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ON THE GROWTH OF COMPOSITE ENTIRE FUNCTIONS

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Abstract. In this paper, we study the order and growth of composite entire functions f and g, and disscuss the behaviour as $r\rightarrow\infty$ of the ratios: $\frac{\log\log M(r,f\circ g)}{\log\log M(R,f)}$ and $\frac{\log\log M(r,f\circ g)}{\log\log M(R,g)}$, where $R=R(r)$ is an increasing real function.

1. Introduction

Let f be an entire function. We denote the order and lower order of f by $\rho(f)$ and $\lambda(f)$, respectively. A well known theorem of Polya [1] asserts: If f and g are entire functions, then the composite function $f\circ g$ is of infinite order unless (a) f is of finite order and g is a polynomial or (b) f is of order zero and g is of finite order. Since then, many results related to this and some further results (e.g. Clunie [2], [3], Edrei and Fuchs [4], Mori [5], Yang and Urabe [6], Yang [7]) have been obtained. Especially, Song and Yang [8] proved that if $f\circ g$ is of finite lower order, then either f is of finite lower order and g is a polynomial, of f is of zero lower order and g is of finite lower order. An interesting problem is that if $\rho(f)=0(\lambda(f)=0)$, what kind of conditions will ensure $f\circ g$ to be of either finite or infinite (lower) order.

It is well known that for any two transcendental entire functions f, g

$$\lim_{r\rightarrow\infty}\frac{M(r,f\circ g)}{M(r,f)}=\infty,\ \lim_{r\rightarrow\infty}\frac{M(r,f\circ g)}{M(r,g)}=\infty$$

Clunie disscussed the behaviour as $r\rightarrow\infty$ of the ratio $\frac{\log M(r,f\circ g)}{\log M(r,f)}$ and $\frac{\log M(r,f\circ g)}{\log M(r,g)}$. Song and Yang [8] studied $\frac{\log\log M(r,f\circ g)}{\log\log M(r,f)}$ and $\frac{\log\log M(r,f\circ g)}{\log\log M(r,g)}$, and proved

Theorem A. *Suppose that* $0<\lambda(f)<\rho(f)<\infty$. *Then*

$$\lim_{r\rightarrow\infty}\frac{\log\log M(r,f\circ g)}{\log\log M(r,f)}=\infty$$

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Theorem B. *Suppose that f and g are transcendental entire functions such that $\lambda(f) > 0$ and $\rho(g) < \infty$. Then*

$$\lim_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(r,g)} = \infty$$

Singh and Baloria [9] raised the question whether for sufficiently large $R = R(r)$

$$\limsup_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(R,f)} < \infty, \quad \limsup_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(R,g)} < \infty,$$

and proved [9] that if f and g are two transcendental entire functions of positive lower orders and finite order, then for each positive constant A

$$\limsup_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(r^A,f)} = \infty = \limsup_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(r^A,g)}$$

Lahiri and Sharma [11] proved

Theorem C. *Let f, g be two entire functions of finite orders and $\lambda(f) > 0$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$*

$$\lim_{r\to\infty} \frac{\{\log\log M(r, f\circ g)\}^{1+\alpha}}{\log\log M(\exp(r^p),f)} = 0 \quad \textit{if} \quad p > (1+\alpha)\rho(g)$$

We study this question further and obtain that

$$\limsup_{r\to\infty} \frac{\{\log\log M(r, f\circ g)\}^{1+\alpha}}{\log\log M(R,f)} \quad \text{and} \quad \limsup_{r\to\infty} \frac{\{\log\log M(r, f\circ g)\}^{1+\alpha}}{\log\log M(R,g)}$$

are finite or infinite depending on what kind of sets of $R = R(r)$. Finally we study the comparative growths of the composition of the forms $h\circ k$ and $f\circ g$. Some of results improve and extend earlier results, e.g. Singh and Baloria [9], [10], Lahiri and Sharma [11]. We may assume that the reader is familiar with the standard notation employed in Nevanlinna's value-distribution theory. Now we recall that the order $\rho(F)$ and lower order $\lambda(F)$ of an entire function F are defined as

$$\rho(F) = \limsup_{r\to\infty} \frac{\log\log M(r,F)}{\log r}, \quad \lambda(F) = \limsup_{r\to\infty} \frac{\log\log M(r,F)}{\log r}$$

