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UNIVERSAL SPACES FOR ZERO-DIMENSIONAL CLOSED IMAGES OF METRIC SPACES

— Dedicate to my mother on her 77th birthday —

By

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Abstract. Using an iterative method due to Stephen Watson, we shall construct universal spaces for 0-dimensional Lašnev spaces. We can show that Watson method can be applied not only for all classes of large cardinalities, but also to make their universal spaces homogeneous for certain classes. We also study relationship between universal spaces made from complete metric spaces and those made from σ-discrete metric spaces.

1. Introduction

a) Iterative constructions and Watson method. The purpose of this paper is to show that a *simple* (but effective) *principle* together with its *iteration* yields *universal spaces* for some classes of 0-dimensional *Lašnev* spaces (i.e. closed continuous images of some metric spaces). Namely, let M be a fixed 0-dimensional metric space (we call this space a *model space*) with a certain universality (e.g. 0-dimensional Baire spaces in [13] or Medvedev's universal spaces for σ-discrete metric spaces in [9]). Then, we can show that there exists a universal space for all 0-dimensional closed images of M.

One of the advantages of our construction is that we can show not only their universality, but also their *homogeneity* (i.e. for every pair x, y of points in X there exists an autohomeomorphism $h: X \to X$ such that $h(x) = y$) in certain cases.

Suppose that a *clopen* (i.e. simultaneously, closed and open) subset G of a model space M is given. Then we adopt the following statement as a simple principle:

Take a nowhere dense closed subset A of G, and *devide* the open set $G\backslash A$ into a collection $\mathcal{U}$ of mutually disjoint clopen subsets of G, which satisfies the following condition ($\mathcal{U}$ is called a *semi-canonical cover* for the pair (G, A) after

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D.M. Hyman [6]):
(SC) for each $a \in A$ and for each neighborhood V of a in G, there exists a neighborhood W of a in G such that $st(W,\mathcal{U}) = \{U \in \mathcal{U} : U \cap W \neq \emptyset\} \subset V$. Then, after iterating this principle infinitely many times (i.e. starting form $G = M$ and applying the principle to each member of $\mathcal{U}$, so on), the universal space is *the decomposition space* defined on the set M, and the set of its non-trivial elements of the decomposition consists of all chosen closed nowhere dense subsets A's. The above construction of the universal space is called *Watson construction*, since it is originated by Stephen Watson (see [17]). We call it *Watson method* that is the whole mechanism of showing universality of the decomposition space together with Watson construction. By Watson method we already proved its universality for all closed images of *rationals* Q [22], and for all 0-dimensional closed images of *irrationals* P [16], using $A \approx M = Q$, and $A \approx M = P$, respectively.

In this paper we extend these results (together with an improvement of [18, Theorem 0]) for the following classes of (not necessarily separable) 0-dimensional Lašnev spaces (see Theorems 1.1 and 1.2 for exact statements):

Definition 1.1. For every infinite cardinals κ and λ with $\omega_1 \leq \lambda \leq \kappa^+$, let $\mathcal{W}_{\kappa,\lambda}$ (respectively, $\mathcal{K}_{\kappa,\lambda}$) be the class of all spaces Y for which there are σ-discrete (respactively, complete) metric space M of $w(M) \leq \kappa$ and a closed onto map $f : M \to Y$ such that $\forall y \in Y[|f^{\leftarrow}(y)| < \lambda]$ (respectively, $[|w(f^{\leftarrow}(y))| < \lambda]$).

In particular, put

$$\mathcal{W}_\kappa = \mathcal{W}_{\kappa,\kappa^+}, \quad \text{and} \quad \mathcal{K}_\kappa = \mathcal{K}_{\kappa,\kappa^+}.$$

Note that every space in $\mathcal{W}_{\kappa,\lambda}$ (in this case we call it a *Watson space*) is 0-dimensional, while spaces in $\mathcal{K}_{\kappa,\lambda}$ (in this case we call it *van Douwen-complete space*) need not be 0-dimensional. Hence, let $D_{\kappa,\lambda}$ be the collection of all 0-dimensional members of $\mathcal{K}_{\kappa,\lambda}$.

We believe that our study in the realm of 0-dimensional spaces will be a *prototype* of universal spaces for higher dimensional Lašnev spaces, since Watson construction can be applied to any dimensional Lašnev spaces, using *locally finite* (instead of disjoint) semi-canonical cover for a pair (G, A), where A is a closed nowhere dense subset of a given *open* (instead of clopen) subset G. In present, however, we succeeded in showing its *universalities* only for certain classes of finite-dimensional Lašnev spaces in [7], which contain every finite-dimensional upper semi-continuous decomposition spaces of open manifolds. In other words, it reminds the author that Menger's n-dimensional universal space and the Cantor set are provided by a unified iterative method (see Remark 1.1), and that Watson method must be that kind of method for Lašnev spaces, since it works quite well in 0-dimensional case. Hence, we also believe that it is important to improve Watson method in order to make the essential point clear for further study of higher dimensional case.

This paper covers all the results in [16, 20], and is an extended revision of [21]. In particular, we publish here a proof of a homeomorphism extension theorem 3.1, which is a key to apply Watson method for non-separable Lašnev space.

Remark 1.1. *Iteration of a simple principle* is one of the standard ways to produce universal spaces (even in classical cases of separable metric spaces). Let us review here some of them. For metric spaces, we adopt the following statement as a simple principle:
Devide the model space into its contiguous closed copies, and *exclude* some of them (see [3, Fig. 13] for details).
Then after iterating this principle infinitely many times and taking their *intersection* (instead of decomposition space) we can produce (1) the Cantor set, (2) Sierpiński's carpet, and (3) Menger sponge, when we use the closed interval I, I^2, and I^3 as a model space M, respectively. In these cases we can show not only their universality for certain subclasses of separable metric spaces, but also their *homogeneity* in cases (1) and (3) (on the other hand, it is also known that Sierpiński's carpet in (2) is never homogeneous).

b) Homogeneity. As is observed in Remark 1.1 iterative constructions for metric spaces sometimes yield nice universal spaces which are *homogeneous*. Hence, we can ask the homogeneity in Watson method. Note that if there were a homogeneous universal space, it must be *nowhere first-countable* in this case. On the other hand, there is a space W_ω due to S. Watson in [22], which is nowhere first-countable, and is a universal space for the class $\mathcal{W}_\omega$ of all closed images of rationals. Moreover, there is a nowhere first-countable universal space W_κ for every class $\mathcal{W}_\kappa$ (see Remark 3 in [18]). Hence, we can reformulate our problem as follows.

Problem 1. *Can we make every W_κ homogeneous by Watson method?*

We can solve this problem affirmatively as follows (see also [18, Remark 8]).

Theorem 1.1. *By using Watson method, for each $\mathcal{W}_{\kappa,\lambda}$, there exists a universal space $W_{\kappa,\lambda}$, which satisfies that each non-empty open subset is homeomorphic each other. Moreover, in case $\lambda = \kappa^+$ we can make our universal space W_κ homogeneous.*

We abbreviate the first property in this theorem as HO. To make a Watson space homogeneous is closely related to the homogeneity of maximal nonmetrizable subspace of a universal space for van Douwen-complete spaces. Hence, for the sake of completeness, we shall prove in this paper, the following theorem, which was proved in [20] and was announced in [18, Theorem 1].

Theorem 1.2. *For each $\mathcal{D}_{\kappa,\lambda}$, there exists a universal space $D_{\kappa,\lambda}$, which satisfies HO.*

Using this theorem, we can strengthen our universality to its *closed embeddability* (i.e. embedding to a closed subset of our universal spaces) in Theorem 4.1.

c) Comparison between earlier results and our new ones. Using *semi-canonical* (*s. c.* for short) covers to investigate Lašnev spaces is not new (e.g. [6]), but it is no doubt to believe that it is effective (e.g. see Lemma 4.3). In [18] and [22] we did not specify *s. c.* covers, which were used to construct universal spaces, so that we showed only their universality. Hence, we could not even show that whether or not given two universal spaces are homeomorphic, where they are produced by the method presented there. Therefore, we must *restrict* their *flexibility* to obtain some *useful topological information* about them (e.g. their homogeneity and property HO).
One of the strategies for that purpose (and that to produce homogeneous spaces) is the use of *standard s. c.* covers defined in §3 and a homeomorphism extension theorem 3.1, which produces a standard *s. c.* covers-preserving homeomorphism. By this theorem we can show the universality of our decomposition spaces, also. We believe that our restriction on the class of *s. c.* covers is *moderate*, since we can show by our method that the important Lašnev space L in [8] is homogeneous. Its original definition (see Example 2.3) is rather *rigid* (here we do not mean the rigidity defined in d) below). Namely, its definition needs non-topological terms (e.g. binary rational points and its rank) essentially. Hence, it seems that it is not easy to see its homogeneity by its definition. The author also believes that it is hard to see that L has the property HO directly (see Proposition 4.2).

d) Relationships between universal spaces. For any class $\mathcal{D}_{\kappa,\lambda}$ there *cannot be a homogeneous* universal space, since every $X \in \mathcal{D}_{\kappa,\lambda}$ has a non-empty metrizable subspace, at which every point is first-countable by [6, Theorems 2 and 4]. We will construct in this paper, however, a universal space $\mathcal{D}_\kappa$ for each class $\mathcal{D}_\kappa$, whose maximal non-metrizable subset Z_κ is nowhere first-countable, and homogeneous.

On the other hand, the space P of irrational numbers not only is a universal space for separable 0-dimensional complete metric spaces, but also contains a copy of the space Q of rational numbers which is a universal space for countable metric spaces. Hence, the second problem is:

Problem 2. *Are there any universal space for $\mathcal{W}_\omega$, which can be embedded in some universal space for $\mathcal{D}_\omega$? In particular, can we embed a Watson space W_ω into Z_ω?*

We shall answer this problem negatively as follows:

Theorem 1.3. *There is a space in $\mathcal{W}_\omega$ which cannot be embedded in any van Douwen-complete space.*

Corollary 1.1. *There is no universal space for $\mathcal{W}_\omega$, which can be embedded in some van Douwen-complete space.*

It is Lašnev [8] who showed earlier that there exists a countable, nowhere first-countable Lašnev space L. For his space we have the following:

Theorem 1.4. *The space L can be embedded in a space $\tilde{L}$ in $\mathcal{D}_\omega$. On the other hand, L cannot be embedded in any space in $\mathcal{W}_{\kappa,\lambda}$. Hence, the space L cannot be a universal space for $\mathcal{W}_\omega$.*

Moreover we can show the following theorem (its proof is given in §4), using a characterization theorem (Theorem 4.3) of our universal space D_ω:

Theorem 1.5. *The space $\tilde{L}$ in Theorem 1.4 is homeomorphic to D_ω, and hence, L is homeomorphic to Z_ω, and is homogeneous.*

On the other hand, we can show the following theorem (we call a space *rigid* when it satisfies that its identity map is the only its autohomeomorphism):

Theorem 1.6. *By using Watson method, for every uncountable cardinal α, there exists $\beta \geq \alpha$ such that there exists a Lašnev space W_β in $\mathcal{W}_\beta$, which is rigid.*

All spaces in this paper are assumed to be *Tychonoff*, and all maps are assumed to be continuous. For a space X its dimension $\dim X$ means the covering dimension. In particular, we call space X *strongly 0-dimensional* when it holds that $\dim X = 0$. See [3, 4] for undefined terminologies.

2. Watson spaces versus van Douwen-complete spaces

We state here two fundamental results which will be used frequently.

Fact 2.1. ([8, Theorem 4]). *For any closed map $f : X \to Y$ from a paracompact space X onto a Fréchet-Urysohn space Y, there exists a closed subset Z of X such that $f|Z : Z \to Y$ is irreducible.*

Fact 2.2. ([4, Theorem 2.4.13]). *If M is a closed subset of X and $\mathcal{E}$ is an upper semicontinuous decomposition of M, then the decomposition of X into elements of $\mathcal{E}$ and one-point sets $\{x\}$ with $x \in X \backslash M$ is upper simicontinuous.*

At first, we propose the following notion of mappings, which is useful for distinguishing two given Lašnev spaces.

Definition 2.1. A closed onto map $f : X \to Y$ is called an *EE-map* when it satisfies that the set X_f of all poionts in X, where f is one to one, is dense in X.

Note that every EE-map is *irreducible* [3, Exercise 3.1.C(c)]. On the other hand every closed irreducible map with complete metric domain X is an EE-map [14, 23]. The following proposition, where we do not assume any completeness of the domains of irreducible maps, is a refinement of [19, Theorem 1]. We leave its parallel proof to the reader.

Proposition 2.1. *Let $h : X \to Y$ be an arbitrary closed irreducible onto map. Suppose that both of M and X are metric spaces, and that $f : M \to Y$ is an EE-map. Then, there exists a metric space Z and two perfect onto mappings $\alpha : Z \to M$ and $\beta : Z \to X$ such that $f\alpha = h\beta$.*

Example 2.1. We present here an example, which shows that the above proposition is effective. Let M be a copy of the irrationals P and let A be its closed nowhere dense subset also homeomorphic to P. Then, let Y be the Lašnev space, which is obtained from M by collapsing the set A to a point (say, y_0). Take a subset B of A, which is homeomorphic to rationals Q, and put

$$X = M \backslash (A \backslash B).$$

Note that B is closed nowhere dense in X, and hence X is not completely metrizable. Let W be the Lašnev space, which is obtained from X by collapsing B to a point (say, w_0). Then, we may ask the question whether or not Y is homeomorphic to W. For example, the use of cardinal invariants seems not to be effective in this case, since $Y \backslash \{y_0\} \approx W \backslash \{w_0\} \approx P$ and $\chi(y_0, Y) = \chi(w_0, W)$ by [1, Theorem 8.13 (c)] (note that both of A and B are F_σ in M and X, respectively).

On the other hand, it is ready to see that both of the natural quotient maps f :

$M \to Y$ and $h : X \to W$ are EE-maps. Hence, X must be completely metrizable by the above proposition if we assume that Y and W are homeomorphic (see also [19, Theorem 1]). This is a contradiction. Hence, Y is not homeomorphic to W.

We shall discuss one more example of EE-map, which plays the key rôle to prove Theorems 1.3 and 1.4.

Example 2.2. Take a closed nowhere dense subset A of a metric space X with $\dim X = 0$, and fix a disjoint clopen *s. c.* cover $\mathcal{U}$ for the pair (X, A). Take a point p_U for every $U \in \mathcal{U}$. Put

$$P_A = \{p_U : U \in \mathcal{U}\} \quad \text{and} \quad Y_A = A \cup P_A.$$

Then, note that P_A is an open dense discrete subset in Y_A. Take a point $p_A \notin P_A$, and let

$$X_A = \{p_A\} \cup P_A.$$

Define a function $\varphi_A : Y_A \to X_A$ as

$$\varphi_A(A) = p_A \quad \text{and} \quad \varphi_A(p) = p \quad \text{for every} \quad p \in P_A.$$

We topologize the set X_A as

$$U \quad \text{is open in} \quad X_A \quad \text{if} \quad \varphi_A^{\leftarrow}(U) \quad \text{is open in} \quad Y_A.$$

Then by Definition 2.1 and the condition (SC), it holds that φ_A is an EE-map for any nowhere dense closed subset A of X.

Proof of Theorem 1.3. Let A be a closed nowhere dense subset of X, where both X and A are homeomorphic to Q. Take a clopen disjoint *s. c.* cover $\mathcal{U}$ for (X, A), and consider the space X_A in Example 2.2. We shall show that X_A cannot be embedded in any space of $\mathcal{K}_{\kappa,\lambda}$. Assume contrary, and suppose that $X_A \subset Y \in \mathcal{K}_{\kappa,\lambda}$, and let $q : M \to Y$ be a closed onto map from a complete metric space M. For the closed map $q|S : S \to X_A$, where $S = q^{\leftarrow}(X_A)$, there exists a closed subset T in S such that $h = q|T : T \to X_A$ is irreducible by Fact 2.1. Note that $h^{\leftarrow}(p_A)$ is completely metrizable, since it is a closed subset of $q^{\leftarrow}(p_A)$. Therefore, we can apply Proposition 2.1 to $f = \varphi_A$ and h so that it holds that

$$K = \alpha^{\leftarrow}\varphi_A^{\leftarrow}(p_A) = \beta^{\leftarrow}h^{\leftarrow}(p_A).$$

Note that K is completely metrizable, since $h^{\leftarrow}(p_A)$ is completely metrizable and β is perfect. Hence, $A = \alpha(K)$ must be also completely metrizable. This contradicts the fact that the set $\varphi^{\leftarrow}(p_A) = A \approx Q$ cannot be completely metrizable. □

In this place we review the construction of the space L due to N. Lašnev [8] for the sake of completeness.

