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ANOTHER CONSTRUCTION OF ALLISON'S GRADED LIE ALGEBRAS OBTAINED FROM STRUCTURABLE ALGEBRAS

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Abstract. An explicit construction of the Kantor's graded Lie algebra for a *structurable algebra* is given. We prove that the graded Lie algebra is isomorphic to the Allison's graded Lie algebra. Also we show that simplicity of a structurable algebra and that of the associated generalized Jordan triple system are equivalent.

Introduction

In his paper [1], Allison introduced a new class of nonassociative algebras, called *structurable algebras*, which contains Jordan algebras. To be more precise, let $(A,^{-})$ be a unital algebra with involution $^{-}$, and let us consider the trilinear operation on A, $B_A(x,y,z)=(x\overline{y})z+(z\overline{y})x-(z\overline{x})y$. Then $(A,^{-})$ is, by definition, a *structurable algebra*, if the pair (A, B_A) is a generalized Jordan triple system (or shortly GJTS). Starting from a structurable algebra $(A,^{-})$, Allison [2] constructed a graded Lie algebra (or shortly GLA) of the second kind (or a 5-graded Lie algebra) $\mathcal{K}(A,^{-})=\sum_{i=-2}^{2}\mathcal{K}_i$, whose Lie bracket operation was defined only by using the inner structure of that algebra. On the other hand, there is the most general method of constructing GLA's, starting from triple system. In fact, Kantor [6] constructed the so-called Kantor functor which assigned a GLA to a GJTS (see also Kaneyuki-Asano [4]). In more detail, let (U_{-1}, B) be a GJTS. Then one can construct an infinite dimensional GLA $\mathcal{L}_0(U_{-1},B)=\sum_{i=-\infty}^{\infty}U_i$ whose negative part $\mathcal{L}_0(U_{-1},B)_-=\sum_{i\leq -1}U_i$ is the free Lie algebra generated by U_{-1} and whose positive part $\mathcal{L}_0(U_{-1},B)_+=\sum_{i\geq 0}U_i$ is completely determined by the operation B (see § 1). The Kantor functor $\mathcal{L}$ is, by definition, the functor which assigns to a GJTS (U_{-1},B) a GLA $\mathcal{L}(U_{-1},B)=\mathcal{L}_0(U_{-1},B)/\mathcal{D}$, where $\mathcal{D}$ is the maximum graded ideal of $\mathcal{L}_0(U_{-1},B)$ contained in $\sum_{i\leq -2}U_i$. (cf. [4]).

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The purpose of this paper is to show that the Allison's GLA's can be constructed by using the above-mentioned Kantor's method. More precisely, we construct the Kantor's GLA $\mathcal{L}(A, B_A)$ for a structurable algebra $(A,^-)$, and that the two GLA's $\mathcal{L}(A, B_A)$ and $\mathcal{K}(A,^-)$ are identical (cf. Theorem 2.5). Also we prove that the GJTS (A, B_A) is simple if and only if a structurable algebra $(A,^-)$ is simple (cf. Proposition 2.6).

Throughout this paper we assume that all vector spaces are finite-dimensional over a field F of characteristic 0.

1. Preliminaries.

We give here a quick review on structurable algebras introduced by Allison [1] and generalized Jordan triple systems introduced by Kantor [6].

1.1. Let A be a unital nonassociative algebra with an involution (involutive anti-automorphism) $^-$. Let us define $V_{x,y}$, $T_x \in \mathrm{End}\,(A)$ by

$$\begin{aligned} V_{x,y}(z) &:= (x\overline{y})z + (z\overline{y})x - (z\overline{x})y, \\ T_x &:= V_{x,e}, \end{aligned} \tag{1}$$

where e denotes the identity element of A.