Furthermore, if $\rho(F) = 0$, we define $\rho^*(F)$ and $\lambda^*(F)$ as follows

$$\rho^*(F) = \limsup_{r\to\infty} \frac{\log\log M(r,F)}{\log\log r}, \quad \lambda^*(F) = \limsup_{r\to\infty} \frac{\log\log M(r,F)}{\log\log r}$$

2. Lemmas

In order to prove our results we need several Lemmas as follow

Lemma 1. (Clunie [3]) *If f and g are entire functions, then for all sufficiently large values of r*

$$M\left(\frac{1}{8}M\left(\frac{r}{2},g\right)-|g(0)|,f\right)\leq M(r,f\circ g)\leq M(M(r,g),f)$$

Lemma 2. (Song and Yang [8]) *Let g be a transcendental entire function of finite lower order. Then for any $\delta>0$, we have*

$$M(r^{1+\delta},f\circ g)\geq M(M(r,g),f)(r\geq r_0)$$

Lemma 3. *Let $0<\lambda(g)\leq\rho(g)<\infty$, Then for every positive number A, we have*

(1) $$\limsup_{r\to\infty}\frac{\log\log M(r,f\circ g)}{\log\log M(r^A,g)}\leq\frac{\rho(f\circ g)}{A\lambda(g)}$$

(2) $$\liminf_{r\to\infty}\frac{\log\log M(r,f\circ g)}{\log\log M(r^A,g)}\leq\frac{\rho(f\circ g)}{A\rho(g)}$$

(3) $$\liminf_{r\to\infty}\frac{\log\log M(r,f\circ g)}{\log\log M(r^A,g)}\leq\frac{\lambda(f\circ g)}{A\lambda(g)}$$

Proof. When $\rho(f\circ g)=\infty$, the inequality (1) is obvious. So we assume $\rho(f\circ g)<\infty$. From the definition of order and lower order, we have for $\epsilon>0$ and sufficiently large r

(4) $$\log\log M(r,f\circ g)\leq(\rho(f\circ g)+\epsilon)\log r$$

(5) $$\log\log M(r,g)\geq(\lambda(g)-\epsilon)\log r$$

combining (4) and (5), and choosing $\epsilon\to 0$ inequality (1) follow from that.
By an analogical argument, we can get (2) and (3)

Similarly, we have following Lemma.

Lemma 4. *Let $0<\lambda(f)\leq\rho(f)<\infty$, Then for every positive number A, we have*

(6) $$\liminf_{r\to\infty}\frac{\log\log M(r,f\circ g)}{\log\log M(r^A,f)}\leq\frac{\lambda(f\circ g)}{A\lambda(f)}$$

Lemma 5. *Let f and g be two entire functions such that $\rho(g) < \infty$. If $\rho(f \circ g) = \infty$, then for every positive number A*

$$\limsup_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(r^A, g)} = \infty \tag{7}$$

If $\lambda(f\circ g) = \infty$, then for every positive number A

$$\lim_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(r^A, g)} = \infty \tag{8}$$

Proof. Assume (7) does not hold. Then there exists a constant $B > 0$ such that for all sufficiently large values of r

$$\log\log M(r, f\circ g) \le B \log\log M(r^A, g) \tag{9}$$

Again from the definition of $\rho(g)$, it follows that

$$\log\log M(r^A, g) \le (\rho(g) + \epsilon) A \log r \tag{10}$$

From (9) and (10), we have

$$\log\log M(r, f\circ g) \le (\rho(g) + \epsilon) AB \log r \tag{11}$$

which implies $\rho(f\circ g) < \infty$. This is a contraction. We prove (7).

By an analogical argument, we can prove (8)

Remark 1. If we take $\rho(f) < \infty$, the lemma remains valid with g replaced by f in the denominator.