Example 2.3. ([8, §3. Example 2]). Let P_0 and Q_2 be the set of all irrationals and that of all binary rational points, respectively, in the unit interval $(0,1)$. We consider the following decomposition of the product space $Q_2 \times P_0$. For every $x \in Q_2$ we say that x has *rank* n if it is represented in a unique manner in the form $m/2^n$, where m is odd. Then, the set of all the points $x \in Q_2$ of rank $\leq n$ partitions the set P_0 into 2^n pairwise disjoint sets $\Delta_1^n, \Delta_2^n, \ldots, \Delta_{2n}^n$, where each Δ_i^n is the set of all irrationals lying between $(i-1)/2^n$ and $i/2^n$. Hence we have the final decomposition of $Q_2 \times P_0$ consisting of the disjoint closed sets $\{\Delta_{i,j}^n : n = 1, 2, \ldots; 1 \leq i \leq 2^n : 1 \leq j \leq 2^{n-1}\}$, where $\Delta_{i,j}^n = \{j/2^n\} \times \Delta_i^n$.

Let L be the decomposition space with respect to this collection. Then, it is known that the natural quotient map $f_L : Q_2 \times P_0 \to L$ is closed [8].

Proof of Theorem 1.4. We note that we can extend the decomposition in the above example to a trivial extension given by Fact 2.2 on the product $M = (P_0 \cup Q_2) \times P_0$ so that the natural quotient map $\tilde{f} : M \to \tilde{L}$, satisfying that every fiber $\tilde{f}^{\leftarrow}(y)$ is one point for $y \in M \backslash Q_2 \times P_0$, is also a closed map (see ¶4.14 and its proof in §4).

We also note that M is homeomorphic to P and that there exists a disjoint clopen *s. c.* cover $\mathcal{U}$ for the pair (M, A), where $A = \Delta_{1,1}^1$, which is *$\tilde{f}$-saturated* (i.e. there exists a disjoint clopen *s. c.* cover $\mathcal{V}$ for the pair $(\tilde{L}, \tilde{f}(A))$ such that $\mathcal{U} = \tilde{f}^{\leftarrow}(\mathcal{V})$, and see ¶4.15 and Lemma 4.6 for the details). The map $\tilde{f}$ is an EE-map, since the set $M \backslash (Q_2 \times P_0)$ is dense in M.

It holds that $\dim \tilde{L} = 0$, since $\operatorname{ind} \tilde{L} = 0$ and $\tilde{L}$ is Lindelöff. This shows that the first half of the statement in our theorem is valid.

For each $U \in \mathcal{U}$ in Example 2.2, where we put $X = M$ and $A = \Delta_{1,1}^1$, we can take a point

$$p_U \in \Delta_{i,j}^n \subset U \quad \text{for some } n, i, j,$$

since L is dense in $\tilde{L}$ and $\mathcal{U}$ is $\tilde{f}$-saturated. Then, consider the spaces X_A and Y_A in Example 2.2. Note that $A \approx P$, and that $\varphi_A = f|Y_A = \tilde{f}|Y_A$. Now, assume that L can be embedded in some space $W \in \mathcal{W}_{\kappa,\lambda}$ so that X_A embeds in W, since $X_A \subset L$.

Let $q : Q_\kappa \to W$ be a closed map, and let $X_A^* \subset W$ be a copy of X_A. For the closed map $q|S : S \to X_A^*$, where $S = q^{\leftarrow}(X_A^*)$, there exists a closed subset T in S such that $h = q|T : T \to X_A^*$ is irreducible by Fact 2.1. Since the set $P_A^* = X_A^* \backslash \{p_A^*\}$ is discrete and h is irreducible and closed, it holds that each $h^{\leftarrow}(y)$ is one point, where $y \in P_A^*$. Hence, h is also an EE-map and T is separable. Therefore, we can apply Proposition 2.1 to $f = \varphi_A$ and h so that it holds that

$$K = \alpha^{\leftarrow} f^{\leftarrow}(p_A^*) = \beta^{\leftarrow} h^{\leftarrow}(p_A^*).$$

Note that K is σ-compact, since $h^{\leftarrow}(p_A^*) \subset Q_\kappa$ is σ-discrete (hence countable in T) and β is perfect. Hence, $A = \alpha(K)$ must be also σ-compact. This contradicts the fact that P cannot be σ-compact. □

3. Semi-cononical covers and a homeomorphism extension theorem

a) Some properties of *s. c.* **covers.** It is known [6, Lemma 1] that every pair (X, A), where A is a closed subset of a metric space X, has a *s. c.* cover. Moreover, it holds that

Fact 3.1. ([6, Lemma 3]). *Let $f : X \to Y$ be a closed surjection, and suppose that B is a closed subset of Y. Let $\mathcal{V}$ be a s. c. cover for $(X, f^{\leftarrow}B)$, and for each $y \in Y \backslash B$, let $\mathcal{G}_y$ be a f-saturated open neighborhood of $f^{\leftarrow}(y)$ (i.e. $G_y = f^{\leftarrow}f(G_y)$) such that $G_y \subset st(f^{\leftarrow}(y), \mathcal{V})$. Then $\mathcal{G} = \{f(G_y) : y \in Y\}$ is a s. c. cover for (Y, B).*

Fact 3.2. *Let $\mathcal{U}$ be a s. c. cover for (X, A), where X is a metric space, and for a neighborhood G of A in X, put*

$$\mathcal{U}_{<G>} = \{U \in \mathcal{U} : U \cap G \neq \emptyset\}.$$

Then, for any given $\varepsilon > 0$, there exists a neighborhood G of A in X such that the mesh of $\mathcal{U}_{<G>} < \varepsilon$.

Fact 3.3. *Suppose that $\mathcal{U}$ is a s. c. cover for the pair (X, A), and for any subset Y of X, put*

$$\mathcal{U}|Y = \{U \cap Y : U \in \mathcal{U}\}.$$

Then, $\mathcal{U}|Y$ is a s. c. cover for the pair $(Y, A \cap Y)$.

Fact 3.4. ([18, Lemma 1]). *Let $\mathcal{U}$ be a clopen disjoint s. c. cover for (X, A), and suppose that, for each $U \in \mathcal{U}$, a non-empty clopen set $O_U \subset U$ are given. Then, the collection $\mathcal{O} = \{O_U : U \in \mathcal{U}\}$ satisfies that, for each neighborhood V of A in S_A, there exists a neighborhood W of A in S_A such that $st(W, \mathcal{O}) \subset V$, where $S_A = A \cup (\cup\mathcal{O})$.*

b) Standard *s. c.* **covers.** Let (X, ρ) be a metric space with a bounded metric $\rho \leq 1$, and suppose that A is its nowhere dense closed subset. Assume that $\dim X = 0$ and X has no isolated points. Let $\mathcal{F}$ be a collection of non-empty clopen subsets of A, satisfying that

¶ **3.1.** $\mathcal{F} = \cup_{i\geq 0}\mathcal{F}_i, \{\mathcal{F}_i\}$ is a *refining sequence* (i.e. $\mathcal{F}_i$ refines $\mathcal{F}_{i-1}$ for each i), and each $\mathcal{F}_i$ is a discrete clopen covering of A with its mesh $\leq 1/2^i$ with respect to ρ.

Since $\dim X = 0$ and each $\mathcal{F}_i$ is discrete, we can expand it to a discrete clopen collection $\mathcal{W}_i$ of X satisfying the following condition.

¶ **3.2.** For each $F \in \mathcal{F}_i$, there exists $W \in \mathcal{W}_i$ such that $F = W \cap A$, and mesh $\mathcal{W}_i \leq 1/2^{i-1}$ with respect to ρ.

By the assumptions that no points are isolated and A is nowhere dense in X, we can assume that

¶ **3.3.** $\{\mathcal{W}_i\}$ is a refining sequence with $X = \cup\mathcal{W}_0$ and that for each $i \geq 0$ and every $W \in \mathcal{W}_i$ it holds that $W \cap A \neq \emptyset$, and that $W \setminus \cup \mathcal{W}_{i+1} \neq \emptyset$.

Put

¶ **3.4.** $\mathcal{U} = \cup_{i\geq 0}\mathcal{U}_i$, where $\mathcal{U}_i = \{W \setminus \cup \mathcal{W}_{i+1} : W \in \mathcal{W}_i\}$.

Then it is not difficult to see that $\mathcal{U}$ is a disjoint clopen *s. c.* cover for (X, A). Therefore,

Lemma 3.1. *Under the above assumptions of X and A in this section, for any given $\mathcal{F}$, which satisfies* ¶3.1, *there exists a disjoint clopen s. c. cover $\mathcal{U}$ for (X, A) satisfying* ¶3.2 – ¶3.4 *with suitable collecions $\mathcal{W}_i$.*

We call $\mathcal{U}$ a *standard s. c.* cover (with respecto to $\mathcal{F}$) for (X, A) that is provided by this lemma. We need the following property of standard *s. c.* covers in the next section (see the proof of Theorem 1.2). See Remark 3.1 for an example of *s. c.* cover which is *not* standard.

Proposition 3.1. *Let $\mathcal{U}$ be a standard s. c. cover for a pair (X, A), and suppose that G is a clopen neighborhood of A such that $G = A \cup (\cup\mathcal{U}_{<G>})$ where $\mathcal{U}_{<G>}$ is defined in Fact 3.2. Then, $\mathcal{U}_{<G>}$ is a standard s. c. cover for the pair (G, A).*

Proof. Let $\{\mathcal{W}_i\}$ be a sequence of discrete collection used in ¶3.2. For each $W \in \mathcal{W}_i$ let

$$U(W) = W \setminus \cup \mathcal{W}_{i+1} \in \mathcal{U}.$$

Then, we shall show that we can rearrange $\mathcal{W}_G$ so that the collection $\mathcal{F}_G = \{W \cap A : W \in \mathcal{W}_G\}$ constitutes a refining sequence for A. At first, we shall show:

Assertion 1. *For each $W \in \mathcal{W}_i$ there exists a collection $\mathcal{W}_W \subset \mathcal{W}_G \cap (\cup_{j>i}\mathcal{W}_j)$ such that $\mathcal{F}_W = \{W \cap A : W \in \mathcal{W}_W\}$ is a disjoint clopen cover of $W \cap A$.*

For any $x \in W \cap A$ there exists a $W' \in \mathcal{W}_j$, for some $j > i$, such that $x \in W' \subset G$, since G is a neighborhood of A and $\{W \in \cup_i \mathcal{W}_i : x \in W\}$ is an open base of x by ¶3.2. Then, $W' \subset W$, since $\mathcal{W}_j$ refines $\mathcal{W}_i$, and $U(W') \subset G$ by the definition of $U(W')$. Hence,

$$\mathcal{W}^* = \{W' \in \cup_{j>i}\mathcal{W}_j : U(W') \subset G, W' \subset W\}$$

is a covering of W. Let

$$\mathcal{W}_W = St(\mathcal{W}^*) = \{st(W', \mathcal{W}^*) : W' \in \mathcal{W}^*\}.$$

Then, $\mathcal{W}_W$ satisfies the required property, since it holds that $W' \subset W''$ or $W' \supset W''$ if $W' \cap W'' \neq \emptyset$ for $W', W'' \in \mathcal{W}^*$ so that $st(W', \mathcal{W}^*) \in \mathcal{W}^*$. Let ρ_G be the same metric of X and put

$$\mathcal{W}_{G,0} = \cup_{W \in \mathcal{W}_0}\mathcal{W}_W \quad \text{and} \quad \mathcal{F}_{G,0} = \cup_{W \in \mathcal{W}_0}\mathcal{F}_W.$$

Then, $\mathcal{F}_{G,0}$ is a disjoint clopen cover of A with its mesh ≤ 1. For each $i \geq 1$ let

$$\mathcal{W}_{G,i} = \cup_{W \in \mathcal{W}_{G,i-1}}\mathcal{W}_W \quad \text{and} \quad \mathcal{F}_{G,i} = \cup_{W \in \mathcal{W}_{G,i-1}}\mathcal{F}_W.$$

Then, the mesh $\mathcal{W}_{G,i} \leq 1/2^{i-1}$ and the mesh $\mathcal{F}_{G,i} \leq 1/2^i$. By their constructions we see that

$$\mathcal{W}_G = \cup_{i \geq 0}\mathcal{W}_{G,i} \quad \text{and} \quad \mathcal{U}_{<G>} = \{U(W) : W \in \cup_{i \geq 0}\mathcal{W}_{G,i}\}.$$

Hence two collections $\{\mathcal{F}_{G,i}\}$ and $\{\mathcal{W}_{G,i}\}$ satisfy ¶3.2 and ¶3.3, and $\mathcal{U}_{<G>}$ is a standard *s. c.* cover for the pair (G, A) □

c) A homeomorphism extension theorem and applications to strongly homogeneous spaces.

Definition 3.1. Assume that $\mathcal{U}$ and $\mathcal{V}$ are *s. c.* covers for (X, A) and (X, B), respectively. Then, a $(\mathcal{U}, \mathcal{V})$ - *preserving* homeomorphism $h : (X, A) \to (Y, B)$ is providend when $h(A) = B$ and h induces a bijection $h^* : \mathcal{U} \to \mathcal{V}$, defined by $h^*(U) = h(U)$ for every $U \in \mathcal{U}$.

On the other hand, a bijection $\varphi : \mathcal{U} \to \mathcal{V}$ is called *piecewise homeomorphic* (*p. h.* for short) when it holds that $U \approx \varphi(U)$ for each $U \in \mathcal{U}$. A *p. h.* bijection $\varphi : \mathcal{U} \to \mathcal{V}$ is *realized by a homeomorphism* $\tilde{h} : X \to Y$ when it holds that $\varphi = \tilde{h}^*$.

Example 3.1. Let A be a one point set $\{a\}$ in $X = P$, and suppose that $\{W_i\}_{i \geq 0}$ is its countable decreasing neighborhood base of a with $W_0 = X$. Then, the collection $\mathcal{U} = \{U_i = W_i \backslash W_{i+1}\}$ is a standard *s. c.* cover for (X, A). On the other hand, let $\mathcal{V} = \cup_{i \geq 0} \mathcal{V}_i$, where each $\mathcal{V}_i$ is a countable infinite discrete clopen cover of U_i. Then, $\mathcal{V}$ is a *s. c.* cover for (X, A), which is *not* standard, and it holds that there are no $(\mathcal{U}, \mathcal{V})$ - preserving autohomeomorphisms of (X, A).

We need one more fundamental lemma to state our homeomorphism extension theorem.

Lemma 3.2. *Let $h : G \to H$ be a homeomorphism between two closed subsets G and H of metric spaces X and Y, respectively. Then there exist two compatible bounded metrics $\rho_G \leq 1$ and $\rho_H \leq 1$ of X and Y, respectively, such that $h : (G, \rho_G) \to (H, \rho_H)$ is an isometric mapping.*

Proof. It is obvious from the fact that $mim\{\rho, 1\}$ is always a metric for every metric ρ, and from a theorem of Hausdorff [3, Problem 4.5.21.(c)]. □

Now, we can state our homeomorphism extension theorem:

Theorem 3.1. *Let X and Y be strongly 0-dimensional metric spaces without isolated points, and let G and H be their closed nowhere dense subsets, respectively. Suppose that ρ_G and ρ_H are two compatible metrics of X and Y, and $h : (G, \rho_G) \to (H, \rho_H)$ is an isometric homeomorphism. Assume that $\mathcal{F}$ is a refining sequence satisfying ¶3.1 for $A = G$ with respect to ρ_G. Let $\mathcal{U}$ and $\mathcal{V}$ be two standard s. c. covers with respect to $\mathcal{F}$ and $h(\mathcal{F})$ for the pairs (X, G, ρ_G) and (Y, H, ρ_H), respectively. Then there exits a bijection $\varphi : \mathcal{U} \to \mathcal{V}$ with the following property: if φ is p. h. , then φ is realized by a homeomorphism $h : X \to Y$ which is an extension of h.*

d) Navigations sets. We begin with the following notion of navigating sets, which is shown effective to prove our homeomorphism extension theorem.