Definition 1.1. ([1]) A unital nonassociative algebra $(A,^-)$ with an involution is called a *structurable algebra* if the following equality is satisfied:

$$[T_z, V_{x,y}] = V_{T_z(x),y} - V_{x,T_z(y)}, \quad x, y, z \in A. \tag{2}$$

The equality (2) is equivalent to the following

$$[V_{x,y}, V_{u,v}] = V_{V_{x,y}(u),v} - V_{u,V_{y,x}(v)}. \tag{3}$$

Put

$$S := \{x \in A | \overline{x} = -x\}.$$

We define a skew-symmetric bilinear map $\psi : A \times A \to S$ by

$$\psi(x, y) := \{x, y\} := x\overline{y} - y\overline{x}.$$

We denote by L_a (resp. R_a) the left (resp. right) multiplication by $a \in A$. Also we put

$$\begin{aligned} T^\delta &:= T + R_{\overline{T(e)}}, \qquad T \in \mathrm{End}\,(A) \\ T^\epsilon &:= T - L_{T(e)+\overline{T(e)}}. \end{aligned}$$

Then we have the following:

Lemma 1.1. ([1], [2]) *For any $x, y, a, b \in A$ and $s, t \in S$, the following equalities are valie*:

(i) $V_{x,y}{}^{\varepsilon} = -V_{y,x}$,

(ii) $(L_s L_t)^{\varepsilon} = -L_t L_s$,

(iii) $L_s L_{\{x,y\}} = V_{sx,y} - V_{sy,x}$,

(iv) $V_{x,y}{}^{\delta}(\{a,b\}) = \{V_{x,y}(a), b\} + \{a, V_{x,y}(b)\}$,

(v) $V_{x,y}{}^{\delta}(s) = \{sy, x\}$.

For a structurable algebra $(A, ^{-})$, let us put

$$\mathcal{K}_{-2} := S,$$
$$\mathcal{K}_{-1} := A,$$
$$\mathcal{K}_0 := \{V_{x,y} \in \mathrm{End}\,(A) | x, y \in A\}_{\mathrm{span}},$$
$$\mathcal{K}_i (i = 1, 2) \text{ is an isomorphic copy of } \mathcal{K}_{-i}$$

and consider the following vector space direct sum

$$\mathcal{K}(A, ^{-}) := \sum_{i=-2}^{2} \mathcal{K}_i.$$

In the following, $\widetilde{a}$ denotes the element in $\mathcal{K}_i$ $(i = 1, 2)$ corresponding to an element $a \in \mathcal{K}_{-i}$ under the isomorphism.

Theorem 1.2. ([2]) *The vector space $\mathcal{K}(A, ^{-}) = \sum_{i=-2}^{2} \mathcal{K}_i$ becomes a graded Lie algebra of the second kind with respect to the following anti-commutative bracket relations*:
for $T_1, T_2 \in \mathcal{K}_0$, $x, y \in \mathcal{K}_{-1}$, $s, t \in \mathcal{K}_{-2}$,

$$(4)\qquad \begin{aligned} &[T_1, T_2] := T_1 T_2 - T_2 T_1, \\ &[T, x] := T(x), && [T, s] := T^{\delta}(s), \\ &[T, \widetilde{x}] := \widetilde{T^{\varepsilon}(x)}, && [T, \widetilde{s}] := \widetilde{(T^{\varepsilon})^{\delta}(s)} \\ &[x, y] := \psi(x, y), && [\widetilde{x}, \widetilde{y}] := \widetilde{\psi(x, y)}, \\ &[x, \widetilde{y}] := V_{x,y}, && [s, \widetilde{t}] := L_s L_t, \\ &[x, \widetilde{t}] := -\widetilde{tx}, && [s, \widetilde{y}] := sy \end{aligned}$$

and the remaining brackets are zero.

We say that the GLA $\mathcal{K}(A,^-)$ is *the Allison's GLA corresponding to a structurable algebra* $(A,^-)$.