Lemma 6. (Bergweiler [12] *If f is meromorphic and g is entire, then for all sufficiently large values of r*

$$T(r, f\circ g) \le \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f)$$

Lemma 7. (Bergweiler [13] *Let f be meromorphic and g entire, and suppose that $0 < \mu < \rho(g) \le \infty$. Then for a sequence of values of r tending to infinity*

$$T(r, f\circ g) \ge T(\exp(r^\mu), f)$$

3. Main Results

Theorem 1. *Let f and g be two transcendental entire functions such that $\rho(f)=0$, $\rho(g)<\infty$. (1) If f and g satisfy either (a) $\rho^*(f)=\infty$, $\lambda(g)>0$ or (b) $\lambda^*(f)=\infty$, $\rho(g)>0$ then $\rho(f\circ g)=\infty$. (2) If $\rho^*(f)<\infty$, then $\rho(f\circ g)<\infty$.*

Proof. The proof of part 1, we disscuss case (a) and (b) respectively.
case (a): By Lemma 2, we have for $A>0$

$$\limsup_{r\to\infty}\frac{\log\log M(r,f\circ g)}{\log\log M(r^A,g)}=\limsup_{r\to\infty}\frac{\log\log M(r^{1+\delta},f\circ g)}{\log\log M(r^{A(1+\delta)},g)}$$

$$\geq\limsup_{r\to\infty}\frac{\log\log M(M(r,g),f)}{\log\log M(r,g)}\frac{\log\log M(r,g)}{\log\log r}\frac{\log\log r^{A(1+\delta)}}{\log\log M(r^{A(1+\delta)},g)}$$

$$\geq\frac{\rho^*(f)\lambda(g)}{\rho(g)}=\infty \tag{12}$$

combining (1) and (12), it follow that $\rho(f\circ g)=\infty$.
case (b): It follows from $\lambda^*(f)=\infty$ and Lemma 2 that

$$\liminf_{r\to\infty}\frac{\log\log M(r,f\circ g)}{\log\log M(r^{\frac{1}{1+\delta}},g)}=\liminf_{r\to\infty}\frac{\log\log M(R^{1+\delta},f\circ g)}{\log\log M(R,g)}$$

$$\geq\liminf_{r\to\infty}\frac{\log\log M(M(R,g),f)}{\log\log M(R,g)}=\lambda^*(f)=\infty \tag{13}$$

which and (2) give $\rho(f\circ g)=\infty$. We complete the proof of part 1.

Proof of Part 2: Since $\rho^*(f)<\infty$, applying Lemma 1, we deduce that

$$\limsup_{r\to\infty}\frac{\log\log M(r,f\circ g)}{\log\log M(r,g)}\leq\limsup_{r\to\infty}\frac{\log\log M(M(r,g),f)}{\log\log M(r,g)}$$

$$=\rho^*(f)<\infty \tag{14}$$

From (14) and Lemma 5, we get $\rho(f\circ g)<\infty$. Theorem is proved.

Remark 2. From (3) and (13), we can obtain that if $\lambda^*(f)=\infty$ and $\lambda(g)>0$, then $\lambda(f\circ g)=\infty$. This conclusion was proved by Yang and Urabe [6].

Theorem 2. *Let f and g be two entire functions of order zero. If either $\rho^*(f)=\infty$ or $\rho^*(g)=\infty$, then $\rho^*(f\circ g)=\infty$. If $\rho^*(f)<\infty$ and $\rho^*(g)<\infty$, then $\rho^*(f\circ g)<\infty$.*

Proof. If $\rho^*(f) = \infty$, from Lemma 2 we have

$$\rho^*(f \circ g) = \limsup_{r\to\infty} \frac{\log\log M(r^{1+\delta}, f \circ g)}{\log\log r^{1+\delta}}$$

$$\geq \limsup_{r\to\infty} \frac{\log\log M(M(r,g),f)}{\log\log M(r,g)} \liminf_{r\to\infty} \frac{\log\log M(r,g)}{\log\log r}$$

$$= \rho^*(f)\lambda^*(g) = \infty \quad (\delta > 0) \tag{15}$$

If $\rho^*(g) = \infty$, we exchange "lim sup" and "lim inf" in inequality (15), we obtain $\rho^*(f \circ g) = \infty$.