Definition 3.2. Let $\{\mathcal{F}_i\}_{i \geq 0}$ be a refining sequence of a space X. Then a set $\{p_F : F \in \mathcal{F} = \cup_{i \geq 0} \mathcal{F}_i\}$ is called a *navigating set* for $\mathcal{F}$ when it satisfies that $p_F = p_K$ if $F \in \mathcal{F}_{i+1}$, $K \in \mathcal{F}_i$ for some i, and $p_K \in F \subset K$.

Remark 3.1. Note that there *always* exists a navigating set for a given refining sequence $\{F_i\}$ by defining p_F inductively (with respect to i) as far as all the elements of every $\mathcal{F}_i$ are non-empty.

We show here one simple application of navigating sets and standard *s. c.* covers.

Lemma 3.3. *Assume that A is nowhere dense in X, where the metric space X has no isolated points and $\dim X = 0$. Then, for any closed subset $B \subset A$, there exists a closed subset T_B of X and a subcollection $\mathcal{U}_B$ of $\mathcal{U}$ such that $\mathcal{U}_B$ is a s. c. cover for (T_B, B), where $T_B = B \cup (\cup \mathcal{U}_B)$ and $\mathcal{U}$ is any given disjoint clopen standard s. c. cover for (X, A).*

Proof. Let $\mathcal{F}$ be a refining sequence with which $\mathcal{U}$ is standard. Take a navigating set $\{p_F : F \in \mathcal{F}\}$, satisfying that $p_F \in B$ whenever $F \cap B \neq \emptyset$. Then, define $r_A^B : X \to A$ as follows:

$$\begin{aligned} r_A^B(x) &= x \quad \text{for every} \quad x \in A \\ &= p_F \quad \text{for every} \quad x \in W \setminus \cup \mathcal{W}_{i+1} \in \mathcal{U}_i \quad \text{and} \quad F = W \cap A. \end{aligned}$$

It is easy to see that $r_A^B : X \to A$ is a continuous retraction (cf. [3, Problem 4.1.G(a)]). Put

$$\mathcal{U}_B = \{U \in \mathcal{U} : U \cap r_A^{B\leftarrow}(B) \neq \emptyset\}.$$

Then, it is not difficult to see that $\mathcal{U}_B$ satisfies the required properties by the condition (SC) and the definition of navigating sets. □

Remark 3.2. Lemma 3.3 is valid for *arbitrary s. c.* cover if we restrict our case when the space X is either B_κ or Q_κ, though in this case we need some characterization of X and another homeomorphism extension theorem (see [18, Lemma 0]).

e) Proof of Theorem 3.1.

Proof. We consider two clopen collections

$$\mathcal{G} = \cup_{i \geq 0} \mathcal{G}_i, \quad \text{and} \quad \mathcal{H} = \cup_{i \geq 0} \mathcal{H}_i$$

of X and Y for the refining sequence $\{\mathcal{F}_i\}$ of $A = G$, and for the refining sequence $\{h(\mathcal{F}_i)\}$ of $A = H$, which satisfies ¶3.2 – ¶3.4 for $\mathcal{W} = \mathcal{G}$ of $\mathcal{H}$, respectively. Put, for each $S \in \mathcal{G}_i$ and $T \in \mathcal{H}_i$,

$$U(S) = S \setminus \cup \mathcal{G}_{i+1} \quad \text{and} \quad V(T) = T \setminus \cup \mathcal{H}_{i+1}.$$

Then, by ¶3.4 it holds that $\mathcal{U} = \cup_{i\geq 0}\mathcal{U}_i$ and $\mathcal{V} = \cup_{i\geq 0}\mathcal{V}_i$, where $\mathcal{U}_i = \{U(S) : S \in \mathcal{G}_i\}$ and $\mathcal{V}_i = \{V(T) : T \in \mathcal{H}_i\}$.
Take an arbitrary navigating set $N(\mathcal{F}) = \{p_F : F \in \mathcal{F}\}$ for $\mathcal{F}$. Then, note that $h(N(\mathcal{F}))$ is a navigating set for $\{f(\mathcal{F}_i)\}$ by the Definition 3.1 and the fact that h is one to one. Note also that since each $\mathcal{F}_i$ is a covering of G, we have:

¶ 3.5. For each $p = p_F$ that set $I(p) = I(h(p)) = \{j : p_F = p_K$ for some $K \in \mathcal{F}_i\}$ is just the set $\{j : j \geq j_p\}$ for some integer j_p.

Hence, for each $p = p_F$ put

$$\mathcal{G}(p) = \{S : p \in S \in \mathcal{G}_j, j \geq j_p\} \text{ and } \mathcal{H}(h(p)) = \{T : h(p) \in T \in \mathcal{H}_j, j \geq j_p\}.$$

For every $j \geq j_p$ let S_j and T_j be the unique elements of the collections $\mathcal{G}_j \cap \mathcal{G}(p)$ and $\mathcal{H}_i \cap \mathcal{H}(h(p))$, respectively. Let $\phi_p : \mathcal{G}(p) \to \mathcal{H}(h(p))$ be the bijection defined by $\phi_p(S_j) = T_j$.
Note that

¶ 3.6. ϕ_p induces a *level preserving* bijection (i.e. it induces a bijection between $\mathcal{G}_i$ and $\mathcal{H}_i$ for each i) by its definition and the facts that

$$\mathcal{G} = \cup_{F\in\mathcal{F}}\mathcal{G}(p_F), \ \mathcal{H} = \cup_{F\in\mathcal{F}}\mathcal{H}(h(p_F)),$$

and $\mathcal{G}(p_F) \cap \mathcal{G}(p_K) = \emptyset$, $\mathcal{H}(h(p_F)) \cap \mathcal{H}(h(p_K)) = \emptyset$ if $p_F \neq p_K$.

By ¶3.6 we see that $\phi : \mathcal{G} \to \mathcal{H}$ defined by $\phi(S) = \phi_p(S)$ for each $S \in \mathcal{G}(p)$, is a bijection. Define $\varphi : \mathcal{U} \to \mathcal{V}$ by

$$\varphi(U(S)) = V(\phi(S)).$$

Let us show that if φ is $p.h.$, then it is realized by a homeomorphism $\tilde{h} : X \to Y$.
Let us define $\tilde{h} : X \to Y$ as follows.
(a) Put $\tilde{h}|G = h$.
(b) For each $U \in \mathcal{U}_j$, let $\tilde{h}|U$ be an arbitrary homeomorphism between U and V, where $V = \varphi(U)$, given by the assumption that φ is $p.h.$
Then it is evident that $h^* = \varphi$ (hence, $\tilde{h}$ is $(\mathcal{U}, \mathcal{V})$ - preserving).

(i) Continuity of $\tilde{h}$. It suffices to show that $\tilde{h}$ is continuous for every point $x \in G$. Assume that a sequence $\{x_n \in X\backslash G\}$ converges to x. For any ε - neighborhood of $h(x) = \tilde{h}(x)$ with respect to the metric ρ_H take a sufficiently large integer i such that $1/2^{i-3} < \varepsilon$. Since $\{x_n\}$ converges to x, there exists n_0 such that $x_n \in S$ for every $n \geq n_0$, where S is the unique element of $\mathcal{G}_i$ which contains x. Then by the definition of $\tilde{h}(x_n)$, for each $n \geq n_0$, there exist unique $F_n \in \mathcal{F}$, $p_n = p_{F_n}$, and $j_n \geq j_{p_n}$ such that

$$x_n \in U(S_{j_n}), \quad \text{and} \quad S_{j_n} \in \mathcal{G}(p_n).$$

Note that the following ¶3.7 holds, by ¶3.6, the fact that $x_n \in S \in \mathcal{G}_i$, and the part (b) of the definition of $\tilde{h}$.

¶ **3.7.** $j_n \geq i$ and $h(p_n), \tilde{h}(x_n) \in T_{j_n}$ for some $T_{j_n} \in \mathcal{H}_{j_n}$, and hence $\rho_H(\tilde{h}(x_n), h(p_n)) \leq 1/2^{j_n-1}$.

On the other hand,

$$\rho_G(x, p_n) \leq \rho_G(x, x_n) + \rho_G(x_n, p_n) \leq 1/2^{i-1} + 1/2^{j_n-1} \leq 1/2^{i-2}.$$

Since ¶3.7 holds and $\rho_G(x, p_n) = \rho_H(h(x), h(p_n))$ by the assumption, we have

$$\begin{aligned}\rho_H(h(x), \tilde{h}(x_n)) &\leq \rho_H(h(x), h(p_n)) + \rho_H(h(p_n), \tilde{h}(x_n)) \\ &\leq 1/2^{i-2} + 1/2^{j_n-1} \leq 1/2^{i-3} < \varepsilon.\end{aligned}$$

(ii) Continuity of $\tilde{h}^{\leftarrow}$. It can be seen without difficulties by a parallel argument given above, using ¶3.2 and the assumption that h is isometric. □

We have the following applications of this theorem (we call a metric space *strongly homogeneous* (abbr. *s. h.*) when all its non-empty clopen subspaces are homeomorphic [10] (see also Remark 3.4(a)):

Corollary 3.1. *Let X be s. h. strongly 0-demensional matric space and let G and H be its closed nowhere dense subsets. If $h : G \to H$ is a homeomorphism then h can be extended to an autohomeomorphism $\tilde{h}$ of X.*

Corollary 3.2. ([12, Lemma 1]) *Let G and H be closed nowhere dense subsets of strongly 0-dimensional metric spaces X and Y, respectively. Assume that $X \backslash G$ and $Y \backslash H$ are homeomorphic s. h. spaces. Then, any homeomorphism $h : G \to H$ can be extended to a homeomorphism $\tilde{h} : X \to Y$.*

Simultaneous prrof of Corollaries 3.1 and 3.2. By Lemmas 3.1 and 3.2 we have two compatible bounded metrics $\rho_G \leq 1$ and $\rho_H \leq 1$ such that h is an isometric mapping and two standard *s. c.* covers $\mathcal{U}$ and $\mathcal{V}$ with respect to them. Then, we can apply Theorem 3.1 to get a bijection φ such that it is realized by an autohomeomorphism $\tilde{h}$, which is an extension of h, since h is *p. h.* by the strong homogeneity of the assumptions. □

Remark 3.3. (a) Cantor set C, the space of irrationals P (moreover, every 0-dimentional Baire sapce B_κ of weight κ), and that of rationals Q (moreover, the universal σ-discrete space Q_κ of weight κ in [9]) are all *s. h.* spaces. On the other hand, the space $C^* = C \backslash \{p\}$ for any $p \in C$ is no longer *s. h.* (hence, Corollary 3.1 does not follow from Corollary 3.2), but it still has the homeomorphism

extension property guaranteed in Theorem 3.1 as follows. Since every closed nowhere dense subset of C^* has a standard *s. c.* cover $\mathcal{U}$, consisting of clopen disjoint subsets of C^* suth that each $U \in \mathcal{U}$ is homeomorphic to C. Hence every homeomorphism between them can be extended to an autohomeomorphism of C^*.
(b) Corollary 3.1 generalized [10, Theorem 3.1] to non-separable metric space. The *s. c.* cover preserving property of $\tilde{h}$ in our theorem is important, because we shall use it to construct an embedding between *u. s. c.* decompositions with respect to given *s. c.* covers (see also Theorem 4.3).
(c) For *dense subsets* of complete metric spaces we can prove the following theorem, which is a generalization of [3, Problem 1.3 H(b), (c)] (we reserve its proof ot §5):

Theorem 3.2. *Let D and E be two σ-discrete dense subsets of a s. h. complete metric space X of* $\dim X = 0$. *Then there exists an autohomeomorphism $h : X \to X$ suth that $h(D) = E$.*

Remark 3.4. (a) The assumption of completeness in Theorem 3.2 is essential, since every dense subset of Q is homeomorphic to Q by [3, Problem 1.3 H(d)] (e.g. there are no autohomeomorphism h of Q such that $h(D) = E$, if D and E are chosen to satisfy that $|Q \backslash D| = 1$ and $|Q \backslash E| = 2$).
(b) The strong homogeneity relates with *self similarity* of fractals as follows (see [5] for its definition and its topological properties). We propose to say that a space X is *topologically self similar* (abbr. *t. s. s.*) if every non-empty open subset contains a topological copy of X, which is a closed subset of X. Then, it is evident that every *s. h.* space X is *t. s. s.* when $\dim X = 0$. By well-known classical characterization theorems of C, P, Q, B_κ, and Q_κ we have the following theorem, which says that *s. h.* spaces are rather restrictive when $\dim X = 0$ (hence, a complete metric space X in Theorem 3.2 is homeomorphic to either C or B_κ, where $\kappa = w(X)$):

Theorem 3.3. *Every t. s. s. complete metric space X of* $\dim X = 0$ *is homeomorphic to either C (when X is compact), or B_κ (when X is non-compact). Every t. s. s. σ-discrete metric space is homeomorphic to Q_κ. In particular, every countable t. s. s. metric space is homeomorphic to Q.*

We conclude this section with the following proposition, which is one more application of Theorem 3.1 and will be used to prove Theorem 4.2 (b) and Corollary 4.1.

Proposition 3.2. *Let $\mathcal{U}$ be a standard s. c. cover for a pair (X, A) in The-*

orem 3.1, and suppose that G is a dense subset of X, containing A. Then, $\mathcal{U}|G = \{U \cap G : U \in \mathcal{U}\}$ is a standard s. c. cover for the pair (G, A), and the bijection $\varphi : \mathcal{U} \to \mathcal{U}|G$, defined by $\varphi(U) = U \cap G$ for each $U \in \mathcal{U}$, satisfies all the conditions of Theorem 3.1.

Proof. Indeed, since G is dense and X has no isolated point by the assumption, it holds that G has no isolated points, $U \cup G \neq \emptyset$ for each $U \in \mathcal{U}$, and ¶3.2 – ¶3.4 hold for $\mathcal{U}|G$. Hence, $\mathcal{U}|G$ is a standard *s. c.* cover for the pair (G, A). Letting $h = id_A$ and $\rho_G = \rho_X$, we see that φ satisfies all the conditions of Theorem 3.1. □

4. Universal spaces for 0-dimensional van Douwen-complete spaces

a) Preliminaries. In this section we shall show the following theorem, which is a refinement of a theorem in [16]. We remind that a (not necessarily first countable) space is called *h-homogeneous* when any non-empty clopen subsets are homeomorphic (e.g. [9]).

Theorem 4.1. *For each $\mathcal{D}_\kappa$, there exists a space D_κ, which is h-homogeneous, such that every $X \in \mathcal{D}_\kappa$ can be embedded in D_κ as a closed subset. Moreover, it is also HO, and the subset Z_κ, consisting of all non-first-countable points in D_κ, is homogeneous.*

Recall that B_κ is the *Baire's* 0-dimensional space of a given weight $\kappa \geq \omega$ (i.e. $B_\kappa =^\omega \kappa$ the countable Cartesian product of a discrete space of size κ). The following facts are well-known.

Fact 4.1. (a characterization theorem of B_κ [13, Theorem 1]). *Let X be a complete metric space of* $\dim X = 0$. *If it has a dense set of cardinality κ, and if any non-empty open subset contains a discrete subset of cardinality κ, then X is homeomorphic to B_κ. Hence, every dense G_δ (respectively, non-empty open) subset of B_κ is homeomorphic to B_κ.*

Fact 4.2. ([13]). *Every completely metrizable space X of $w(X) \leq \kappa$ can be embedded in B_κ as a nowhere dense closed subset.*

Definition 4.1. We call a refining sequence $\{\mathcal{B}_i\}_{i \geq 0}$ of a space X *κ-complete* for an infinite cardinal number κ when it satisfies the following conditions ¶4.1 and ¶4.2.

¶ **4.1.** Each $\mathcal{B}_i$ is a clopen disjoint cover of X with the property that, for every decreasing collection $\{B_i\}$, where $B_i \in \mathcal{B}_i$, there exists a point $x \in X$ such that $\{B_i\}$ is a neighborhood base of x.

