\begin{aligned} &\operatorname{Tr}\ell\, C_2 + \eta(Q\varphi X)(A_2 - A_1) = 0, \\ &\frac{\operatorname{Tr}\ell}{2} C_1 + \eta(Q\varphi X)(B_1 + B_2) = 0, \\ &\eta(Q\varphi X)C_1 + \frac{\operatorname{Tr}\ell}{2}(2B_2 - B_1) = 0. \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\frac{\operatorname{Tr}\ell}{2} C_2 + \eta(QX)(B_1 + B_2) = 0, \\ &\operatorname{Tr}\ell\, C_1 + \eta(QX)(A_1 - A_2) = 0, \\ &\eta(QX)C_2 + \frac{\operatorname{Tr}\ell}{2}(2B_1 - B_2) = 0. \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\eta(Q\varphi X)(A_1 - A_2) = 0, \\ &\frac{\operatorname{Tr}\ell}{2} A_1 - \eta(Q\varphi X)C_2 = 0. \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\eta(QX)(A_1 - A_2) = 0, \\ &\frac{\operatorname{Tr}\ell}{2} A_2 - \eta(QX)C_1 = 0. \end{aligned} \tag{3.10}$$

The main result of the present paper is

Theorem 3.1. *A 3-τ-manifold satisfying $R(X,\xi)\cdot C=0$, is flat or locally isometric to $S^3(1)$.*

To prove this theorem we shall use the theorem 4.1 of [1], i.e. we shall show that if M^3 is a 3-τ-manifold with $R(X,\xi)\cdot C=0$, then ξ belongs to the k-nullity distribution of M^3. For this reason we shall prove two lemmas and a proposition.

Lemma 3.2. *If on a 3-τ-manifold we have $A_1=A_2$, then $A_1=A_2=0$.*

Proof. Using the well known relation: $(\nabla_X Q)X+(\nabla_{\varphi X}Q)\varphi X+(\nabla_\xi Q)\xi=\frac{1}{2}\operatorname{grad} S$, we have:

$$\eta((\nabla_X Q)X)+\eta((\nabla_{\varphi X}Q)\varphi X)+\eta((\nabla_\xi Q)\xi)=\frac{1}{2}\xi\cdot S.$$

From $\nabla_\xi\xi=0$, (2.3), the Corollary A and the first of (2.4), we have:

$$\eta((\nabla_\xi Q)\xi)=0.$$

Using (2.5), we obtain:

$$\eta((\nabla_X Q)X)=\eta(\nabla_X QX)-\frac{1}{2\lambda}\left[\varphi X\cdot\lambda+\eta(QX)\right]\eta(Q\varphi X),$$
$$\eta((\nabla_{\varphi X}Q)\varphi X)=\eta(\nabla_{\varphi X}Q\varphi X)-\frac{1}{2\lambda}\left[X\cdot\lambda+\eta(Q\varphi X)\right]\eta(QX).$$

The above four equations give: $A_1+A_2=0$. After this remark we can easily obtain the seeking result.

Lemma 3.3. *Let M^3 be a 3-τ-manifold with $R(Y,\xi)\cdot C=0$, $\ell=0$, and $\eta(QX)=0$. Then, ξ belongs to the k-nullity distribution.*

Proof. The assumption $\ell=0$ and (2.3) give $\lambda^2=1$. Because of $R(Y,\xi)\cdot C=0$ we obtain the relations (3.7), therefore the second of (3.7) gives:

$$\eta(Q\varphi X)\left\{\frac{1}{2\lambda}\left[\eta(Q\varphi X)\right]^2+\eta(\nabla_X Q\varphi X)\right\}=0.$$

(i) We assume that: $\eta(Q\varphi X)=0$. Substituting Y by X and φX in (2.8), we obtain respectively:

$$\varphi QX=a\varphi X+\eta(QX)\xi\quad\text{and}\quad Q\varphi X=a\varphi X+\eta(Q\varphi X)\xi.$$

The above last equations and [4] give the seeking result.

(ii) We assume that:

$$\eta(\nabla_X Q\varphi X)=-\frac{1}{2\lambda}\left[\eta(Q\varphi X)\right]^2. \tag{3.11}$$

From (2.1) we have that $R(X,\varphi X)\varphi X = \frac{S}{2}X$ and by direct computation using (2.5), (2.6) and the first of (2.4), we obtain

$$\begin{aligned} R(X,\varphi X)\varphi X &= \nabla_X \nabla_{\varphi X}\varphi X - \nabla_{\varphi X}\nabla_X \varphi X - \nabla_{[X,\varphi X]}\varphi X \\ &= \frac{1}{4\lambda^2}\left\{ [\eta(Q\varphi X)]^2 + 2\lambda\eta(\nabla_X Q\varphi X) - 2\lambda(\lambda+1)S \right\} X. \end{aligned}$$

Hence, from the last two equations we have

$$\eta(\nabla_X Q\varphi X) = (2\lambda+1)S - \frac{1}{2\lambda}[\eta(Q\varphi X)]^2. \tag{3.12}$$

Comparing (3.11) and (3.12) we have $S = 0$. Hence, the third of (3.7), because of $\ell = 0$, $\lambda^2 = 1$ and the fifth of (3.3) gives $\eta(Q\varphi X) = 0$. Therefore, as in the case (i) we obtain that ξ belongs to the k-nullity distribution.

