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BOUNDEDNESS OF SUBLINEAR OPERATORS AND COMMUTATORS ON $L^{p,\omega}(\mathbb{R}^n)$

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Abstract. In this paper, the authors establish the boundedness of some rough operators and their commutators with BMO ($\mathbb{R}^n$) functions on generalized Morrey spaces, $L^{p,\omega}$ ($\mathbb{R}^n$).

1. Introduction

To study the local behaviour of solutions to second order elliptic partial differential equations, Morrey in [6] introduced some spaces of functions which are called to be the classical Morrey spaces $L^{p,\omega}(\mathbb{R}^n)$ ($1\leq p<\infty$, $0<\lambda<n$) now. Since then, a series of results relative to these spaces have been obtained. Moreover, these spaces and the theories of singular integrals and commutators on them are proved to be very useful in studying the regularity of solutions to partial differential equations; see [1] and [3]. Recently, Mizuhara [5] introduced a kind of generalized spaces $L^{p,\phi}(\mathbb{R}^n)$ and investigated the behaviour of maximal operators and singular operators on $L^{p,\phi}(\mathbb{R}^n)$. In [4], Lu, Yang and Zhou established the boundedness of some rough operators and their commutators with BMO ($\mathbb{R}^n$) functions on the spaces $L^{p,\omega}(\mathbb{R}^n)$, which are introduced by Nakai in [7].

For this purpose, we first recall the definition of $L^{p,\omega}(\mathbb{R}^n)$ in [7]. Let ω be a nonnegative function on $\mathbb{R}^n\times\mathbb{R}_+$, and $I(a,r)=\{x\in\mathbb{R}^n : |x_i-a_i|\leq r/2,\ i=1,2,\cdots,n\}$ for any $a=(a_1,\cdots,a_n)\in\mathbb{R}^n$ and $r>0$. For $I=I(a,r)$, let $kI\equiv I(a,kr)$ and $\omega(I)\equiv\omega(a,r)$.

Definition. ([7]) Let $1\leq p<\infty$ and ω be as above. We denote by $L^{p,\omega}\equiv L^{p,\omega}(\mathbb{R}^n)$ the space of locally integrable functions f which satisfy $\|f\|_{L^{p,\omega}(\mathbb{R}^n)}<\infty$, where

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$$\|f\|_{L^{p,\omega}(\mathbb{R}^n)} \equiv \sup_I \left(\frac{1}{\omega(I)} \int_I |f(x)|^p \, dx \right)^{1/p},$$

and the supremum is taken over all the cubes with the edges parallel to the axes.

It is clear that $L^{p,\omega}(\mathbb{R}^n)$ is a Banach space. Moreover, if $\omega(a,r) \equiv 1$, then $L^{p,\omega} = L^p$; if $\omega(a,r) = r^\lambda$, $0 < \lambda < n$, then $L^{p,\omega}$ is just the standard Morrey space $L^{p,\lambda}$; and if $\omega(a,r)$ is independent of a, then $L^{p,\omega}$ is the generalized Morrey space introduced by Mizuhara in [5].

In what follows, we shall write $p' = p/(p-1)$ for any $p \in (1,\infty)$ and $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$. Suppose that T represents a linear or a sublinear operator, which satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,

$$|Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| \, dy, \tag{1}$$

where $C > 0$ is an absolute constant, Ω is homogeneous of degree zero and $\Omega \in L^q(\Sigma)$ for some $q \in [1,\infty]$. Similarly, we assume that $\widetilde{T}$ represents a linear or a sublinear operator satisfying that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,

$$|\widetilde{T} f(x)| \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| \, dy \tag{2}$$

for some $\alpha \in (0,n)$, where C is as in (1).

Theorem 1. *Assume that there is a constant $C > 0$ such that for any $a \in \mathbb{R}^n$ and any $r > 0$,*

$$r \leq t \leq 2r \Longrightarrow C^{-1} \leq \omega(a,t)/\omega(a,r) \leq C, \tag{3}$$

$$\int_r^\infty \frac{\omega(a,t)}{t^{n+1}} \, dt \leq C \frac{\omega(a,r)}{r^n}. \tag{4}$$

Let $p \in (1,\infty)$ and $q \geq p'$. If a sublinear operator T satisfying (1) with $\Omega \in L^q(\Sigma)$ is bounded on $L^p(\mathbb{R}^n)$, then T is also bounded on $L^{p,\omega}(\mathbb{R}^n)$.

Our Theorem 1 obviously generalizes (i) of Theorem 2 in [7].

Now let T be a linear operator and a be a BMO function. We define the commutator $[a,T]$ by letting

$$[a,T] f(x) \equiv a(x) T f(x) - T(af)(x)$$

for any suitable function f.

Theorem 2. *Let ω satisfy (3) and (4). Let $p \in (1,\infty)$, and $q > p'$. Suppose that a linear operator T satisfies (1). If $[a,T]$ is bounded from $L^p(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, then $[a,T]$ is also bounded from $L^{p,\omega}(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ to $L^{p,\omega}(\mathbb{R}^n)$.*

Theorem 3. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. Assume that ω satisfies (3) and*

$$\int_r^\infty \frac{\omega(a,t)}{t^{n-\alpha p+1}}\,dt \le C\frac{\omega(a,r)}{r^{n-\alpha p}}. \tag{5}$$

If a sublinear operator $\widetilde{T}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ and satisfies (2) with $\Omega \in L^\beta(\Sigma)$ and $\beta > p'$, then $\widetilde{T}$ is also bounded from $L^{p,\omega}(\mathbb{R}^n)$ to $L^{q,\omega^{q/p}}(\mathbb{R}^n)$.