Because we have

$$\rho^*(f \circ g) = \limsup_{r\to\infty} \frac{\log\log M(r, f \circ g)}{\log\log r}$$

$$\leq \limsup_{r\to\infty} \frac{\log\log M(M(r,g),f)}{\log\log M(r,g)} \limsup_{r\to\infty} \frac{\log\log M(r,g)}{\log\log r} = \rho^*(f)\rho^*(g) \tag{16}$$

we get if $\rho^*(f) < \infty$ and $\rho^*(g) < \infty$, then $\rho^*(f \circ g) < \infty$. The proof of Theorem 2 is completed.

Theorem 3. *Let f and g be two transcendental entire functions. If $0 < \lambda(f) \leq \rho(f) < \infty$, then for every positive number A*

$$\lim_{r\to\infty} \frac{\log\log M(r, f \circ g)}{\log\log M(r^A, f)} = \infty \tag{17}$$

Further, if $\lambda(f) > 0$, $\rho(g) < \infty$, then for every positive number A

$$\lim_{r\to\infty} \frac{\log\log M(r, f \circ g)}{\log\log M(r^A, g)} = \infty \tag{18}$$

Proof. Suppose that the equality (17) does not hold. Then there exist a constant $B > 0$ and a sequence of r_n tending to infinity which satisfy that

$$\log\log M(r_n, f \circ g) \leq B \log\log M(r_n^A, f) \tag{19}$$

So

$$\log M(r_n, f \circ g) \leq r_n^{AB(\rho(f)+\epsilon)} \tag{20}$$

On the other hand, we have for all sufficiently large values of r

$$M(r, f \circ g) \geq M\left(\frac{1}{16} M\left(\frac{r}{2}, g\right), f\right) \tag{21}$$

Because $g(z)$ is a transcendental entire function, we have

$$\frac{1}{16}M\left(\frac{r}{2},g\right) \geq r^K \tag{22}$$

for $K = AB(\frac{2\rho(f)}{\lambda(f)}+1)$ and for all sufficiently large values of r. From Maximum Modulus Principle, combining (21) and (22), it follows that

$$\log M(r_n, f\circ g) \geq \log M(r_n^K, f) > r_n^{(\lambda(f)-\epsilon)K} \tag{23}$$

Choosing $\epsilon = \frac{\lambda(f)}{2} > 0$ and combining (20) and (23) we get $AB(\rho(f)+\frac{\lambda(f)}{2}) \geq \frac{\lambda(f)}{2}K$, which contradicts with the choice of K. This complete the proof of (17)

By an analogous argument, we can prove (18).

Remark 3. From Lemma 4 and (17), we get the conclusion that if f, g are transcendental entire functions such that $\lambda(f) > 0$, then $\lambda(f\circ g) = \infty$. This conclusion was proved by Song and Yang [8].

Remark 4. If we take $\rho(f) > 0$ instead of $\lambda(f)$, (17) and (18) remain true with 'lim' replaced by 'lim sup'.

Remark 5. If we take $\lambda(g) > 0$ instead of $\lambda(f) > 0$, (17) remain true. But Lemma 4 and Theorem 1 show that (18) may not hold.

Theorem 4. *Let f and g be two entire functions such that $0 < \lambda(f) \leq \rho(f) < \infty$ and $\lambda(g) \leq \rho(g) < \infty$. Then for each $\alpha \in [0,\infty)$ and $0 < p < (1+\alpha)\rho(g)$*

$$\limsup_{r\to\infty} \frac{\{\log\log M(r, f\circ g)\}^{1+\alpha}}{\log\log M(\exp(r^p), f)} = \infty \tag{24}$$

$$\limsup_{r\to\infty} \frac{\{\log\log M(r, f\circ g)\}^{1+\alpha}}{\log\log M(\exp(r^p), g)} = \infty \tag{25}$$

Proof. By Lemma 1, we have for all sufficiently large values of r

$$M(r, f\circ g) \geq M\left(\frac{1}{16}M\left(\frac{r}{2},g\right), f\right)$$

For any $\epsilon > 0$, there exists a sequence of $r_n (r_n \to \infty)$ such that

$$M\left(\frac{r_n}{2},g\right) \geq \exp\left\{\left(\frac{r_n}{2}\right)^{\rho(g)-\epsilon}\right\}$$