Proposition 3.4. *Let M^3 be a 3-τ-manifold satisfying $R(Y,\xi)\cdot C = 0$ and $A_1 = A_2 = C_1 = C_2 = 0$. Then ξ belongs to the k-nullity distribution.*

Proof. Differentiating the equations: $C_1 = 0$ and $C_2 = 0$ with respect to φX and X respectively and using $[X,\varphi X]\cdot S = X\cdot\varphi X\cdot S - \varphi X\cdot X\cdot S$ we obtain

$$\begin{aligned} [X,\varphi X]\cdot S = &-8\lambda[X,\varphi X]\cdot\lambda + 4(X\cdot\lambda)\eta(QX) - 4(\varphi X\cdot\lambda)\eta(Q\varphi X) \\ &+ 4(\lambda-3)\eta(\nabla_X QX) - 4(\lambda+3)\eta(\nabla_{\varphi X}Q\varphi X). \end{aligned}$$

Using the Corollary A, (2.3), (2.6) and $A_1 = A_2 = 0$, the above equation takes the form

$$\begin{aligned} [X,\varphi X]\cdot S = &-\frac{12}{\lambda}\eta(QX)\eta(Q\varphi X) + 6\frac{\lambda-1}{\lambda}(X\cdot\lambda)\eta(QX) \\ &- 6\frac{\lambda+1}{\lambda}(\varphi X\cdot\lambda)\eta(Q\varphi X) - 6\xi\cdot S. \end{aligned}$$

Since $\nabla_X\varphi X - \nabla_{\varphi X}X = [X,\varphi X]$, using (2.6) and $C_1 = C_2 = 0$ we obtain

$$\begin{aligned} [X,\varphi X]\cdot S = &-\frac{12}{\lambda}\eta(QX)\eta(Q\varphi X) + 6\frac{\lambda-1}{\lambda}(X\cdot\lambda)\eta(QX) \\ &- 6\frac{\lambda+1}{\lambda}(\varphi X\cdot\lambda)\eta(Q\varphi X) + 2\xi\cdot S. \end{aligned}$$

Comparing the above equations we obtain $\xi\cdot S = 0$.

Using the Corollary A, $\xi\cdot S = 0$, $C_1 = 0$ and (2.3) we have

$$(\lambda+1)(\varphi X\cdot S) = -8\lambda(\lambda+1)(\varphi X\cdot\lambda) + 4(\lambda+3)\eta(\nabla_\xi Q\varphi X).$$

From the above equation, $C_1 = 0$ and (2.4) we get

$$\lambda(\lambda+3)(\varphi X\cdot\lambda) = 3(\lambda+1)\eta(QX). \tag{3.13}$$

In a similar way we can obtain

$$\lambda(\lambda-3)(X\cdot\lambda)=-3(\lambda-1)\eta(Q\varphi X). \tag{3.14}$$

Assuming that $\lambda=\pm 3$, we have $X\cdot\lambda=\varphi X\cdot\lambda=0$, and therefore using (3.13), (3.14) and [6] we obtain that ξ belongs to the k-nullity distribution. We now assume that $\lambda\neq\pm 3$. The equations (2.4), (3.13) and (3.14) give

$$\eta(\nabla_\xi QX)=\frac{(\lambda-1)(\lambda+3)}{\lambda-3}\eta(Q\varphi X), \tag{3.15}$$

$$\eta(\nabla_\xi Q\varphi X)=\frac{(\lambda+1)(\lambda-3)}{\lambda+3}\eta(QX). \tag{3.16}$$