1.2. Let U be a vector space, and let us denote by $\mathcal{U}_-$ the free Lie algebra generated by U. We denote by $a \circ b$ the Lie bracket $[a,b]$ of $a,b \in U$ in $\mathcal{U}_-$. We inductively put

$$a_1 \circ \ldots \circ a_{n-1} \circ a_n := [a_1 \circ \ldots \circ a_{n-1}, a_n].$$

Let us put

$$\mathcal{U}_{-n} := \{a_1 \circ a_2 \circ \cdots \circ a_n | a_i \in U\}_{\text{span}},$$
$$\mathcal{U}_n := \{f \mid f \text{ is a } U\text{-valued } (n+1)\text{-linear map on } U\},$$
$$\mathcal{U}_+ := \sum_{i \geq 0} \mathcal{U}_i,$$
$$\mathcal{U} := \mathcal{U}_- \oplus \mathcal{U}_+, \qquad \mathcal{U}_- = \sum_{i<0} \mathcal{U}_i.$$

It was shown by Kantor [5] that $\mathcal{U} = \sum_{i=-\infty}^{\infty} \mathcal{U}_i$ is a GLA under a suitable Lie bracket operation in $\mathcal{U}$. $\mathcal{U}$ is called the *universal graded Lie algebra* $(= UGLA)$ *generated by* $U = \mathcal{U}_{-1}$.

For later use, here we sum up some bracket relations in $\mathcal{U}$.

Lemma 1.3. ([5]) *For $E_i \in \mathcal{U}_i$ $(i = 0,1)$, $a,b,c \in A$, the following equalities are valid:*

(i) $[E_0, a] = E_0(a)$,

(ii) $[E_0, a \circ b] = E_0(a) \circ b - E_0(b) \circ a$,

(iii) $[E_1, a \circ b] = E_1(a,b) - E_1(b,a)$,

(iv) $[E_1, a \circ b \circ c] = E_1(a,b) \circ c - E_1(b,a) \circ c - E_1(c,a) \circ b + E_1(c,b) \circ a$.

A vector space U with a trilinear map $B : U \times U \times U \to U$ is called a *triple system*. We shall use the notation (xyz) instead of $B(x,y,z)$. A triple system (U,B) is called a *generalized Jordan triple system* (shortly, GJTS) if the following equality is valid:

(5) $\quad (xy(uvw)) = ((xyu)vw) - (u(yxv)w) + (uv(xyw)), \quad x,y,u,v,w \in U.$

Putting

(6) $$L_{x,y}(z) = (xyz),$$

we may rewrite (5) as

$$[L_{x,y}, L_{u,v}] = L_{L_{x,y}(u),v} - L_{u,L_{y,x}(v)}. \tag{7}$$

For any element z of a triple system (U, B), let us define a bilinear map $B_z : U \times U \to U$ by

$$B_z(x, y) := B(x, z, y).$$

Let us consider the UGLA $\mathcal{U}$ generated by $U = \mathcal{U}_{-1}$. Let us put

$$U_{-1} := U, \quad U_1 := \{B_a \in \mathcal{U}_1 \mid a \in U\}.$$

Let $\mathcal{L}_0(U, B)$ be the subalgebra generated by $U_{-1} + U_1$ in $\mathcal{U}$. Then it is a graded subalgebra of $\mathcal{U}$ ([6]):

$$\mathcal{L}_0(U, B) = \sum_{-\infty}^{\infty} U_i,$$

where $U_i = [U_{i-1}, U_1]$ $(i \geq 2)$, $U_{-i} = \mathcal{U}_{-i}$ $(i \geq 1)$ and $U_0 = [U_1, U_{-1}]$.

Lemma 1.4. ([6])

(i) $[B_a, b] = L_{b,a}$,

(ii) $[B_a, L_{b,c}] = B_{(cba)}$.

Let $\mathcal{D}$ be a maximal graded ideal of $\mathcal{L}_0(U, B)$ contained in $\sum_{i \leq -2} U_i$. The quotient GLA

$$\mathcal{L}(U, B) := \mathcal{L}_0(U, B)/\mathcal{D}$$

is called *the Kantor's GLA for a GJTS* (U, B).