¶ **4.2.** Each element of $\mathcal{B}_i$ is non-empty, and $\mathcal{B}_i$ can be indexed by

$$\mathcal{B}_i\{B(\alpha_0,\ldots,\alpha_i) : \alpha_0,\ldots,\alpha_i < \kappa\} \text{ and } B(\alpha_0,\ldots,\alpha_i) = \cup_{\beta<\kappa} B(\alpha_0,\ldots,\alpha_i,\beta).$$

For a given κ-complete refining sequence $\{\mathcal{B}_i\}_{i\geq 0}$ we shall use its *induced metric* $\rho(x,y) = 1/2^i$, where $i = \min\{j :$ there exists $B \in \mathcal{B}_j$ such that $x \in B$ but $y \notin B\}$. By Definition 4.1 it holds that:

¶ **4.3.** For any $x \in X$, there exists unique sequence of indexes $\alpha(x) = \{\alpha_i\}_{i\geq 0}$ such that $\{x\} = \cap_i B(\alpha_0,\ldots,\alpha_i)$.

For two given κ-complete refining sequences $\mathcal{B}$ and $\mathcal{B}'$ of spaces G and H respectively, we can define their *induced homeomorphism* $h : G \to H$ as follows.

¶ **4.4.** $h(x) = \cap_i B'(\alpha_0,\ldots,\alpha_i)$, where $\alpha(x) = \{\alpha_i\}_{i\geq 0}$ satisfies ¶4.3.

Note that h satisfies a refining sequence preserving property that $\mathcal{B}' = h(\mathcal{B}) = \{h(B) : B \in \mathcal{B}\}$.

Remark 4.1. By ¶4.1 – 4.2 we see that the existence of a κ-complete refining sequence *characterizes* the Baire's 0-dimensional space B_κ (i.e. $X \approx B_\kappa$). This simple fact is closely related to the HO property of our universal space D_κ and the homogeneity of its non-first countable subset Z_κ. On the other hand, we do not have any characterization, using only its refining sequences, for the universal σ-discrete space Q_κ of weight κ in [9]. Hence, we shall prove the homogeneity of our universal space W_κ by using restrictions of the following $\mathcal{P}_\kappa$ - *s. c.* covers of B_κ to σ-discrete pairs (see §5).

Definition 4.2. Let $\mathcal{P}_\kappa$ be the class of all the pairs (X, A), where A is a nowhere dense closed subset of X and $A \approx X \approx B_\kappa$. Then, for each pair $(X, A) \in \mathcal{P}_\kappa$, take a κ-complete refining sequence $\mathcal{B}_A$ of A. A standard *s. c.* cover with respect to $\mathcal{B}_A$ is called a $\mathcal{P}_\kappa$ - *s. c.* cover for the pair (X, A).

The following theorem is one of main tools to prove Theorem 4.1.

Theorem 4.2. *Every $\mathcal{P}_\kappa$ - s. c. cover has the following properties:*
(a) *For each (X, A), and $(Y, B) \in \mathcal{P}_\kappa$, there exists a $(\mathcal{U}, \mathcal{V})$ - preserving homeomorphism $h : (X, A) \to (Y, B)$, where $\mathcal{U}$ and $\mathcal{V}$ are any given $\mathcal{P}_\kappa$ - s. c. covers for (X, A) and (Y, B), respectively.*

(b) *Suppose that* $\mathcal{U}$ *is a* $\mathcal{P}_\kappa$ *cover for* (X, A)*, and assume that* G *is a* G_δ *dense subset of* X*, containing* A*. Then,* $\mathcal{U} \,|\, G$ *defined in Fact 3.3 is a* $\mathcal{P}_\kappa$ *- s. c. cover for the pair* (G, A)*, and the bijection* $\varphi : \mathcal{U} \to \mathcal{U}\backslash G$*, defined by* $\varphi(U) = U \cap G$ *for each* $U \in \mathcal{U}$*, is realized by a homeomorphism* $h : (X, A) \to (G, A)$*.*

(c) *Suppose that* $\mathcal{U}$ *is a* $\mathcal{P}_\kappa$ *cover for* (X, A)*, and assume that* G *is a clopen neighborhood of* A *such that* $G = A \cup (\cup \mathcal{U}_{<G>})$*, where* $\mathcal{U}_{<G>}$ *is defined in Fact 3.2. Then,* $\mathcal{U}_{<G>}$ *is a* $\mathcal{P}_\kappa$ *- s. c. cover for the pair* (G, A)*.*

Proof. (a) It is a direct consequence of Theorem 3.1, using the induced homeomorphism defined in ¶4.4 and the induced metric defined after ¶4.2.
(b) follows from Proposition 3.2, since it holds that $U \approx \varphi(U)$ for each $U \in \mathcal{U}_A$ by Fact 4.1.
(c) We can apply Proposition 3.1 to get a refining sequence $\mathcal{F}$ of A. Note that in this case $\mathcal{F}$ satisfies ¶4.1 and ¶4.2, since $|\mathcal{F}_W| = \kappa$ for each $W \in \mathcal{W}_i$ by the fact that $\mathcal{F}_W \subset \cup_{j>i} \mathcal{W}_i$. Hence, $\mathcal{F}$ is a κ-complete refining sequence of A and $\mathcal{U}_G$ is a $\mathcal{P}_\kappa$ - *s. c.* cover for the pair (G, A) □

b) Watson construction. We shall perform the construction for the space B_κ, where it is used as a model space M in the introduction. Let $\mathcal{V}_0$ be a clopen disjoint cover of B_κ with $|\mathcal{V}_0| = \kappa$, and the mesh $\mathcal{V}_0 \leq 1$ with respect to some complete metric of B_κ. Then by Fact 4.2, for each $V \in \mathcal{V}_0$, take a nowhere dense closed subset $F_V \approx B_\kappa$ of V. Put $\mathcal{F}_0 = \{F_V : V \in \mathcal{V}_0\}$.
By recursion we shall construct two collections $\mathcal{F}_i$ and $\mathcal{V}_i$ for each i, satisfying the following ¶4.5 – ¶4.6.

¶ 4.5. $\mathcal{V}_i$ is a clopen disjoint collection with $|\mathcal{V}_i| = \kappa$ and $\mathcal{F}_i = \{F_V : V \in \mathcal{V}_i\}$, where F_V is a closed nowhere dense subset of V, and is homeomorphic to B_κ.

¶ 4.6. The mesh $\mathcal{V}_i \leq 1/2^i$ with respect to some complete metric of B_κ, and $\mathcal{V}_i = \cup_{U \in \mathcal{V}_{i-1}} \mathcal{V}_U$, where $\mathcal{V}_U$ is a $\mathcal{P}_\kappa$ - *s. c.* cover for the pair (U, F_U).

Put also
$\mathcal{C}_\infty = \cup_{j \geq 0} \mathcal{C}_j$, for $\mathcal{C} = \mathcal{F}$, or $\mathcal{V}$, $M_0 = B_\kappa \backslash \cup \mathcal{F}_\infty$, and $E_i = \cup_{j \leq i} F_j$, where $F_i = \cup \mathcal{F}_i$.
Note that:

¶ 4.7. The collection $\mathcal{V}_i$ refines $\mathcal{V}_{i-1}$, each E_i is nowhere dense closed in the space B_κ, $\mathcal{V}_i$ is a *s. c.* cover for (B_κ, E_i), hence $\mathcal{V}_\infty$ is a π - *base* (i.e. any non-empty open set contains some element of $\mathcal{V}_\infty$), and M_0 is dense in B_κ.

For each F_V, where $V \in \mathcal{V}_i$, let $\mathcal{B}_V$ be the collection of all clopen subsets B

in V such that, for each $B \in \mathcal{B}_V$, there exists a subcollection $\mathcal{V}_B \subset \mathcal{V}_V \subset \mathcal{V}_{i+1}$ satisfying that $B = \cup\mathcal{V}_B \cup F_V$. Then one can show the following lemma without difficulties by the property (SC) of *s. c.* covers and our construction.

Lemma 4.1. *The collection $\mathcal{B}_V$ is a clopen neighborhood base of F_V in V (hence in B_κ).*

Now, let $D_{\kappa,\mathcal{F}_\infty}$ be the decomposition space defined by the following identification on B_κ:

we identify $r, s \in B_\kappa$ if $r, s \in F \in \mathcal{F}_i$ for some i.

From our construction this definition of identification makes sense, and let $q : B_\kappa \to D_{\kappa,\mathcal{F}_\infty}$ be the natural quotient map. Put $\mathcal{W}(x) = \{q(B) : B \in \mathcal{B}_V\}$ for $x = q(F_V)$, where $V \in \mathcal{V}_i$,
and put
$\mathcal{W}(x) = \{q(V) : q^{\leftarrow}(x) \in V \in \mathcal{V}_i, i \in \omega\}$ for $x \in D_{\kappa,\mathcal{F}_\infty}$ such that $q^{\leftarrow}(x) \in M_0$.
Then, we have the following fundamental property of our quotient topology (see [22, Lemmas 2 and 3]).

Lemma 4.2. *For each $x \in D_{\kappa,\mathcal{F}_\infty}$ the collection $\mathcal{W}(x)$ is its clopen neighborhood base.*

By this lemma we know that each point x, where $q^{\leftarrow}(x) \notin \mathcal{F}_\infty$, has a countable neighborhood base, and has a *s. c.* cover for the pair $(D_{\kappa,\mathcal{F}_\infty}, \{x\})$. On the other hand, each pair $(B_\kappa, q^{\leftarrow}(x))$, where $q^{\leftarrow}(x) \in \mathcal{F}_\infty$, has a q-saturated *s. c.* cover by ¶4.6 since V is clopen. Hence, by virtue of the following lemma it holds that q is a closed map.

Lemma 4.3. *Let $p : X \to Y$ be a quotient map, where X is normal, and suppose that every $y \in Y$ has an open collection $\mathcal{W}$ such that $p^{\leftarrow}\mathcal{W}$ is a s. c. cover for the pair $(X, p^{\leftarrow}(y))$. Then, p is a closed map.*

Proof. Let F be an arbitrary closed subset of X, and suppose that $F \cap p^{\leftarrow}(y) = \emptyset$. Take disjoint neighborhoods U and V of F and $p^{\leftarrow}(y)$, respectively. Since $p^{\leftarrow}\mathcal{W}$ is a *s. c.* cover for the pair $(X, p^{\leftarrow}(y))$, there exists a subcollection $\mathcal{W}_V$ of $\mathcal{W}$ such that $G = p^{\leftarrow}(y) \cup p^{\leftarrow}(\cup\mathcal{W}_V)$ is a neighborhood of $p^{\leftarrow}(y)$, satisfying $G \subset V$. Then, $p(G)$ is a neighborhood of y, since p is quotient. Hence, $p(F)$ is a closed set, since $p(G) \cap p(F) = \emptyset$. □

Lemma 4.4. *The quotient map $q : B_\kappa \to D_{\kappa,\mathcal{F}_\infty}$ is an EE-map. Hence, the decomposition space $D_{\kappa,\mathcal{F}_\infty}$ is a member of $\mathcal{K}_\kappa$.*

Proof. By Lemmas 4.1 – 4.3 it holds that q is closed. The map q is an EE-map, since M_0 is dense by ¶4.7, and $q|M_0$ is a homeomorphism. □

The next lemma shows that the space $D_{\kappa,\mathcal{F}_\infty}$ is actually a member of $\mathcal{D}_\kappa$.

Lemma 4.5. *Every non-empty open subset of $D_{\kappa,\mathcal{F}_\infty}$ is a countable union of its clopen subsets, and hence* $\dim D_{\kappa,\mathcal{F}_\infty} = 0$.

Proof. Let O be its arbitrary non-empty open subset. Then, by Lemma 4.2, for each $x \in O$, it holds that either there exists
$B_x \in \mathcal{B}_{V_x}$, where $V_x \in \mathcal{V}_i$, such that $x \in q(B_x) \subset O$, or $V_x \in \mathcal{V}_i$ such that $x \in q(V_x) \subset O$.
For each $i, j \geq 0$ put

$$\mathcal{G}_{ij} = \{V_x \in \mathcal{V}_i : \text{diam } V_x \geq 1/2^j\}, \text{ and}$$
$$\mathcal{H}_{ij} = \{B_x \in \mathcal{B}_{V_x} : V_x \in \mathcal{V}_i, \text{ and diam } B_x \geq 1/2^j\}.$$

Note that two collections $\mathcal{G}_{ij}$ and $\mathcal{H}_{ij}$ are discrete in B_κ by our construction.
Put

$$G_{ij} = \cup\mathcal{G}_{ij}, \ H_{ij} = \cup\mathcal{H}_{ij}, \text{ and } O_{ij} = q(G_{ij} \cup H_{ij}).$$

Then, O_{ij} is clopen in $D_{\kappa,\mathcal{F}_\infty}$, since $q^{\leftarrow}(O_{ij}) = G_{ij} \cup H_{ij}$ is clopen in B_κ, and q is a quotient map. This completes the proof, since we have $O = \cup_{ij\geq 0} O_{ij}$, and hence $\dim D_{\kappa,\mathcal{F}_\infty} = 0$ by [4, Exercise 6.2.C.(b)]. □

We shall show that the topology of our space $D_{\kappa,\mathcal{F}_\infty}$ does not depend on the choice of a collection $\mathcal{F}_\infty$.

Definition 4.3. A family $\mathcal{F} = \cup_{j\geq 0}\mathcal{F}_j$ of closed subset of B_κ is called *universal* when there exists a collection $\mathcal{V} = \cup_{j\geq 0}\mathcal{V}_j$, satisfying the above conditions ¶4.5 – 4.6 as well as $\mathcal{V}_0$ is a cover of B_κ.

Theorem 4.3. *Every decomposition space $D_{\kappa,\mathcal{F}}$ with respect to some universla closed collection $\mathcal{F}$ is homeomorphic each other.*

Proof. Let $\mathcal{F}'$ be another universal collection, and let $\mathcal{V}'$ satisfy ¶4.5 – 4.6. Since $|\mathcal{F}_0| = |\mathcal{F}'_0| = \kappa$, there exists a bijection $\phi_0 : \mathcal{F}_0 \to \mathcal{F}'_0$. For each $V \in \mathcal{V}_0$, let $h_{0,V} : (V, F_V) \to (V', \phi_0(F_V))$ be a $(\mathcal{V}_V, \mathcal{V}'_{V'})$ - preserving homeomorphism given by Theorem 4.2 (a), where V' is the unique $V' \in \mathcal{V}'_0$, which contains the set $\phi_0(F_V)$ by ¶4.5. Let $h_0 : B_\kappa \to B_\kappa$ be the homeomorphism defined by $h_0|V = h_{0,V}$. By recursion, for each $i \geq 1$ we shall define a homeomorphism $h_i : (B_\kappa \backslash E_{i-1}, F_i) \to (B_\kappa \backslash E'_{i-1}, F'_i)$, and a bijection $\phi_i : \mathcal{F}_i \to \mathcal{F}'_i$, which satisfy that, for each $V \in \mathcal{V}_U \subset \mathcal{V}_i$, where $U \in \mathcal{V}_{i-1}$,

¶ **4.8.** $h_i|V = h_{i,V} : (V, F_V) \to (V', \phi_i(F_V))$ is a $(\mathcal{V}_V, \mathcal{V}'_{V'})$ - preserving homeomorphism;

¶ **4.9.** V' is the unique element of $\mathcal{V}'_i$, which satisfies that $V' \supset \phi_i(F_V)$;

¶ **4.10.** $\phi_i|\mathcal{F}_{i,U} : \mathcal{F}_{i,U} \to \mathcal{F}'_{i,U'}$ is bijective, where $\mathcal{F}_{i,U} = \{F \in \mathcal{F}_i : F \subset U\}$.

By ¶4.8 and ¶4.10 it holds that

¶ **4.11.** $h_{i+1}(V_{i+1}) \subset h_i(V_i)$, when $V_{i+1} \subset V_i$ and $V_k \in \mathcal{V}_k$ for $k = i, i+1$.

Define $h_\infty : B_\kappa \to B_\kappa$ as follows.

$$h_\infty|F = h_i|F, \text{ where } F \in \mathcal{F}_i, \text{ and}$$
$$h_\infty(x) = \cap\{h_j(V_j) : x \in V_j \in \mathcal{V}_j, j \geq 0\} \text{ for each } x \in M_0.$$

Then, this definition is well-defined, because the point $h_\infty(x)$ exists, ¶4.6 holds and $h_i(V) \in \mathcal{V}'_i$. Note that by Theorem 4.2, ¶4.8, and ¶4.11 we have

¶ **4.12.** h_∞ is a $(\mathcal{V}_i, \mathcal{V}'_i)$ - preserving autohomeomorphism for each $i \geq 0$.