So we have for sufficiently large r_n

$$\log\log M(r_n, f\circ g) \geq \log\log M\left(\frac{1}{16}M\left(\frac{r_n}{2}, g\right), f\right)$$

$$\geq (\lambda(f)-\epsilon)\log\left\{\frac{1}{16}M\left(\frac{r_n}{2}, g\right)\right\}$$

(26)
$$\geq (\lambda(f)-\epsilon)\log\frac{1}{16} + (\lambda(f)-\epsilon)\left(\frac{r_n}{2}\right)^{\rho(g)-\epsilon}$$

On the other hand, we get for all sufficiently large values of r

(27)
$$\log\log M(\exp(r^p), f) \leq (\rho(f)+\epsilon)r^p$$

Now we choose $0 < \epsilon < \lambda(f)$ such that $p < (1+\alpha)\{\rho(g)-\epsilon\}$, then we have

(28)
$$\lim_{r\to\infty}\frac{\left[(\lambda(f)-\epsilon)\log\frac{1}{16} + (\lambda(f)-\epsilon)(\frac{r_n}{2})^{\rho(g)-\epsilon}\right]^{1+\alpha}}{(\rho(f)+\epsilon)r_n^p} = \infty$$

By combining (25), (26) and (27), it follows that

$$\limsup_{r\to\infty}\frac{\{\log\log M(r, f\circ g)\}^{1+\alpha}}{\log\log M(\exp(r^p), f)} = \infty$$

Replacing f with g in inequality (27), we can obtain (25).

Remark 6. If the condition $0 < p < (1+\alpha)\rho(g)$ is replaced with the condition $0 < p < (1+\alpha)\lambda(g)$, then 'limsup' in (24) and (25) can be changed to 'lim'.

Theorem 5. *Let f and g be two entire functions of finite order and $\lambda(f) > 0$, $\lambda(g) > 0$. If $\sigma(g) < \infty$ then*

(29)
$$\limsup_{r\to\infty}\frac{\log\log M(r, f\circ g)}{\log\log M(\exp(r^{\rho(g)}), f)} < \infty;$$

(30)
$$\limsup_{r\to\infty}\frac{\log\log M(r, f\circ g)}{\log\log M(\exp(r^{\rho(g)}), g)} < \infty.$$

If $\sigma(g) = \infty$ then

(31)
$$\limsup_{r\to\infty}\frac{\log\log M(r, f\circ g)}{\log\log M(\exp(r^{\rho(g)}), f)} = \infty;$$

(32)
$$\limsup_{r\to\infty}\frac{\log\log M(r, f\circ g)}{\log\log M(\exp(r^{\rho(g)}), g)} = \infty.$$

where $\sigma(g)$ is the type of g.

Proof. If $\sigma(g) < \infty$ then by Lemma 1

$$\limsup_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(\exp(r^{\rho(g)}), f)} \le \limsup_{r\to\infty} \frac{\log\log M(M(r,g), f)}{\log\log M(\exp(r^{\rho(g)}), f)}$$

$$\le \limsup_{r\to\infty} \frac{\log\log M(M(r,g), f)}{\log M(r,g)} \limsup_{r\to\infty} \frac{\log M(r,g)}{r^{\rho(g)}}$$

$$\limsup_{r\to\infty} \frac{\log\{\exp(r^{\rho(g)})\}}{\log\log M(\exp(r^{\rho(g)}), f)}$$

$$= \rho(f)\sigma(f)\frac{1}{\lambda(f)} < \infty$$

Similarly, we can get

$$\limsup_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(\exp(r^{\rho(g)}), g)} \le \rho(f)\sigma(g)\frac{1}{\lambda(g)} < \infty$$

Thus, we prove (29) and (30).