We say that (U, B) *satisfies the condition* (A) if $B_z = 0$ implies $z = 0$.

Lemma 1.5. ([4]) *Let* (U, B) *be a GJTS satisfying the condition* (A). *Then there exists the canonical grade-reversing involutive automorphism of* $\mathcal{L}(U, B)$.

2. Coincidence of the two graded Lie algebras

Let $(A, ^-)$ be a structurable algebra, and let us consider a trilinear map:

$$B_A(x, y, z) := V_{x,y}(z) \qquad x, y, z \in A.$$

In the following a bilinear map $(B_A)_z$, $z \in A$, is denoted by B_z.

Lemma 2.1. *Let* $(A, ^-)$ *be a structurable algebra. Then the triple system* (A, B_A) *is a GJTS satisfying the condition* (A).

Proof. Note that $V_{x,y} = L_{x,y}$ (cf. [6]). Then, since (3) implies (7), (A, B_A) is a GJTS. Now assume that $B_z = 0$. Then we have

$$B_z(x,y) = (xzy) = (x\bar{z})y + (y\bar{z})x - (y\bar{x})z = 0, \qquad x, y \in A.$$

Putting $x = y = e$, we have $2\bar{z} - z = 0$. Operating the involution $^-$ to this equality, we have $2z - \bar{z} = 0$. Thus we obtain $z = 0$. □

The above GJTS (A, B_A) is called *the GJTS obtained from a structurable algebra* $(A, ^-)$. We are going to construct the Kantor's GLA $\mathcal{L}(A, B_A)$ corresponding to the GFTS (A, B_A), which is called *the Kantor's GLA obtained from a structurable algebra* $(A, ^-)$. Let us consider the subspace of $\mathcal{U}_{-2}$:

$$V_{-2} := \left\{ \sum_i a_i \circ b_i \mid \sum_i \{a_i, b_i\} = 0 \right\}.$$

Lemma 2.2. *The following relations are valid:*

(i) $[U_{-3}, U_1] \subset V_{-2}$, (ii) $[V_{-2}, U_0] \subset V_{-2}$, (iii) $[V_{-2}, U_1] = \{0\}$.

Proof. From Lemma 1.3 (iv), we have

$$[B_a, b \circ c \circ d] = (bac) \circ d - (cab) \circ d - (dab) \circ c + (dac) \circ b, \quad a, b, c, d \in A.$$

Hence, to prove (i) it is enough to show the following equality:

$$\{(bac), d\} - \{(cab), d\} - \{(dab), c\} + \{(dac), b\} = 0. \tag{8}$$

Using (1), we have easily

$$\begin{aligned} (bac) - (cab) &= (b\bar{a})c + (c\bar{a})b - (c\bar{b})a - (c\bar{a})b - (b\bar{a})c + (b\bar{c})a \\ &= (b\bar{c} - c\bar{b})a = \{b, c\}a. \end{aligned} \tag{9}$$

Using this relation, we have

$$\begin{aligned} &\{(bac), d\} - \{(cab), d\} - \{(dab), c\} + \{(dac), b\} \\ &\quad = \{(bac) - (cab), d\} - \{(dab), c\} + \{(dac), b\} \\ &\quad = \{\{b, c\}a, d\} - \{(dab), c\} - \{b, (dac)\}. \end{aligned} \tag{10}$$

On the other hand, since $\{b, c\} \in S$, it follows from Lemma 1.1 (v), (iv) that

$$\begin{aligned} \{\{b, c\}a, d\} &= V_{d,a}{}^{\delta}(\{b, c\}) = \{V_{d,a}(b), c\} + \{b, V_{d,a}(c)\} \\ &= \{(dab), c\} + \{b, (dac)\}. \end{aligned} \tag{11}$$