Let $q' : B_\kappa \to D_{\kappa,\mathcal{F}'_\infty}$ be the canonical quotient map. Then, define $g : D_{\kappa,\mathcal{F}_\infty} \to D_{\kappa,\mathcal{F}'_\infty}$ as follows.

$$g(y) = q' \circ h_\infty \circ q^{\leftarrow}(y).$$

Then, $g(y)$ is well-defined and bijective by the definition of h_∞. Since both q and q' are quotient, both g and $g^{\leftarrow} = q \circ h_\infty{}^{\leftarrow} \circ g'^{\leftarrow}$ are continuous. □

Hereafter we shall abbreviate the space $D_{\kappa,\mathcal{F}_\infty}$ as D_κ by virtue of Theorem 4. Finally let Z_κ be the subspace of D_κ consisting of non-trivial equivalent classes (i.e. $Z_\kappa = q(\cup\mathcal{F}_\infty)$).

Remark 4.2. (a) We may think that the above homeomorphism g is a prototype of our general embedding, which shall be given in the final part of this section (see Remark 4.3).

(b) By ¶4.12 we have proved the following proposition, which sounds interesting by itself (compare it with Theorem 3.2).

Proposition 4.1. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two universal collections. Then there exists an autohomeomorphism $h : B_\kappa \to B_\kappa$, which induces a bijection $h^* : \mathcal{F} \to \mathcal{F}'$, defined by $h^*(F) = h(F) \in \mathcal{F}'$ for each $F \in \mathcal{F}$.*

We shall show that Z_κ is homogeneous. For that purpose we begin with:

Proposition 4.2. *Spaces D_κ and Z_κ are h-homogeneous. Moreover, they satisfy HO.*

Proof. Let G be an arbitrary non-empty open subset of $Z_\kappa appar$, and let U be an open subset of D_κ such that $G = U \cap D_\kappa$. Put

$$\mathcal{C}_i^* = \{K \in \mathcal{C}_i : K \subset q^{\leftarrow}(U)\} \text{ and } \mathcal{C}^* = \cup_{i\geq 0}\mathcal{C}_i^* \text{ for } \mathcal{C} = \mathcal{F} \text{ or } \mathcal{V}.$$

We shall show that $\mathcal{F}^*$ is a universal collection for the set $q^{\leftarrow}(U) \approx B_\kappa$. For that purpose, by rearranging the collection $\mathcal{V}^*$ we will construct a collection $\mathcal{V}' = \cup_{i\geq 0}\mathcal{V}_i'$, satisfying ¶4.6 for $q^{\leftarrow}(U)$. For each $x \in U$, there exists a clopen subset $W(x) \in \mathcal{W}(x)$ such that $W(x) \subset U$ by Lemma 4.2. By ¶4.6 every such $W(x)$ is contained in a *maximal* $W(y)$, since any two of them are disjoint or one contains the other. Let $\mathcal{W}_0$ be the collection (moreover, a covering of U) of all maximal $W(y)$'s, and let $\mathcal{V}_0' = q^{\leftarrow}\mathcal{W}_0$. Note that $\mathcal{V}_0'$ is a clopen disjoint covering of $q^{\leftarrow}(U)$, and by the definition of $\mathcal{W}(x)$ it satisfies that:

¶ 4.13. If $H \in \mathcal{V}_0'$, then it holds, for some i, that either $H = V \in \mathcal{V}_i$ or $H = \cup\mathcal{V}_H \cup F_V$ for some $\mathcal{V}_H \subset \mathcal{V}_V$ where $V \in \mathcal{V}_i$.

Let $F_H = F_V$ for each $V \in \mathcal{V}_0'$ in either case of ¶4.13, and let $\mathcal{F}_0' = \{F_H : H \in \mathcal{V}_0'\}$. Note that in the second case in ¶4.13 the collection $\mathcal{V}_H$ is a $\mathcal{P}_\kappa$ - *s. c.* cover $\mathcal{V}_H$ for the pair (H, F_H) by Theorem 4.2 (c), and that it corresponds to the collection $\mathcal{V}_U$ in ¶4.6.

Consider a complete metric ρ^* of $q^{\leftarrow}(U)$, which is defined as follows:

$$\rho^*(x,y) = \begin{cases} \rho(x,y) & \text{if there exists } V \in \mathcal{V}_0' \text{ such that } x, y \in V, \text{ and} \\ 1 & \text{otherwise,} \end{cases}$$

where ρ is any complete metric of B_κ. Then, for each $i \geq 1$, put

$$\mathcal{V}_i' = \{V \in \mathcal{V}^* : V \in \mathcal{V}_U \text{ and } F = F_U \in \mathcal{F}_{i-1}', \text{ where } U \in \mathcal{V}_{i-1}', \text{ and}$$
$$\mathcal{F}_i' = \{F \in \mathcal{F}^* : F = F_V \text{ for some } V \in \mathcal{V}_i'\}.$$

Then $\mathcal{V}_i'$ satisfies ¶4.5 – 4.6 since $\mathcal{V}_\infty$ satisfies them (note that $\mathcal{V}_0'$ satisfies them by Theorem 4.2 (c)). By Theorem 4.3 there exists a homeomorphism $h : U \to D_\kappa$. Hence, the corresponding subspaces, consisting of all non first countable points G in U, and Z_κ are homeomorphic by the restriction $h|G$ of h. □

Proposition 4.3. *The space Z_κ is homogeneous.*

Proof. For a pair of distinct points $x, y \in q(\cup\mathcal{F}_0)$, it is easy to see that there exists a homeomorphism $g_0 : Z_\kappa \to Z_\kappa$ such that $g_0(x) = y$. In general case, let

G and H be clopen disjoint neighborhoods of $q(x)$ and $q(y)$, respectively, given by Lemma 4.2. Then, there exists a required homeomorphism, since $Z_\kappa \backslash G$ and $Z_\kappa \backslash H$ are non-empty clopen subsets (and hence are homeomorphic), and there exists a homeomorphism $g_0 : G \to H$ such that $g_0(x) = y$ by the first case. □

Before proving Corollary 4.1, which is a key to construct closed embeddings, we present here a proof of Theorem 1.5, since it is more concrete, and it uses the same idea, handling G_δ subsets (hence, it helps a lot to imagine the heart of the proof of the corollary).

Proof of Theorem 1.5. Let $\tilde{f} : M \to \tilde{L}$ be the EE-map, defined in Example 2.3, where $M = P^* \times P_0$ and $P^* = (P_0 \cup Q_2)$. We will show that $\tilde{L}$ is homeomorphic to D_ω (hence, the maximal non-first countable subset L is homeomorphic to Z_ω, which is homogeneous by the above proposition). It suffices to show that the collection $\mathcal{F} = \{\Delta^n_{i,j} : n = 1, 2, \ldots ; 1 \leq i \leq 2^n : 1 \leq j \leq 2^{n-1}\}$ is a universal collection of M. We begin with the following lemma.

Lemma 4.6. *Let $\mathcal{G}$ be a countable collection of $\tilde{f}$-saturated clopen subsets, which covers a clopen subset H of M. Then, there exists a disjoint refinement $\mathcal{H}$ of $\mathcal{G}$, which covers H and consists of clopen $\tilde{f}$-saturated subsets.*

Proof. For each $G_i \in \mathcal{G}$ put $H_i = G_i \backslash \cup_{k<i} G_k$. Then, one can shows that $\{H_i\}$ satisfies all the conditions of the lemma. □

From the construction of $\tilde{L}$ we can show that:

¶ 4.14. For each point $x \in M$, there exists a neighborhood base $\mathcal{B}$ in M such that each B (respectively, $B\backslash\Delta^n_{i,j}$), where $B \in \mathcal{B}$, is $\tilde{f}$-saturated when $x \notin \mathcal{F}$ (respectively, when $x \in \Delta^n_{i,j}$).

Indeed, for each two numbers $m \in \omega$ and $q \in Q\backslash Q_2$, take open intervals $(p_x - q, p_x + q)$ and (r''_x, r'_x), and put

$$B_m(q, x) = M \cap ((p_x - q, p_x + q) \times (r''_x, r'_x)),$$

where $x = \{p_x\} \times \{r_x\}$ and r''_x, r'_x are the unique subsequent binary rational numbers with rank at most m such that $r''_x < r_x < r'_x$. Then, $\mathcal{B} = \{B_m(q, x) : m \in \omega, q \in Q\backslash Q_2\}$ satisfies ¶4.14. Indeed, for any neighborhood U of x, take open intervals H' and O' such that $x \in G = M \cap (H' \times O') \subset U$. Take a sufficiently large $n_0 (n_0 > n$ when $x \in \Delta^n_{i,j})$ such that it satisfies that

$$O' \supset \Delta^m_{i,j} \text{ if } \Delta^m_{i,j} \cap (r''_x, r'_x)) \neq \emptyset \text{ and } m \geq n_0.$$

In case $x \notin \mathcal{F}$ (respectively, $x \in \Delta^n_{i,j}$) take open intervals $H \subset H'$ and $O \subset O'$

such that

$$(H \times O) \cap (\cup_{m \leq n_0}(\cup_{i,j}\Delta^n_{i,j})) = \emptyset$$
$$(\text{respectively, } (H \times O\backslash\Delta^n_{i,j}) \cap (\cup_{m \leq n_0}(\cup_{i,j}\Delta^n_{i,j}) = \emptyset).$$

Finally, take $l \geq n_0$ and $q \in Q\backslash Q_2$ such that $B_l(q,x) \subset H \times O$. Then, it is ready to see that $B_l(q,x) \subset U$ and it satisfies the remaining conditions in ¶4.14. Moreover, we have:

¶ 4.15. For each $F = \Delta^n_{i,j} \in \mathcal{F}$, there exists a clopen neighborhood base in M consisting of $\tilde{f}$-saturated subsets.

Indeed, for any neighborhood U of F, let $\{B_i\}$ be a countable clopen collection of $\tilde{f}$-saturated subsets such that $F \subset \cup B_i \subset U$ by ¶4.14 and the Lindelöffness of F. Note that we may assume that it holds that $q_i > q_{i+1}$ and $m_i \geq n$ for each i, where $B_i = B_{m_i}(q_i, x_i)$. Then, the set

$$H = \cup B_i$$

is $\tilde{f}$-saturated and is clopen in M, since it holds that $H \cap (\{j/2^n\} \times P^*) = F$.

Let ρ^* and ρ_E be a bounded complete metric ≤ 1 of P^*, and the usual Euclidean metric of the closed unit interval $I = [0,1]$, respctively. Put

$$d = \sqrt{(\rho^*)^2 + (\rho_E)^2}.$$

Note that:

¶ 4.16. For any $\varepsilon > 0$, there are only finitely many $\Delta^n_{i,j}$ such that their diameter $> \varepsilon$, where we use the metric d of $P^* \times I$.

At first we shall construct an ω-complete refining sequence $\{\mathcal{H}_i\}_{i \geq 0}$ of M such that there exists a decomposition $\{\mathcal{F}_i\}$ of $\mathcal{F}$ such that mesh $\mathcal{F}_i < 1/2^i$ with respect to the metric induce by $\{\mathcal{H}_i\}_{i \geq 0}$. Let $\{x_i\}_{i \geq 0}$ be an enumeration of $I\backslash P_0$, and put $K_i = \cup_{n \leq i} P^* \times \{x_n\}$. Put also

$$\mathcal{F}'_i = \{F \in \mathcal{F} : cl_d(F) \cap K_i \neq \emptyset\}.$$

Using ¶4.16 we see that:

¶ 4.17. Every $\mathcal{F}'_i$, where $i \geq 0$, is infinite discrete in M.

On the other hand, for each $H \in \mathcal{F}\backslash\mathcal{F}'_i$, we have:

¶ 4.18. For the d-closed set $K = cl_d(H)$ it holds that $K \cap K_i = \emptyset$.

Let

$$\mathcal{F}_0^* = \mathcal{F}_0' \cup \{F \in \mathcal{F} : d(F) \geq 1/2\}.$$

Then, $\mathcal{F}_0^*$ is discrete by ¶4.16 and ¶4.17. Hence, by ¶4.14 and 4.15 there exists a discrete $\tilde{f}$-saturated clopen collection $\mathcal{H}_0' = \{H_F : F \in \mathcal{F}_0^*\}$ such that $F \subset H_F$ and $d(H_F) \leq 1$ (note that $d(F) \leq 1/2$ by the definitions of $\Delta_{i,j}^n$ and d).

Let $\mathcal{H}_0^*$ be a countable cover of the clopen subset $M \backslash \cup \mathcal{H}_0'$ by ¶4.14 and ¶4.15, which consists of clopen $\tilde{f}$-saturated subsets of diameter ≤ 1 with respect to d, and which also satisfies ¶4.18 for each $H \in \mathcal{H}_0^*$. By Lemma 4.6 we can assume that all members of $\mathcal{H}_0^*$ are disjoint. Put

$$\mathcal{H}_0 = \mathcal{H}_0' \cup \mathcal{H}_0^*.$$

Assume that we have defined a discrete collection $\mathcal{F}_{i-1}^*$, and a discrete clopen cover $\mathcal{H}_{i-1}$. Let

$$\mathcal{F}_i^* = \mathcal{F}_i' \cup \{F \in \mathcal{F} : d(F) \geq 1/2^{i+1}\}.$$

Note that $\mathcal{F}_i^*$ is discrete, $\mathcal{F}_{i-1}^* \subset \mathcal{F}_i^*$ and $\mathcal{F} = \cup_i \mathcal{F}_i^*$. By the collectionwise normality of M take a clopen discrete collection $G_i = \{G_F : F \in \mathcal{F}_i^*\}$ such that $F \subset G_F$ for each $F \in \mathcal{F}_i^*$. For each $F \in \mathcal{F}_{i-1}^*$, take a countable infinite discrete $\tilde{f}$-saturated clopen collection $\mathcal{H}_F$ such that, for each $H \in \mathcal{H}_F$:

¶ 4.19. It holds that $d(H) \leq 1/2^i$, $H \cap F \neq \emptyset, H \backslash F$ is $\tilde{f}$-saturated, satisfying ¶4.18, and $F \subset \cup \mathcal{H}_F \subset G_F \cap H_F$, where $H_F \in \mathcal{H}_{i-1}$.