If $\sigma(g) = \infty$ then by Lemma 1 we can get

$$\limsup_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(\exp(r^{\rho(g)}), f)} \le \limsup_{r\to\infty} \frac{\log\log M(\frac{1}{16}M(\frac{r}{2},g), f)}{\log\log M(\exp(r^{\rho(g)}), f)}$$

$$\ge \liminf_{r\to\infty} \frac{\log\log M(\frac{1}{16}M(\frac{r}{2},g), f)}{\log\{\frac{1}{16}M(\frac{r}{2},g)\}} \limsup_{r\to\infty} \frac{\log M(\frac{r}{2},g)}{(\frac{r}{2})^{\rho(g)}}$$

$$\liminf_{r\to\infty} \frac{(\frac{1}{2})^{\rho(g)}\log\{\exp(r^{\rho(g)})\}}{\log\log M(\exp(r^{\rho(g)}), f)}$$

$$= \lambda(f)\sigma(g)\frac{1}{\rho(f)}\left(\frac{1}{2}\right)^{\rho(g)} = \infty$$

Replacing f with g in the denominator, we can get (32) at once.

Singh and Baloria [10] proved that if g, h, k are transcendental entire functions of finite order such that $\lambda(h) > 0$ and $0 < \rho(g) < \rho(k)$, then for every transcendental entire function f of finite order

$$\limsup_{r\to\infty} \frac{T(r, h\circ k)}{T(r, f\circ g)} = \infty$$

Lahiri and Sharma [11] consider the case when h is meromorphic. We improve their results and prove the following theorem.

Theorem 6. *Let k, g be entire functions, and h meromorphic such that $0 < \lambda(h)$, $\rho(g) < \rho(k)$, then for every $\nu(1 \leq \nu < \frac{\rho(k)}{\rho(g)})$ and every meromorphic function f of finite order, we have*

$$\limsup_{r\to\infty} \frac{T(r, h\circ k)}{T(r^\nu, f\circ g)\log M(r^\nu, g)} = \infty \tag{33}$$

If $\lambda(f) > 0$, $\nu > \frac{\rho(k)}{\rho(g)}(0 < \rho(g) < \rho(k) < \infty)$, then

$$\liminf_{r\to\infty} \frac{T(r, h\circ k)}{T(r^\nu, f\circ g)} = 0 \tag{34}$$

Proof. By Lemma 7, we have for a sequence of values of r_n tending to infinity

$$T(r_n, h\circ k) \geq T(\exp(r_n^\nu), h)$$

where $0 < \nu\rho(g) < \mu < \rho(k)$. Noticing $\lambda(h) > 0$, we get

$$T(r_n, h\circ k) \geq \exp\{(\lambda(h) - \delta)r_n^\mu\} \tag{35}$$

where we choose $\delta > 0$ such that $\lambda(h) - \delta > 0$. From Lemma 6, we have that

$$\begin{aligned} T(r^\nu, f\circ g)\log M(r^\nu, g) &\leq \{1 + o(1)\}T(r^\nu, g)T(M(r^\nu, g), f) \\ &< (1 + o(1))r^{\nu(\rho(g)+\epsilon)}\exp\left\{(\rho(f) + \epsilon)r^{\nu(\rho(g)+\epsilon)}\right\} \end{aligned} \tag{36}$$

choose $\epsilon > 0$ so that $\nu(\rho(g) + \epsilon) < \mu$. Then (33) follows (35) and (36).

Similarly, from Lemma 6 we have for all sufficiently large r

$$\begin{aligned} T(r, h\circ k) &\leq (1 + o(1))\frac{T(r, k)}{\log M(r, k)}T(M(r, k), h) \\ &\leq (1 + o(1))\frac{r^{(\rho(k)+\epsilon)}}{r^{\lambda(k)-\epsilon}}\exp\left\{(\rho(h) + \epsilon)r^{(\rho(k)+\epsilon)}\right\} \end{aligned} \tag{37}$$

On the other hand, from Lemma 7 for a sequence of $r_n (r_n \to \infty)$

$$T(r_n^\nu, f\circ g) \geq T(\exp r_n^{\nu(\rho(g)-\epsilon)}, f) \geq \exp\left\{(\lambda(f) - \epsilon)r_n^{\nu(\rho(g)-\epsilon)}\right\} \tag{38}$$

(34) follows from (37) and (38) because we can choose $\epsilon > 0$ such that $\nu > \frac{\rho(k)+\epsilon}{\rho(g)-\epsilon}$. Thus, we have proved this theorem.

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