Substituting (11) into (10), we see that (8) is valid, which proves (i). Next, from Lemma 1.3 (ii) and (11), we see

$$[V_{x,y}, a \circ b] = V_{x,y}(a) \circ b - V_{x,y}(b) \circ a, \quad a, b, x, y \in A, \tag{12}$$

$$\{V_{x,y}(a), b\} - \{V_{x,y}(b), a\} = \{\{a, b\}y, x\}. \tag{13}$$

Now choose an element $\sum_i a_i \circ b_i \in V_{-2}$ ($a_i, b_i \in U_{-1}$) and an element $V_{x,y} \in U_0$. Then $\sum_i \{a_i, b_i\} = 0$ holds. Therefore we have from (13)

$$\sum_i \left(\{V_{x,y}(a_i), b_i\} - \{V_{x,y}(b_i), a_i\}\right) = \left\{\left(\sum_i \{a_i, b_i\}\right)y, x\right\} = 0. \tag{14}$$

Consequently it follows from (12) and (14) that

$$[V_{x,y}, \sum_i a_i \circ b_i] = \sum_i \left(V_{x,y}(a_i) \circ b_i - V_{x,y}(b_i) \circ a_i\right) \in V_{-2},$$

which implies (ii). Finally we will show (iii). Let us choose an element $\sum_i a_i \circ b_i \in V_{-2}$. Using Lemma 1.3 (iii) and (9), we have

$$\begin{aligned} [B_x, \sum_i a_i \circ b_i] &= \sum_i \left(B_x(a_i, b_i) - B_x(b_i, a_i)\right) \\ &= \sum_i \left((a_i x b_i) - (b_i x a_i)\right) \\ &= \left(\sum_i \{a_i, b_i\}\right)x = 0, \end{aligned} \tag{15}$$

which implies (iii). □

Proposition 2.3. *The Kantor's GLA $\mathcal{L}(A, B_A)$ obtained from a structurable algebra $(A, ^-)$ is of the second kind.*

Proof. Let us consider the subspace $\mathcal{D} := \sum_{i \leq -3} U_i + V_{-2}$ of $\mathcal{L}_0(A, B_A) = \sum_{-\infty}^{\infty} U_i$. Then is follows from Lemma 2.2 that $\mathcal{D}$ is a graded ideal of $\mathcal{L}_0(A, B_A)$. Furthermore $\mathcal{D}$ is a maximal graded ideal of $\mathcal{L}_0(A, B_A)$ contained in $\sum_{i \leq -2} U_i$. In fact, let $\mathcal{E} = \sum_{i \leq -2} W_i$ be a graded ideal of $\mathcal{L}_0(A, B_A)$ such that $\mathcal{D} \subset \mathcal{E} \subset \sum_{i \leq -2} U_i$. Then we have $W_i = U_i$ $(i \leq -3)$ and $W_{-2} \supset V_{-2}$. Now choose any element $x = \sum_i a_i \circ b_i \in W_{-2}$. Then it follows from (15) that

$$\sum \{a_i, b_i\} = \sum \{a_i, b_i\}e = [B_e, \sum a_i \circ b_i] \in [U_1, W_{-2}] \subset U_1 \cap \mathcal{E} = \{0\},$$

which implies $x \in V_{-2}$. Hence we have $\mathcal{E} = \mathcal{D}$, which proves the maximality of $\mathcal{D}$. Thus the Kantor's GLA can be written in the form

$$\begin{aligned} \mathcal{L}(A, B_A) &= \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots\cdots, \\ \mathfrak{g}_{-2} &= U_{-2}/V_{-2}, \quad \mathfrak{g}_k = U_k \ (k \geq -1). \end{aligned} \tag{16}$$