For each $F \in \mathcal{F}_i^* \backslash \mathcal{F}_{i-1}^*$, take a $\tilde{f}$-saturated clopen set H_F of diameter $\leq 1/2^i$ with respect to d, which also satisfies that $H_F \subset G_F$. Put

$$\mathcal{H}_i' = \{H_F : F \in \mathcal{F}_i^* \backslash \mathcal{F}_{i-1}^*\} \cup \{\mathcal{H}_F : F \in \mathcal{F}_{i-1}^*\}$$

Let $\mathcal{H}_i^*$ be a disjoint cover of the clopen subset $M \backslash \cup \mathcal{H}_i'$, which consists of clopen $\tilde{f}$-saturated subsets of diameter $\leq 2^i$ with respect to d, and which also satisfies ¶4.18 for each $H \in \mathcal{H}_i^*$. Put

$$\mathcal{H}_i = \mathcal{H}_i' \cup \mathcal{H}_i^*.$$

By ¶4.19 it is ready to see that $\{\mathcal{H}_i\}$ is a κ-complete refining sequence of M, and that mesh $\mathcal{F}_i^* \leq 1/2^i$ with respect to its induced metric. Note that it is not difficult to see that we can define $\mathcal{F}_i$ and $\mathcal{V}_i$ so that $\mathcal{F}$ is a universal collection by using the above constructions of $\mathcal{F}_i^*$ and $\mathcal{H}_i^*$. □

Corollary 4.1. *Let G be a G_δ subset in D_κ such that $G \supset q(\cup \mathcal{F}_\infty)$. Then, G is homeomorphic to D_κ.*

An outline of the proof. Put $K = B_\kappa \backslash q^{\leftarrow}(G)$ and let $K = \cup_{j\geq 0} K_j$, where each K_j is non-empty closed set in B_κ and $K_j \subset K_{j+1}$. Note that each K_j is nowhere dense, since $K \cap (\cup \mathcal{F}_\infty) = \emptyset$ and $\cup \mathcal{F}_\infty$ is dense by ¶4.7. Then, for each open subset $M \backslash K_j$, by a parallel argument with the proof of Proposition 4.2, we can construct a clopen disjoint collection $\mathcal{V}_j \subset \mathcal{V}'_\infty$, which corresponds to the collection $\mathcal{V}'_0$ in there. Put $\mathcal{V}^*_j = \mathcal{V}'_j | q^{\leftarrow}(G)$. Then, by Theorem 4.2 (b) $\mathcal{V}^*_j$ consists of $\mathcal{P}_\kappa$-*s. c.* covers. Moreover, by a same idea, avoiding the set K_j, in the above proof of Theorem 1.5, we can make that $\mathcal{V}^*_{j+1}$ refines $\mathcal{V}^*_j$ for each $j \in \omega$, and that the mesh $\mathcal{V}^*_j \leq 1/2^j$ with respect to the induced metric of some κ-complete refining sequence of $q^{\leftarrow}(G)$. We leave the details to the reader. Hence, the corollary follows from Theorem 4.3, since $\mathcal{F}_\infty$ is a universal collection of $q^{\leftarrow}(G) \approx B_\kappa$. □

c) Universality of D_κ. Let $f : M \to Y$ be a closed onto map from a complete metric space M of $w(M) \leq \kappa$. By Facts 2.1 and 2.2 we can assume that $M = B_\kappa$ and f is irreducible (see also [2]). In this section we shall show that Y can be embedded in D_κ as a closed subset. For that purpose we shall use the control due to S. Watson in [22]. In the present case, however, we have a different situation from there, since we have a dense subset Y_0 of Y, consisting of first countable points (in other words, Y_0 is the maximal metrizable part of Y). Moreover, we must see that there exists a *closed* embedding. For that purpose, we shall construct a closed embedding $g : Y \to G$, where G is some dense G_δ subset G of D_κ, and shall show that G is homeomorphic to D_κ.

By [8, 14, 23] it holds that $Y \backslash Y_0$ is σ-discrete. Put $Y \backslash Y_0 = \cup_{i\geq 0} D_i$, where each D_i is closed discrete in Y and pairwise disjoint. Considering $Y \oplus D_\kappa$, if necessary, we can assume that $|D_i| = \kappa$ and that any $f^{\leftarrow}(y)$ is non-compact, whenever $y \in D_i$ and $i \geq 0$.
Put $\mathcal{Z}_i = \{Z_y = f^{\leftarrow}(y) : y \in D_i\}$, $Z_i = \cup \mathcal{Z}_i$, and $M_i = M \backslash \cup_{j<i} Z_j$.

Remark 4.3. The author explains here for the reader the brief explanation of how to produce a desired embedding g. Our embedding is a kind of limit of (not necessarily continuous) mappings. For each i-th step, we construct a homeomorphism $\tilde{h}_i$ between open sets M_i and $T_i \backslash \cup_{j<i} \tilde{h}_j(Z_j)$, where T_i is a closed subset of B_κ. By means of $\tilde{h}_i$, we can correspond neighborhoods of each set Z_y in M and those of closed set $\tilde{h}_i(Z_y) \in \mathcal{F}_\infty$ in T_i freely. But, it does not induce a mapping between open subsets of Y and $q(T_i) \subset D_\kappa$, since it is not *s. c.* cover preserving. Hence, we should adjust it to a correspondence $\varphi_i : M_i \to S_i$, where S_i is a closed subset of T_i, which preserves certain *s. c.*-covers about Z_i and $\tilde{h}_i(Z_i)$. Though it must adjust neighborhoods between each point of D_i in Y and that of $q(\tilde{h}_i(Z_i))$ in $q(T_i)$, we remind the reader again that φ_i needs not induce a continuous mapping between subsets of our Lašnev spaces either,

because it controls nothing about points in D_j for $j > i$. Thus, by recursion with respect to i, we can adjust topology about all the points in $Y\backslash Y_0$ and those in $g(Y\backslash Y_0)$. By choosing *s. c.* covers for S_i carefully, we can also adjust topology about the remaining metrizable points in Y_0 and those in $g(Y_0)$.

Since it holds that $|D_i| = \kappa$ and $w(f^{\leftarrow}(y)) \leq \kappa$ for each $y \in Y$, we can apply Fact 4.2 and ¶4.5, so that the following property holds.

¶ 4.20. For any Z_y, where $y \in D_0$, there exists $F_y \in \mathcal{F}_0$, and a closed embedding $h_y : Z_y \to F_y$, with $F_y \cap F_{y'} = \emptyset$ if $y \neq y' \in D_0$.

Let $\mathcal{G}_{-1}$ be a clopen disjoint cover of B_κ, which *separates* D_0
(i.e. $\forall_y \in D_0[|\{G \in \mathcal{G}_{-1} : y \in G\}| = 1]$).
Let $h_0 : Z_0 \to B_\kappa$ be the closed embedding defined by $h_0|Z_y = h_y$. Applying Lemma 3.3 for $A = F_0$ and $B = h_0(Z_0) = H_0$, we have a closed subspace T_0 of B_κ and a subcollection $\mathcal{H}_0$ of $\mathcal{V}_0$, which is a *s. c.* cover for (T_0, H_0). Note that by Fact 4.1 the space T_0 is homeomorphic to B_κ and H_0 is its nowhere dense closed subset. Applying Corollary 3.1 for $G = Z_0$, $H = H_0$, we have a homeomorphism $\tilde{h}_0 : M \to T_0$ which is an extension of h_0. For each $y \in Y\backslash D_0$ let

$$G_y = f^{\leftarrow}(Y\backslash f(M\backslash U_y)),$$

where $U_y = st(Z_y, \tilde{h}^{\leftarrow}(\mathcal{H}_0))$. Put

$$\mathcal{G}' = \{G_y : y \in Y\backslash D_0\}.$$

Since $\dim Y = 0$ and Fact 3.1 holds for Y, take a clopen disjoint refinement $\mathcal{G}_0$ of $\mathcal{G}'$, which is a *s. c.* cover for (Y, D_0) such that

$$f^{\leftarrow}\mathcal{G}_0 \text{ refines } \{B_0(Z_y) : y \in Y\backslash D_0\},$$

where $B_i(A)$ is the $1/2^i$-neighborhood of A. Note that the following ¶4.21 is valid for $i = 0$, since ¶4.7 holds.

¶ 4.21. For each $G \in \mathcal{G}_i$ we can take a $U_G \in \cup_{k \geq i} \mathcal{V}_k$, such that $U_G \subset \tilde{h}_i(f^{\leftarrow}(G))$.

Let $\mathcal{O}_0 = \{U_G : G \in \mathcal{G}_0\}$. Then by Fact 3.4 we have a closed set $S_0 = S_A$, where $A = H_0$.
Let $\varphi : M \to S_0$ be a one to one correspondence defined by $\varphi_0|Z_0 = \tilde{h}|Z_0$, and $\varphi_0|f^{\leftarrow}(G)$ is an arbitrary homeomorphism onto U_G for each $G \in \mathcal{G}_0$. By recursion we have, for each $i \geq 1$, two closed subsets T_i and S_i in B_κ, a *s. c.* cover $\mathcal{G}_i$, two subcollections $\mathcal{H}_i, \mathcal{O}_i \subset \mathcal{V}_\infty$, a homeomorphism satisfying the following ¶4.22 – 4.28.

¶ **4.22.** $\mathcal{G}_i$ refines $\mathcal{G}_{i-1}$, $\mathcal{G}_i$ separates D_{i+1}, and $\mathcal{G}_i$ is a clopen disjoint *s. c.* cover for $(Y, \cup_{j\leq i} D_j)$.

¶ **4.23.** $f^{\leftarrow}\mathcal{G}_i$ refines $\{B_i(Z_y) : y \in Y \setminus \cup_{j\leq i} D_j\}$, and $\tilde{h}_i(Z_i) = H_i \subset \cup\mathcal{F}_\infty$.

¶ **4.24.** $\mathcal{H}_i$ refines $\mathcal{H}_{i-1}$, and $S_i = \cup\mathcal{O}_i \cup (\cup_{j\leq i} H_j) \subset T_i = \cup\mathcal{H}_i \cup (\cup_{j\leq i} H_j) \subset S_{i-1}$.

¶ **4.25.** For each $y \in G \cap D_i$, where $G \in \mathcal{G}_{i-1}$ (by ¶4.22 G is unique and is different for each y), the closed embedding $h_y : Z_y \to F_y = F_{V_y}$ is given by ¶4.5 and Fact 4.2, where it holds that $V_y = U_G \in \mathcal{O}_{i-1}$ and U_G is chosen by ¶4.21. We also apply Lemma 3.3 to get $T_y = T_B$ and $\mathcal{H}_y = \mathcal{U}_B$ for $B = \tilde{h}_i(Z_y) \subset A = F_y$, $X = V_y$, $\mathcal{U} = \mathcal{V}_{V_y}$.

¶ **4.26.** $\mathcal{H}_i = \cup_{y\in D_i}\mathcal{H}_y \cup \{U_G \in \mathcal{O}_{i-1} : G \cap D_i = \emptyset\}$, and $\tilde{h}_i : M_i \to (T_i \setminus \cup_{j<i} H_j)$ is a homeomorphism, satisfying that, for each $G \in \mathcal{G}_i$ with $G \cap D_i \neq \emptyset$, the restriction $\tilde{h}_i | f^{\leftarrow}(G)$ is an extension of h_y onto T_y given by Corollary 3.1, and the restriction in case $G \cap D_i \neq \emptyset$ is an arbitrary homeomorphism onto U_G.

¶ **4.27.** $\mathcal{O}_i$ is defined by ¶4.21 in such a way that $\mathcal{O}_i = \{U_G : G \in \mathcal{G}_i\}$ refines $\mathcal{O}_{i-1}$, and for each $y \in D_i$, the collection $\mathcal{O}_y = \{O \in \mathcal{O}_i : O \subset V_y\}$ satisfies Fact 3.4 for the pair (S_B, B), where $S_B = \cup\mathcal{O}_y \cup B$ and $B = \tilde{h}_i(Z_y)$.

¶ **4.28.** $\varphi_i : M_i \to S_i \setminus \cup_{j<i} H_j$ is a one to one correspondence defined by $\varphi_i | Z_i = \tilde{h} | Z_i$, and $\varphi_i | f^{\leftarrow}(G)$ is an arbitrary homeomorphism onto U_G for each $G \in \mathcal{G}_i$.

The following lemma is used not only for defining a closed embedding, but also for showing its required properties.

Lemma 4.7. (a) *$\{\mathcal{G}_i\}_{i\geq 0}$ is a clopen base of Y_0 in Y.*
(b) *$\mathcal{O}_i \subset \cup_{k\geq i}\mathcal{V}_k$, and for any $G_j \in \mathcal{G}_j$, where $j = i, i-1$, it holds that if $G_i \subset G_{i-1}$, then $O_i = \tilde{h}_i(f^{\leftarrow}(G_i)) \subset O_{i-1} = \tilde{h}_{i-1}(f^{\leftarrow}(G_{i-1}))$.*
(c) *The map φ_i has the following property: Every $\varphi_i(f^{\leftarrow}N)$ is a clopen neighborhood of $\tilde{h}_i(Z_y)$ in S_i, where $N = \{y\} \cup (\cup\mathcal{C})$ is a neighborhood of $y \in D_i$ in Y for some subcollection $\mathcal{C}$ of $\mathcal{G}_i$. On the other hand, for each neighborhood E of $\tilde{h}_i(Z_y)$ in S_i, there exists a subcollection $\mathcal{C}$ of $\mathcal{G}_i$ such that $N = \{y\} \cup (\cup\mathcal{C})$ is a neighborhood of $y \in D_i$ in Y and $\varphi_i(f^{\leftarrow}N) \subset E$.*

Proof. (a) It is shown without difficulties by a parallel argument in [3, Theorem 4.4.15]. The remaining proofs for (b) and (c) are easily obtained by our construction and Fact 3.4. □

Define $g : Y \to D_{\kappa,\lambda}$ as follows.

$$\begin{aligned} g(y) &= q \circ \tilde{h}_i(F_y) \text{ for every } y \in D_i \text{ and,} \\ &= \cap_{i \geq 0} q(O_i) \text{ for each } y \in Y_0, \text{ where } O_i = \varphi_i(f^{\leftarrow}(G_i)) \in \mathcal{O}_i \\ &\quad \text{and } \{G_i \in \mathcal{G}_i : y \in G_i\}. \end{aligned}$$

Note that g is well-defined, since $g(y)$ is uniquely determined by ¶4.6, ¶4.27, and Lemma 4.6 (b) (compare with ¶4.12). For each $y \in D_i$, let $\mathcal{N}(y)$ be the collection of all clopen subsets of Y such that, for each $N \in \mathcal{N}(y)$, there exists a subcollection $\mathcal{G}_N \subset \mathcal{G}_i$ satisfying that $N = \cup \mathcal{G}_N \cup \{y\}$. Put also

$$\mathcal{N}(y) = \{G : x \in G \in \mathcal{G}_i\} \text{ for } y \in Y_0.$$

Then, the following lemma, corresponding to Lemma 4.1, holds. We omit its parallel proof.

Lemma 4.8. *For each $y \in Y$ the collection $\mathcal{N}(y)$ is its clopen neighborhood base.*

Proof of Theorem 4.1. By Propositions 4.2 and 4.3 it suffices to show the following two lemmas. At first we shall show that g is injective. By its definition, Lemma 4.6 (a) and (b), it is ready to see that $g|(Y \backslash Y_0)$ is one to one and *into* $D_\kappa \backslash q(M_0)$ by ¶4.25 and ¶4.26. On the other hand, $g|Y_0$ is one to one and *into* $q(M_0)$, because $\cap_{i \geq 0} O_i \subset M_0$. Hence, g is injective. Next, we show that:

Lemma 4.9. *The map $g : Y \to D_{\kappa,\lambda}$ is an embedding.*

Proof. (i) *Continuity of g.* For arbitrary neighborhood W of $x = g(x) \in D_{\kappa,\lambda}$, we can assume that $W \in \mathcal{W}(x)$ by Lemma 4.2. Hence,
(a) in case $y \in Y_0$, there exists a unique point $q^{\leftarrow}(x)$ such that $q^{\leftarrow}(x) \in V$ and $W = q(V)$ for some $V \in \mathcal{V}_i$. On the other hand, by the definition of g, we have $q^{\leftarrow}(x) \in O_i$, where $O_i = \varphi_i(f^{\leftarrow}(G))$ and $y \in G \in \mathcal{G}_i$. Then, $g(G) \subset W$ by ¶4.21, ¶4.26, and Lemma 4.7 (b).
(b) In case $y \in D_i$ for some $i \geq 0$, we can put $W = q(B)$, where $B \in \mathcal{B}_V$ for some $V \in \mathcal{V}_{k_i}$ and $k_i \geq i$. By Lemma 4.7 (c) we have a neighborhood N of y such that $\varphi_i(N) \subset E = q^{\leftarrow}(W)$. Then $g(N) \subset W$ by the definition of g and Lemma 4.7 (b).
(ii) *Continuity of $g^{\leftarrow}$.* (a) Assume that $y \in Y_0$, and suppose that $G_i \in \mathcal{G}_i$ and that $y \in G_i$. Then, $x = g(y) \in q(O_i)$ and $y \in g^{\leftarrow}(q(O_i)) \subset G_i$ by Lemma 4.7 (b) and $g^{\leftarrow}$ is one to one. Then, $g^{\leftarrow}$ is continuous at x by Lemma 4.8 and by the fact that $q(O_i)$ is clopen in D_κ.