Using the assumption $R(Y,\xi)\cdot C=0$, (3.7), (3.8), (3.13), (3.14) and the lemma 3.4, we have

$$\begin{aligned}
\eta(\nabla_X QX)&=\frac{\lambda^2+6\lambda+3}{2\lambda^2(\lambda+3)}\eta(QX)\eta(Q\varphi X),\\
\eta(\nabla_{\varphi X} Q\varphi X)&=\frac{\lambda^2-6\lambda+3}{2\lambda^2(\lambda-3)}\eta(QX)\eta(Q\varphi X),\\
\eta(\nabla_{\varphi X} QX)&=-\frac{\lambda^2-6\lambda+3}{2\lambda^2(\lambda-3)}\left[\eta(Q\varphi X)\right]^2+2(1-\lambda^2)(\lambda-1),\\
\eta(\nabla_X Q\varphi X)&=-\frac{\lambda^2+6\lambda+3}{2\lambda^2(\lambda+3)}\left[\eta(QX)\right]^2+2(1-\lambda^2)(\lambda+1).
\end{aligned} \tag{3.17}$$

From the above equations, after some calculations we obtain,

$$[\varphi X,\xi]\cdot\eta(QX)=\frac{3\lambda^5-17\lambda^4+18\lambda^3+18\lambda^2-45\lambda+27}{2\lambda^2(\lambda-3)^2(\lambda+3)}\eta(QX)\eta(Q\varphi X).$$

The above equation, the relations (2.3) and the last of (3.17) give

$$\frac{2\lambda^4-9\lambda^3-3\lambda^2+39\lambda-27}{\lambda(\lambda-3)^2(\lambda+3)}\eta(QX)\eta(Q\varphi X)=0. \tag{3.18}$$

Using the equations (2.3), (3.14), (3.16) and (3.17) we obtain

$$[X,\xi]\cdot\eta(Q\varphi X)=\frac{3\lambda^5+17\lambda^4+18\lambda^3+114\lambda^2-45\lambda-27}{2\lambda^2(\lambda-3)(\lambda+3)^2}\eta(QX)\eta(Q\varphi X).$$

The last equation, the relations (2.3) and the second of (3.3) give

$$\frac{2\lambda^4+9\lambda^3-3\lambda^2+27\lambda-27}{\lambda(\lambda-3)(\lambda+3)^2}\eta(QX)\eta(Q\varphi X)=0. \tag{3.19}$$

From (3.18) and (3.19) we obtain $\eta(QX) = 0$, or $\eta(Q\varphi X) = 0$, or

$$2\lambda^4 - 9\lambda^3 - 3\lambda^2 + 39\lambda - 27 = 0 \quad \text{and} \quad 2\lambda^4 + 9\lambda^3 - 3\lambda^2 + 27\lambda - 27 = 0. \tag{3.20}$$

Let $\eta(QX) = 0$. If $\operatorname{Tr}\ell = 0$ or $\operatorname{Tr}\ell = -3$ we have already proved that ξ belongs to the k-nullity distribution. If $\operatorname{Tr}\ell \neq 0, -3$ then from the equation (3.15) and [4] we obtain the seeking conclusion. Using a similar way we have the same result if

$$\eta(Q\varphi X) = 0.$$

Finally, from (3.20) using (3.13), (3.17) we obtain $\eta(QX) = \eta(Q\varphi X) = 0$, therefore ξ belongs to the k-nullity distribution.

Proof of Theorem 3.1. It is known ([4]) that on every 3-dimensional Sasakian manifold ξ belongs to the k-nullity distribution. Therefore, we suppose that M^3 is a non-Sasakian manifold. The first equations of (3.9) and (3.10) restricts our situation into the following cases:

(i) $\eta(QX) = \eta(Q\varphi X) = 0$,
(ii) $A_1 = A_2$.

In the case (i) making use of the proposition B we conclude that ξ belongs to the k-nullity distribution. In the case (ii) because of the lemma 3.2 we have $A_1 = A_2 = 0$. This equation, the first relation of (3.7) and the second relations of (3.8), (3.9) and (3.10) give us the following possibilities:

(a) $C_1 = C_2 = 0$.
(b) $\eta(QX) = \eta(Q\varphi X) = \operatorname{Tr}\ell = 0$,
(c) $\eta(QX) = \operatorname{Tr}\ell = 0$, and $C_2 = 0$,
(d) $\eta(Q\varphi X) = \operatorname{Tr}\ell = 0$, and $C_1 = 0$.

Using the lemmas 3.2 and 3.3 in the case (a) we have that ξ belongs to the k-nullity distribution. We obtain the same result in (b) making use of the proposition B and [4]. Finally, because of the proposition 3.4 in (c) and (d) we also have that ξ belongs to the k-nullity distribution. The above arguments hold for every point of the manifold, (i.e. pointwisely).

In [1] it is proved that: Let M^{2n+1} be a contact metric manifold with ξ belonging to the k-nullity distribution. If $R(X,\xi) \cdot C = 0$, then either M^{2n+1} is locally isometric to a Riemannian product $E^{n+1} \times S^n(4)$ or M^{2n+1} is locally isometric to $S^{2n+1}(1)$. This completes the proof of the theorem.

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