From Lemmas 2.1 and 1.5, we have $\mathfrak{g}_k = \{0\}$, $k \geq 3$. □

We say that a GLA $\mathfrak{g} = \sum_k \mathfrak{g}_k$ is *of type* α_0 if the following conditions are satisfied: $\mathfrak{g}_{-k-1} = [\mathfrak{g}_{-k}, \mathfrak{g}_{-1}]$, $\mathfrak{g}_{k+1} = [\mathfrak{g}_k, \mathfrak{g}_1]$ $(k \geq 1)$. Note that a GLA $\mathfrak{g} = \sum_k \mathfrak{g}_k$ of type α_0 is completely determined by its local part

$$\operatorname{loc}(\mathfrak{g}) := \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1.$$

The GLA's $\mathcal{L}_0(A, B_A)$ and consequently $\mathcal{L}(A, B_A)$ are of type α_0.

Lemma 2.4. *The Allison's GLA $\mathcal{K}(A,^-)$ corresponding to a structurable algebra $(A,^-)$ is of type α_0.*

Proof. The map

$$\phi : s + x + T + \widetilde{y} + \widetilde{t} \to t + y + T^\epsilon + \widetilde{x} + \widetilde{s}$$
$$(s, t \in \mathcal{K}_{-2},\ x, y \in \mathcal{K}_{-1},\ T \in \mathcal{K}_0)$$

is a grade-reversing automorphism of $\mathcal{K}(A,^-) = \sum_{i=-2}^{2} \mathcal{K}_i$ (cf. [2]). Hence, to prove this lemma, it is sufficient to show that $[\mathcal{K}_{-1}, \mathcal{K}_{-1}] = \mathcal{K}_{-2}$. Now we choose any element $s \in \mathcal{K}_{-2}$. Then s is represented as $s/2 = \sum a_i \overline{b_i}$ $(a_i, b_i \in A)$, since we have the equality $AA = A$. Then we have

$$s = s/2 - \overline{s}/2 = \sum_i (a_i\overline{b_i} - b_i\overline{a_i}) = \sum_i \{a_i, b_i\} = \sum_i [a_i, b_i] \in [\mathcal{K}_{-1}, \mathcal{K}_{-1}].$$

It follows that $\mathcal{K}_{-2} \subset [\mathcal{K}_{-1}, \mathcal{K}_{-1}]$. □

Thus we have the main theorem.

Theorem 2.5. *For any structurable algebra $(A,^-)$, the Allison's graded Lie algebra $\mathcal{K}(A,^-)$ coincides with the Kantor's graded Lie algebra $\mathcal{L}(A, B_A)$.*

Proof. Let $\mathcal{K}(A,^-) = \sum_{i=-2}^{2} \mathcal{K}_i$ and $\mathcal{L}(A, B_A) = \sum_{i=-2}^{2} \mathfrak{g}_i$. Then the following equalities are valid:

$$\mathcal{K}_{-1} = \mathfrak{g}_{-1} = A,$$
$$\mathcal{K}_0 = \{V_{a,b} \mid a, b \in A\}, \qquad \mathfrak{g}_0 = \{L_{a,b} \mid a, b \in A\},$$
$$\mathcal{K}_1 = \{\widetilde{a} \mid a \in A\}, \qquad \mathfrak{g}_1 = \{B_a \mid a \in A\}.$$

Now define a linear map f from $\operatorname{loc}(\mathcal{K}(A,^-))$ to $\operatorname{loc}(\mathcal{L}(A, B_A))$ by

$$f(a) := a \qquad a \in \mathcal{K}_{-1},$$
$$f(V_{a,b}) := L_{a,b} \qquad V_{a,b} \in \mathcal{K}_0,$$
$$f(\widetilde{a}) := -B_a \qquad \widetilde{a} \in \mathcal{K}_1.$$