(b) In case $y \in D_i$, using Lemma 4.8, it can be seen that $g^{\leftarrow}$ is continuous at x by a parallel argument for g. □

Lemma 4.10. *The set $g(Y)$ is closed in a G_δ subset G in D_κ, which satisfies Corollary 4.1, hence G is homeomorphic to D_κ.*

Proof. Note that the set $q(\cup_{k\leq i} H_k)$ is closed, since T_i is closed in B_κ and $\cup_{j\leq i} H_j$ is closed in T_i (by Lemma 4.7 (c) and the construction of $\mathcal{O}_i$ in ¶4.27 we can show without difficulties that each $q(H_i)$ is closed discrete). It holds that

$$g(\cap_i S_i) \supset g(Y) \supset \cup_i q(H_i),$$

by the definition of g. Hence, since S_i is closed for each i, it holds that

$$cl(g(Y)) \backslash g(Y) \subset q(M_0).$$

On the other hand, put $X = (q|\cap_i S_i)^{\leftarrow}(g(Y))$ and $h = q|X : X \to Y$. Then h is closed irreducible, since $g(Y_0)$ is dense in $g(Y)$ (note also that $h|h^{\leftarrow}(Y_0)$ is a homeomorphism). Hence, X is a G_δ subset of B_κ, since it is completely metrizable by [19, Theorem 1]. Therefore it holds that

$$K = cl(X) \backslash X \text{ is an } F_\sigma \text{ subset of } B_\kappa \text{ and } K \subset M_0.$$

Let $G = q(B_\kappa \backslash K)$. Then, it is easy to see that G satisfies the required property, since q is a closed map, and that $q|M_0$ is one to one. □

5. Homogeneous universal spaces for Watson spaces

a) Preliminaries. Let us fix a point $* = (0, 0, \ldots) \in B_\kappa$. Let Q_κ be the subspace of B_κ consisting of all points all but finitely many coordinates of which are equal those of $*$. Then, Q_κ is σ-discrete, and $|Q_\kappa| = \kappa$ by its definition. In particular, Q_ω is homeomorphic to Q, and it holds that:

¶ 5.1. For every infinite $\mu \leq \kappa$, there exists a closed nowhere dense subset $F_\mu = Q_\kappa \cap ({}^\omega I_\mu)$ of Q_κ, where $I_\mu \subset \kappa$ satisfies that $|I_\mu| = \mu$ and $|\kappa \backslash I_\mu| = \kappa$ (hence F_μ is homeomorphic to Q_μ and $|F_\mu| = w(F_\mu) = \mu$).

We outline here the method by which we shall produce a homogeneous universal space for Watson spaces. One of the main points is developing a method how to produce a homeomorphism between closed nowhere dense subsets, using only their two preassigned refining sequences (see also Remark 4.1). For example, we must show Lemma 5.2 corresponding to Theorem 4.2. In other words, we need

some method, which produces an isometric homeomorphism $h : G \to H$, satisfying that $\mathcal{B}' = h(\mathcal{B})$, when $G \approx H \approx Q_\kappa$, squeezing from *preassigned* standard *s. c.* covers $\mathcal{B}$ and $\mathcal{B}'$ of (Q_κ, G) and (Q_κ, H), respectively.
For that purpose, we can use the method of restricting the homeomorphism h defined in ¶4.4 to the particular subset A satisfying the following condition, where $\mathcal{B} = \{B(\alpha_0, \ldots, \alpha_i)\}$ is a preassigned κ-complete refining sequence of B_κ.

¶ 5.2. The set A is dense in B_κ, and for each $x \in A$, there exists i and the unique $B(\alpha_0, \ldots, \alpha_i)$ such that

$$\{x\} = \cap_{n>i} B(\alpha_0, \ldots, \alpha_i, 0_{i+1}, \ldots, 0_n).$$

For a given subset A we use the following renaming technique of complete refining sequences.

Lemma 5.1. *Let A be a dense σ-discrete subset of B_κ. Suppose that $\mathcal{B} = \{\mathcal{B}_i\}$ is a complete refining sequence of B_κ. Then, we can rename each member of $\mathcal{B}_i$ so that it satisfies ¶5.2 for the set A.*

Proof. Note that A is homeomorphic to Q_κ by [9, Theorem 5] and, hence put $A = \cup_{i \geq 0} D_i$, where each D_i is closed discrete in B_κ and $D_i \subset D_{i+1}$. For each $B \in \mathcal{B}_0$ let x_B be any point such that $x_B \in B \cap D_i$, where $i = \min\{j : B \cap D_j \neq \emptyset\}$. Assume that we have defined $E_m = \{x_B : B \in \mathcal{B}_m\}$. For each $B \in \mathcal{B}_{m+1}$ let x_B be the unique point such that $x_B \in B \cap D_m$ if $B \cap E_m \neq \emptyset$. Otherwise, let x_B be any point such that $x_B \in B \cap D_i$ and $i = \min\{j : B \cap D_i \neq \emptyset\}$. Using x_B, we shall rename B, if necessary, such that ¶5.2 holds for each $x \in A \cap E_i$. For the members of $\mathcal{B}_0$ we do not change their indices, and we assume that we rename all the members of $\mathcal{B}_m$. For each $B \in \mathcal{B}_{m+1}$, there is the unique element B^* of $\mathcal{B}_m$ such that $B \subset B^* = B(\alpha_0, \ldots, \alpha_m)$. Hence, we change the first k-coordinates of B to that of B^* if necessary. Then, we change its last coordinate α_m in such a way that $B = B(\alpha_0, \ldots, \alpha_{m-1}, 0)$ only when $x_B = x_{B^*}$.
By the above definition of renaming we see that ¶5.2 holds for each $x \in A \cap E_i$. Hence, we shall show the following property, which concludes our proof.

$$A = \cup_{i \geq 0} E_i.$$

Indeed, for each $x \in D_i$, take a sufficiently small $B \in \mathcal{B}_m$, with $m > i$ such that $\{x\} = B \cap D_i$. Then, $x = x_B$ if $x \notin E_{m-1}$. □

By the above lemma we can show Theorem 3.2 as follows.

Proof of Theorem 3.2. Since a proof for the Cantor set is achieved by a parallel way, we specify our proof for the case $X = B_\kappa$. By Lemma 5.1 we can assume that we have κ-complete refining sequences $\mathcal{B}$ and $\mathcal{B}'$ such that ¶5.2 holds for D and E, respectively. Hence, the natural homeomorphism h defined in ¶4.4 satisfies the theorem. □

b) $\mathcal{R}_\kappa$-covers. Now we can define final *s. c.* covers, which will be used in our construction. One of the key points is that it is preserved to certain clopen subsets (see Lemma 5.2).

Definition 5.1. For a fixed $\kappa \geq \omega$ let $\mathcal{R}_\kappa$ be the class of all the pairs (X, A), where A is a nowhere dense closed subset of X, and $A \approx X \approx Q_\kappa$. Then, for each pair $(X, A) \in \mathcal{R}_\kappa$, we can assume that $X = Q_\kappa \subset B_\kappa$. Let $F = cl_{B_\kappa} A$, and note that F is nowhere dense in B_κ and $F \approx B_\kappa$ by a characterization of B_κ (Fact 4.1). Take a κ-complete refining sequence $\mathcal{B}$ of F, satisfying ¶4.1 – ¶4.3. Let $\mathcal{U}$ be a standard *s. c.* cover for the pair (B_κ, F) with respect to $\mathcal{B}$. Then, the *s. c.* cover $\mathcal{U}|X$ defined in Fact 3.3 is called an $\mathcal{R}_\kappa$ - *s. c.* cover for the pair (X, A). By Lemma 5.1 we can assume that ¶5.2 holds for A and $\mathcal{B}$.

Lemma 5.2. (a) *For each (X, A), and $(Y, B) \in \mathcal{R}_\kappa$, there exists a $(\mathcal{V}, \mathcal{V}')$ - preserving homeomorphism $h : (X, A) \to (Y, B)$, where $\mathcal{V}$ and $\mathcal{V}'$ are any given $\mathcal{R}_\kappa$ - s. c. covers for (X, A) and (Y, B), respectively.*
(b) *Let $\mathcal{V}$ be an $\mathcal{R}_\kappa$ - s. c. cover for a pair (X, A). Suppose that G is a clopen neighborhood of A in X such that $G = A \cup (\cup \mathcal{V}_G)$ for some subcollection $\mathcal{V}_G$ of $\mathcal{V}$. Then, $\mathcal{V}_G$ is a $\mathcal{P}_\kappa$ - s. c. cover for the pair (G, A).*

Outline of Proofs. (a) Let $\mathcal{U}$ and $\mathcal{U}'$ be standard *s. c.* covers in Definition 5.1 such that $\mathcal{V} = \mathcal{U}|X$ and $\mathcal{V}' = \mathcal{U}'|Y$. We also assume that $\mathcal{U}$ and $\mathcal{U}'$ are standard *s. c.* covers for the pairs (B_κ, clA) and (B_κ, clB), with respect to κ-complete refining sequences $\mathcal{B}$ of clA and $\mathcal{B}'$ of clB, respectively. Let $h : clA \to clB$ be the isometric homeomorphism given by ¶4.4. Then, by Definition 5.1 it satisfies that $h|A$ is an isometric homeomorphism between A and B, and that $\mathcal{V}' = (h|A)(\mathcal{V})$. On the other hand, $\mathcal{V}$ and $\mathcal{V}'$ satisfy the remaining conditions in Theorem 3.1, since $\mathcal{V}$ and $\mathcal{V}'$ are standard *s. c.* covers with respect to refining sequences $\mathcal{B}|A$ and $\mathcal{B}'|B = (h|A)(\mathcal{B}|A)$, respectively. Since each element of both collections $\mathcal{V}$ and $\mathcal{V}'$ is homeomorphic, we have a $(\mathcal{V}, \mathcal{V}')$ - preserving homeomorphism $h : (X, A) \to (Y, B)$ by Theorem 3.1
(b) Put $\mathcal{V} = \mathcal{U}|X$, where $\mathcal{U}$ is a standard *s. c.* cover for (B_κ, F) with respect to a κ-complete refining sequence $\mathcal{F}$ of $F = clA \approx B_\kappa$. Put $G = O \cap X$ for some open subset O of B_κ, and let $F^* = O \cap F$. Then, let

$$\mathcal{W}_G = \{W \in \cup_{i \geq 1} \mathcal{W}_i : W \subset O\} \text{ and } H = F^* \cup (\cup \mathcal{W}_G),$$

where $\mathcal{W} = \cup_{i \geq 0} \mathcal{W}_i$ is a refining sequence satisfying ¶3.2 and ¶3.3. Note that $H \cap F = F^*$ and we see that H is open in B_κ. Indeed, for each $x \in F^*$, there exists $F \in \mathcal{F}$ such that $x \in F \subset O$. Hence, by taking sufficiently small F, there exists $W \in \mathcal{W}$ such that $F = W \cap A$ and $W \subset O$ by ¶3.2. Then, $x \in W \subset H$, since $W \in \mathcal{W}_G$. Therefore, both sets F^* and H are homeomorphic to B_κ. Put

$$\mathcal{U}_G = \{U \in \mathcal{U} : U = U(W) = W \setminus \cup \mathcal{W}_{i+1} \text{ for some } W \in \mathcal{W}_G \cap \mathcal{W}_i\}.$$

By a parallel argument with that given in Proposition 3.1 we have the following two assertions. We skip their proofs (see the proof of Theorem 4.2 (c)).

Assertion 2. *The set F^* is nowhere dense closed in H, and the collection $\mathcal{U}_G$ is a standard s. c. cover for the pair (H, F^*).*

Put $\mathcal{V}_0 = \mathcal{V}_G \setminus (\mathcal{U}_G|X)$. Let $\mathcal{V}_0 = \mathcal{U}_0|X$ for some subcollection $\mathcal{U}_0$ of $\mathcal{U}$. Then, it holds that $U \cap H = \emptyset$ for every $U \in \mathcal{U}_0$, since $\mathcal{V}_G$ is a disjoint collection and it holds that $U \cap H \subset O$. Assume that $|\mathcal{U}_0| = \kappa$, since $|\mathcal{U}| = \kappa$ and it is easier to manage the case $|\mathcal{U}| < \kappa$.

Assertion 3. *For the standard s. c. cover $\mathcal{U}_G$ of the pair (H, F^*), the collection $\mathcal{O} = \mathcal{U}_0 \cup \mathcal{U}_G$ is also a standard s. c. cover for the pair (S, F^*), hence $\mathcal{V}_G = \mathcal{O}|X$ is an $\mathcal{R}_\kappa$ - cover for the pair (G, A), where $S = H \cup (\cup\mathcal{U}_0)$.*

We see that $\mathcal{V}_G = \mathcal{O}|G$ satisfies the required property, since F is a σ-discrete dense subset of $F^* \approx B_\kappa$, and we have the following assertion, which can be shown by a parallel method in the proof of Lemma 5.1.

Assertion 4. *For any given κ-complete refining sequence $\mathcal{B} = \{\mathcal{B}_i\}$ of F^*, we can rename elements of $\mathcal{B}$ in such a way that ¶5.2 holds for $A = F$.*

□

c) One more Watson construction. We present here one more Watson construction, which produces a *homogeneous* universal space. We explain our method rather than necessary proofs, since our construction is similar to that given for van Douwen-complete spaces in §4. We also remark the *differences* from [18] for the reader who is familiar to the method give in [17, 18, 22].

Put $Q_\kappa = \cup_{i \geq 0} D_i$, where each D_i is closed discrete in Q_κ, $|D_i| = \kappa$, and pairwise disjoint. Let $\mathcal{V}_0$ be a clopen disjoint cover of Q_κ with $|\mathcal{V}_0| = \kappa$, and the mesh $\mathcal{V}_0 \leq 1$ with respect to some metric of Q_κ. Then by ¶5.1, for each $V \in \mathcal{V}_0$, take a nowhere dense closed subset F_V of V, satisfying that $D_i \cap V \subset F_V \approx Q_\kappa$. Put $\mathcal{F}_0 = \{F_V : V \in \mathcal{V}_0\}$.
By recursion we shall construct two collections $\mathcal{F}_i$ and $\mathcal{V}_i$ for each i, satisfying the following ¶5.3 – ¶5.4.

¶ 5.3. $\mathcal{V}_i$ is a clopen disjoint collection with $|\mathcal{V}_i| = \kappa$ and $\mathcal{F}_i = \{F_V : V \in \mathcal{V}_i\}$, where F_V is a closed nowhere dense subset of V, and is homeomorphic to Q_κ.

¶ 5.4. The mesh $\mathcal{V}_i \leq 1/2^i$ with respect to some metric of Q_κ, and $\mathcal{V}_i = \cup_{U\in\mathcal{V}_{i-1}}\mathcal{V}_U$, where $\mathcal{V}_U$ is an $\mathcal{R}_\kappa$ - *s. c.* cover for the pair (U, F_U), and $F_i \supset D_i$, where $F_i = \cup\mathcal{F}_i$.

Put also

$$\mathcal{V}_\infty = \cup_{j\geq 0}\mathcal{V}_i,\ \mathcal{F}_\infty = \cup_{j\geq 0}\mathcal{F}_j,\ \text{and } E_i = \cup_{j\leq i}F_j.$$

Note that the properties ¶5.3 – 5.4 imply that:

¶ 5.5. The collection $\mathcal{V}_i$ refines $\mathcal{V}_{i-1}$, $\cup\mathcal{F}_i = Q_\kappa$, each E_i is nowhere dense closed in the space Q_κ, $\mathcal{V}_i$ is a *s. c.* cover for (Q_κ, E_i), hence $\mathcal{V}_\infty$ is a π - *base* (i.e. any non-empty open set contains some element of $\mathcal{V}_\infty$).

Explanation of above collections. One of the differences form [18] is that we make the set, corresponding first countable points in our universal space, empty by the fact that $\cup\mathcal{F}_i = Q_\kappa$ in ¶5.5. The main difference from §4 is that we use $\mathcal{R}_\mu$-covers in Definition 5.1 so that we can apply Lemma 5.2 to show the homogeneity. Each *s. c.* cover $\mathcal{V}_i$ controls about each point, defined by collapsing each $F \in \mathcal{F}_i$. □

We skip the way how to take the above collections, since we can construct the above collections without any difficulties (see §4, and [18]). Now, let $W_{\kappa,\mathcal{F}_\infty}$ be the decomposition space defined by the following identification on Q_κ:

¶ 5.6. We identify $r, s \in Q_\kappa$ if $r, s \in F \in \mathcal{F}_i$ for some i.

From the above construction this definition of identification makes sense, and let $q : Q_\kappa \to W_{\kappa,\mathcal{F}_\infty}$ be the natural quotient map. Here, we need the following fundamental property of our quotient topology, which easily follows from the construction.