First we want to prove that f is a homomorphism between the local parts of the two GLA's, that is, f satisfies

$$f([X_i, Y_j]) = [f(X_i), f(Y_j)] \quad \text{for} \quad X_i \in \mathcal{K}_i,\ Y_j \in \mathcal{K}_j, \tag{17}$$

where $(i,j) = (0,0), (-1,0), (-1,1)$ and $(0,1)$. In case $(i,j) = (0,0)$, noting that $V_{x,y} = L_{x,y}$, we obtain (17) from (3) and (7). In case $(i,j) = (-1,0)$, let $X_{-1} = a$ and $Y_0 = V_{c,d}$. Since $[a, V_{c,d}] \in \mathcal{K}_{-1}$, we have (cf. [4] and Lemma 1.3 (i))

$$f([a, V_{c,d}]) = [a, V_{c,d}] = -(cda) = [a, L_{c,d}] = [f(a), f(V_{c,d})].$$

In case $(i,j) = (-1,1)$, let $X_{-1} = a$ and $Y_1 = \widetilde{b}$. From (4) and Lemma 1.4 (i) we have

$$f([a, \widetilde{b}]) = f(V_{a,b}) = L_{a,b} = [B_b, a] = [a, -B_b] = [f(a), f(\widetilde{b})].$$

In case $(i,j) = (0,1)$, let $X_0 = V_{a,b}$ and $Y_1 = \widetilde{c}$. From (4), Lemma 1.1 (i) and Lemma 1.4 (iii), we have

$$\begin{aligned} f([V_{a,b}, \widetilde{c}]) &= f(\widetilde{V_{a,b^\epsilon}(c)}) = -f(\widetilde{V_{b,a}(c)}) = -f(\widetilde{(bac)}) = B_{(bac)} \\ &= [B_c, L_{a,b}] = [L_{a,b}, -B_c] = [f(V_{a,b}), f(\widetilde{c})]. \end{aligned}$$

Now suppose $f(\widetilde{a}) = 0$, $\widetilde{a} \in \mathcal{K}_1$. Then, since the GJTS (A, B_A) satisfies the condition (A) (cf. Lemma 2.1), it follows that $a = 0$. Consequently we have proved that f is an isomorphism between the local parts of $\mathcal{K}(A,^-)$ and of $\mathcal{L}(A, B_A)$. Since the two GLA's $\mathcal{K}(A,^-)$ and $\mathcal{L}(A, B_A)$ are of type α_0, the local isomorphism f is naturally extended to an isomorphism between $\mathcal{K}(A,^-)$ and $\mathcal{L}(A, B_A)$. □

At the end we have

Proposition 2.6. *Let (A, B_A) be the GJTS obtained from a structurable algebra $(A,^-)$. Then (A, B_A) is simple if and only if $(A,^-)$ is simple.*

Proof. Since any ideal of algebra $(A,^-)$ is $^-$-invariant by definition, it is also an ideal of the GJTS (A, B_A). Hence, if the GJTS (A, B_A) is simple, then the algebra $(A,^-)$ is simple. Conversely, let us assume that I is an ideal of the GJTS (A, B_A). For any element $a \in I$, we have

$$2a - \overline{a} = V_{a,e}(e) = \{aee\} \in I.$$

It follows that I is $^-$-invariant. Next, for any elements $a \in I$ and $x \in A$, we have

$$a\overline{x} + \overline{x}a - \overline{a}x = \{axe\} \in I, \tag{18}$$

$$a\overline{x} + \overline{x}a - \overline{x}\,\overline{a} = \{ae\overline{x}\} \in I, \tag{19}$$

$$\overline{x}\,\overline{a} + \overline{a}\,\overline{x} - \overline{a}x = \{ex\overline{a}\} \in I. \tag{20}$$

Subtracting (19) from (18), we obtain

$$\overline{x}\,\overline{a} - \overline{a}x \in I. \tag{21}$$

Subtracting (21) from (20), we have

$$\overline{a}\,\overline{x} \in I.$$

Since $a \in I$ and $x \in A$ are arbitrary, this means $IA \subset I$. Operating the involution $^-$, we have $AI \subset I$. Hence I is an ideal of the algebra $(A,^-)$. Therefore if the algebra $(A,^-)$ is simple, then the GJTS (A, B_A) is also simple. □

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