Lemma 5.3. *For a point $q(x) \in W_{\kappa,\mathcal{F}_\infty}$, where $x \in F \in \mathcal{F}_i$, there exists a clopen neighborhood base $\mathcal{W}$ of $q(x)$ such that $\mathcal{W} = \{q(B) : B \in \mathcal{B}_F\}$, where $\mathcal{B}_F$ is the collection of all neighborhoods of F in Q_κ such that, for each $B \in \mathcal{B}$, there exists a collection $\mathcal{U}_B \subset \mathcal{U}_i$ satisfying that $B = \cup\mathcal{U}_B \cup F$.*

Hereafter we fix the above notation of the neighborhood base $\mathcal{W}$, appearing in this lemma.

Lemma 5.4. *The quotient map $q : Q_\kappa \to W_{\kappa,\lambda}$ is closed. Hence, the decomposition space $W_{\kappa,\mathcal{F}_\infty}$ is σ-discrete, and is a member of $\mathcal{W}_\kappa$.*

Proof. The map q is closed by virtue of Lemma 4.3 and Lemma 5.3. □

In the rest of this section we shall show that the space $W_{\kappa,\mathcal{F}_\infty}$ satisfies the second statement in Theorem 1.1.

Lemma 5.5. *The topology of our universal spaces does not depend on the choice of $\mathcal{F}_\infty$ and $\mathcal{V}_\infty$.*

Proof. Let $\mathcal{F}'_\infty$ and $\mathcal{V}'_\infty$ be another pair of collections, satisfying ¶5.3 – 5.4. Then, we can follows the same argument and same notations used in the proof of Theorem 4.3, when we replace the space B_κ by the space Q_κ. In particular, we have homeomorphism $h_i : (Q_\kappa \backslash E_{i-1}, F_i) \to (Q_\kappa \backslash E'_{i-1}, F'_i)$ for each i. By the fact that $\cup \mathcal{F}_i = Q_\kappa$ in ¶5.5 we can define $h_\infty : Q_\kappa \to Q_\kappa$ as follows.

$$h_\infty | F = h_i | F, \text{ where } F \in \mathcal{F}_i.$$

Then, we can proceed the remaining parallel argument with the proof of Theorem 4.3 to obtain the desired homeomorphism g by Lemma 5.3. □

d) Proof of homogeneity. By Lemma 5.5 we abbreviate the space $W_{\kappa,\mathcal{F}_\infty}$ as W_κ. Since W_κ satisfies all the conditions in [18], it is a universal space for $\mathcal{W}_\kappa$. Then, we have the following assertions. We skip its parallel proofs (see those of Propositions 4.2 and 4.3).

Assertion 5. *The space W_κ are h-homogeneous. Moreover, any non-empty open subset of it is homeomorphic each other.*

Assertion 6. *For any $x, y \in W_\kappa$, there exists an autohomeomorphism $h : W_\kappa \to W_\kappa$ such that $h(x) = y$.*

6. Rigid Watson spaces and how to manage two parameters classes

a) Proof of Theorem 1.6. At first we need the following lemma.

Lemma 6.1. *For any $\alpha > \omega$ there exists $\beta \geq \alpha$ such that $\beta = |\{\gamma :$ uncountable cardinal $< \beta\}|$.*

Proof. (due to Professor K. Eda). Let $f : \omega \to ON$ be the function defined inductively by

$$f(0) = \alpha, \text{ and } f(n+1) = \omega_{f(n)}.$$

Let

$$\beta = \sup_{n \in \omega} f(n).$$

Since $\beta = \omega_\beta$, the cardinal β has the required property by the equality $\{\gamma : \text{cardinal} < \beta\} = \{\omega_\eta : \eta < \beta\}$. □

For the above β, put $\Gamma = \{\gamma : \text{uncountable cardinal} < \beta\}$, and let $\{\Gamma_i : i \in \omega\}$ be a mutually disjoint decomposition of Γ satisfying that $\beta = |\Gamma_i|$ for each $i \in \omega$.

By ¶5.1 it holds that:

¶ 6.1. for every $\gamma \in \Gamma$ there exists a closed nowhere dense subset F_γ, satisfying that it is homeomorphic to Q_γ and hence, $|F_\gamma| = w(F_\gamma) = \gamma$.

¶ 6.2. Let $\mathcal{V}_0$ be a clopen disjoint cover of its mesh ≤ 1 for some metric d of Q_β with $|\mathcal{V}_0| = \beta$.

¶ 6.3. For each $V \in \mathcal{V}_0$, let F_V be a closed nowhere dense subset of V. Since $|\mathcal{V}_0| = \beta$ and each V is homeomorphic to Q_β, we can assume that

$$F_V \approx F_\gamma \text{ for some (unique) } \gamma \in \Gamma_0 \text{ in ¶6.1.}$$

Put $\mathcal{F}_0 = \{F_U : U \in \mathcal{V}_0\}$. By recursion, with respect to $i \in \omega$, we shall construct two collections $\mathcal{F}_i$ and $\mathcal{V}_i$ satisfying the following ¶6.4 – ¶6.5.

¶ 6.4. $\mathcal{V}_i$ is a clopen disjoint collection with $|\mathcal{V}_i| = \beta$ and $\mathcal{F}_i = \{F_V : V \in \mathcal{V}_i\}$, where F_V is a closed nowhere dense subset of V, and it satisfies that, for each $F \in \mathcal{F}_i$, it holds that $F \approx F_\gamma$ for the unique $\gamma \in \Gamma_i$, where F_γ is given by ¶6.1.

¶ 6.5. The mesh $\mathcal{V}_i \leq 1/2^i$ with respect to the metric d of Q_β in ¶6.2, the collection $\mathcal{V}_i$ refines $\mathcal{V}_{i-1}$, and $\mathcal{V}_i = \cup_{U \in \mathcal{V}_{i-1}} \mathcal{V}_U$, where $\mathcal{V}_U$ is a *s. c.* cover for the pair (U, F_U).

¶ 6.6. We identify $r, s \in Q_\beta$ if $r, s \in F \in \mathcal{F}_i$ for some i.

From the above construction this definition of the identification makes sense, and let $q : Q_\beta \to T_{\beta, \mathcal{F}_\infty}$ be the natural quotient map. Finally let W_β be the subspace of $T_{\beta, \mathcal{F}_\infty}$ consisting of non-trivial equivalent classes, i.e. $W_\beta = q(\cup \mathcal{F}_\infty)$. From the above construction it follows that $W_\beta \in \mathcal{W}_\beta$. We shall show that it is rigid. On the contrary, assume that there exists an autohomeomorphism $h \neq id$ from W_β onto itself. Then, take two different points $x_\eta = q(F_\eta), x_\gamma = q(F_\gamma)$ such that $h(x_\eta) = x_\gamma$. Assume that $\eta < \gamma$. By ¶6.5 there exist two *s. c.* covers

$\mathcal{V}_\eta$ and $\mathcal{V}_\gamma$ for the pairs (U_η, F_β) and (U_γ, F_γ), respectively. Put

$$P_\eta = \{q(F_V) : V \in \mathcal{V}_\eta\}.$$

Then, let

$$X_\eta = \{x_\eta\} \cup P_\eta, \text{ and } X_\gamma = h(X_\eta).$$

Note that P_η (respectively, $h(P_\eta)$) is open dense discrete in X_η (respectively, in X_γ). Then, by a method parallel to the one in Example 2.2 we have metric space $Y_\eta, Y_\gamma \subset Q_\beta$ satisfying that $F_\eta \subset Y_\eta$, $F_\gamma \subset Y_\gamma$, and that $\varphi_\eta = q|Y_\eta : Y_\eta \to X_\eta$ and $\varphi_\gamma = q|Y_\gamma : Y_\gamma \to X_\gamma$, are EE-maps. Then, by Proposition 2.1 there exist two perfect onto maps $p_\eta : Z \to X_\eta$ and $p_\gamma : Z \to X_\gamma$. Then, the following lemma concludes our proof, since $l(F_\eta) = \eta < l(F_\gamma) = \gamma$, where $l(X)$ denotes the *Lindelöff number* of X (see [4] for its definition).

Lemma 6.2. *Let $p : Z \to X$ be a perfect onto map. Then, it holds that $l(Z) = l(X)$, where $l(X)$ is the Lindelöff number of X.*

We omit its straightforward proof of this lemma.

b) How to manage two parameters classes. For two parameters class $\mathcal{D}_{\kappa,\lambda}$ (respectively, $\mathcal{W}_{\kappa,\lambda}$), we produce its universal space, using a method parallel to the one in [18]. The main difference is that we shall derive its topological properties (e.g. HO (see the discussion in §1 d) also)) by using standard *s. c.* covers in order to define our universal spaces firmly. Comparing with previously defined universla spaces in this paper, we need one more sequence $\{\mathcal{U}_i\}$ of clopen collections, where each $\mathcal{U}_i$ *refines* certain standard *s. c.* cover (see also Example 3.1). More precisely, at first, we should replace each closed nowhere dense subset $F \in \mathcal{F}_\infty$ by that homeomorphic to B_μ (respectively, Q_μ) for some $\omega \leq \mu < \lambda$ (¶6.7). Then, for each $F \in \mathcal{F}_i$ take a standard *s. c.* cover $\mathcal{V}_F$ with respect to some refining sequence of F (¶6.8). Note that the cardinality of $\mathcal{V}_F$ is different for each μ. Hence, we should chop every member of $\mathcal{V}_F$ up κ-times, and name resulting clopen collection $\mathcal{U}_F$ (by this chopping up process in ¶6.9, we can derive not only property HO, but also universality). All $\mathcal{U}_F$ constitute the required clopen collection $\mathcal{U}_i$ of B_κ (respectively, Q_κ).

We begin with the construction for the class $\mathcal{D}_{\kappa,\lambda}$. Instead of ¶4.5 – ¶4.6, by recursion, we shall construct three collections $\mathcal{F}_i, \mathcal{U}_i$, and $\mathcal{V}_i$ for each i, satisfying the following three conditions.

¶ 6.7. $\mathcal{F}_i = \{F_U : U \in \mathcal{U}_i|\}$, and $\mathcal{F}_{i,V} = \{F \in \mathcal{F}_i : F \subset V\}$, where $V \in \mathcal{V}_{i-1}$, consists of κ *many copies* of B_μ (i.e. it holds that $|\{F \in \mathcal{F}_{i,V} : F \approx B_\mu\}| = \kappa$) for each μ satisfying $\omega \leq \mu < \lambda$, and each $F_U \in \mathcal{F}_i$ is a closed nowhere dense subset of $U \in \mathcal{U}_i$.

¶ **6.8.** The mesh $\mathcal{V}_i \leq 1/2^i$ with respect to some complete metric of B_κ, and $\mathcal{V}_i = \cup_{U \in \mathcal{V}_{i-1}} \mathcal{V}_U$, where $F_U \approx B_\mu$ and $\mathcal{V}_U$ is a $\mathcal{P}_\mu$ - *s. c.* cover for the pair (U, F_U) (hence the collection $\mathcal{V}_i$ refines $\mathcal{V}_{i-1}$).

Since the cardinality of $\mathcal{V}_U$ is different for each μ, we chop it to the following collection $\mathcal{U}_V$.

¶ **6.9.** $\mathcal{U}_V$ is a clopen disjoint collection with $|\mathcal{U}_V| = \kappa$, satisfying that $\cup \mathcal{U}_V = V$ for each $V \in \mathcal{V}_i$, and we put $\mathcal{U}_i = \cup_{V \in \mathcal{V}_i} \mathcal{U}_V$ (hence, $\mathcal{U}_i$ refines $\mathcal{V}_i$).

Using $\mathcal{F}_\infty = \cup_i \mathcal{F}_i$, we can define the decomposition space $\mathcal{D}_{\kappa,\lambda,\mathcal{F}_\infty}$ by a parallel way in §4 as follows:

We identify $r, s \in B_\kappa$ if $r, s \in F \in \mathcal{F}_i$ for some i.

Then, it is not difficult to see that $\mathcal{D}_{\kappa,\lambda,\mathcal{F}_\infty} \in \mathcal{D}_{\kappa,\lambda}$ (we can show the corresponding properties in Lemmas 4.1 – 4.5 for $\mathcal{D}_{\kappa,\lambda,\mathcal{F}_\infty}$). We shall show that the topology of our space $\mathcal{D}_{\kappa,\lambda,\mathcal{F}_\infty}$ does not depend on choice of a collection $\mathcal{F}_\infty$.

Definition 6.1. A family $\mathcal{F} = \cup_{j \geq 0} \mathcal{F}_i$ of closed subset of B_κ is called λ-*universal* when there exist two collections $\mathcal{V} = \cup_{j \geq 0} \mathcal{V}_j$ and $\mathcal{U} = \cup_{j \geq 0} \mathcal{U}_j$, satisfying the above conditions ¶6.7 – ¶6.9.

Theorem 6.1. *Every decomposition space $D_{\kappa,\lambda,\mathcal{F}}$ with respect to some λ-universal closed collection $\mathcal{F}$ is homeomorphic each other.*

A sketch of the proof. Let $\mathcal{F}'$ be another λ-universal collection, and let $\mathcal{U}'$ and $\mathcal{V}'$ satisfy ¶6.7 – ¶6.9. We shall follow the notations in the proof of Theorem 4.3. Instead of ¶4.8 – ¶4.9 we can show that, for each $V \in \mathcal{V}_U \subset \mathcal{V}_i$, where $U \in \mathcal{V}_{i-1}$:

¶ **6.10.** $h_i|V = h_{i,V} : (V, F_V) \to (V', \phi_i(F_V))$ is not only a $(\mathcal{V}_V, \mathcal{V}'_{V'})$-preserving, but also $(\mathcal{U}_V, \mathcal{U}'_{V'})$-preserving homeomorphism;

¶ **6.11.** V' is the unique element of $\mathcal{V}'_i$, which satisfies that $V' \supset \phi_i(F_V)$;

¶ **6.12.** $\phi_i|\mathcal{F}_{i,U} : \mathcal{F}_{i,U} \to \mathcal{F}'_{i,U'}$ is not only bijective but also satisfies that $F_V \approx \phi_i(F_V)$, where $U' = h_i(U)$, and $\mathcal{F}_{i,U}$ and $\mathcal{F}'_{i,U'}$ are the collections defined in ¶6.7.

By ¶6.10 – ¶6.12 we can proceed the remaining proof in parallel with that of Theorem 4.3 without any difficulties so that we leave the details to the reader. □

Hereafter we shall abbreviate the space $D_{\kappa,\lambda,\mathcal{F}_\infty}$ as $D_{\kappa,\lambda}$ by virtue of Theorem 6.1. Let $Z_{\kappa,\lambda}$ be the subspace of $D_{\kappa,\lambda}$ consisting of non-trivial equivalent classess (i.e. $Z_{\kappa,\lambda} = q(\cup\mathcal{F}_\infty)$). Now, we have a parallel result with Proposition 4.2 as follows.

Proposition 6.1. *Spaces $D_{\kappa,\lambda}$ and $Z_{\kappa,\lambda}$ are h-homogeneous. Moreover, they satisfy HO.*

We skip its parallel proof (see the proof of Proposition 4.2), and state the corollary, corresponding Corollary 4.1, by which we can show the closed embeddability of any space in $\mathcal{D}_{\kappa,\lambda}$ into $D_{\kappa,\lambda}$.

Corollary 6.1. *Let G be a G_δ subset in $D_{\kappa,\lambda}$ such that $G \supset q(\cup\mathcal{F}_\infty)$. Then, G is homeomorphic to $D_{\kappa,\lambda}$.*

We leave its proof to the reader (see the proof of Lemma 4.10).

The remaining part of a proof of Theorem 1.2. By Proposition 6.1 the remaining thing, which we should show, is universality of our universal space $D_{\kappa,\lambda}$. The argument, however, in §4 b), can be applied in the present case without any difficulties so that we leave the details to the reader. □

The remaining part of a proof of Theorem 1.1. For two parameters class $\mathcal{W}_{\kappa,\lambda}$ we can apply the same argument in this section to the spaces Q_μ (instead of B_μ), using $\mathcal{R}_\mu$-covers (instead of $\mathcal{P}_\mu$-covers), without any difficulties. Hence, one can show that there exists a universal space $W_{\kappa,\lambda}$ for $\mathcal{W}_{\kappa,\lambda}$, which satisfies the property HO. We leave the details to the reader (see also [18]). □

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