Theorem 3 is a generalization of (i) of Theorem 3 in [7]. On the commutator of a such linear operator with any BMO function, we have

Theorem 4. *Let α, p, β, q and ω be as in Theorem 3. If a linear operator $\widetilde{T}$ satisfies (2) with $\Omega \in L^\beta(\Sigma)$, and $[a,\widetilde{T}]$ is bounded from $L^p(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $[a,\widetilde{T}]$ is also bounded from $L^{p,\omega}(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ to $L^{q,\omega^{q/p}}(\mathbb{R}^n)$.*

We point out that the condition (1) was first introduced by Soria and Weiss in [8]. The conditions (1) and (2) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson's maximal operators, Hardy-Littlewood maximal operators, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner-Riesz means and so on; see also [8].

2. Proofs of Theorems

Let us begin with some lemmas on the boundedness of the rough maximal operator M_Ω defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\le r} |\Omega(y) f(x-y)|\,dy,$$

where Ω is a homogeneous function of degree zero on $\mathbb{R}^n \backslash \{0\}$. To do so, we first recall some basic facts relative to ω in [7].

Lemma 1. ([7]) *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. If there is a constant $C > 0$ such that for any $r > 0$,*

$$\int_r^\infty \frac{\varphi(t)}{t}\,dt \le C\varphi(r),$$

then there are constants $\varepsilon > 0$ and $C' > 0$ such that for any $r > 0$,

$$\int_r^\infty \frac{\varphi(t)t^\varepsilon}{t}\,dt \le C'\varphi(r)r^\varepsilon.$$

We remark that the ε in Lemma 1 can be choosen to satisfy $0 < \varepsilon < 1/C$; see [7].

Lemma 2. ([7]) *Let $0 < \delta \le 1$. Assume that ω satisfies (3) and*

$$\int_r^\infty \frac{\omega(a,t)}{t^{n\delta+1}}\,dt \le C\frac{\omega(a,r)}{r^{n\delta}}.$$

Then for $1 \le p < \infty$, there is a constant $C > 0$ such that for any $f \in L^{p,\omega}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)|^p \,(M\chi_I(x))^\delta\,dx \le C\omega(I)\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)},$$

where Mf is the standard Hardy-Littlewood maximal function of f.

In what follows, for any $a \in \mathbb{R}^n$ and $r > 0$, let $I \equiv I(a,r)$. For any complex-valued measurable function $f(y)$ on $\mathbb{R}^n$, we write

$$f(y) = f(y)\chi_{2I(a,r)}(y) + \sum_{k=1}^\infty f(y)\chi_{2^{k+1}I\setminus 2^kI}(y) \equiv \sum_{k=0}^\infty f_k(y). \tag{6}$$

Lemma 3. *Let p, q and ω be as in Theorem 1. Then there is an $\epsilon > 0$ such that for any $k > 0$ and $f \in L^{p,\omega}(\mathbb{R}^n)$,*

$$\int_{I(a,r)} \left|M_\Omega f_k(x)\right|^p dx \le C\omega(I)^{2^{-k\epsilon}}\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)},$$

where C is independent of k, $I(a,r)$ and f.

Proof. Let $I \equiv I(a,r)$. By the properties of A_p weights (see [2, p. 407]), we can easily see that for any $\theta \in (0,1)$,

$$(M\chi_I)^\theta \in A_{p/q'} \quad \text{for} \quad q \ge p'.$$

Then, by the results in [10] we obtain

$$\begin{aligned}\int_{I(a,r)} \left|M_\Omega f_k(x)\right|^p dx &\le \int_{\mathbb{R}^n} \left|M_\Omega f_k(x)\right|^p \left(M\chi_I(x)\right)^\theta dx\\ &\le C(\theta)\int_{\mathbb{R}^n} |f_k(x)|^p \left(M\chi_I(x)\right)^\theta dx.\end{aligned}$$

Note that ω satisfies (4). From Lemma 1 and the remark following it, we deduce that there is an $\epsilon_0 \in (0,n)$ such that

$$\int_r^\infty \frac{\omega(a,t)}{t^{n-\epsilon_0+1}}\,dt \leq C\frac{\omega(a,r)}{r^{n-\epsilon_0}}.$$

Let $\delta = (n-\epsilon_0)/n$. It is clear that $\delta \in (0,1)$. Therefore, if we choose $\theta \in (\delta,1)$, and note that $M\chi_I(x)$ is comparable to 2^{-kn} when $x \in 2^{k+1}I\backslash 2^kI$, then by Lemma 2 we have

$$\begin{aligned}\int_{I(a,r)} |M_\Omega f_k(x)|^p\,dx &\leq C2^{-kn(\theta-\delta)}\int_{\mathbb{R}^n} |f_k(x)|^p\Big(M\chi_I(x)\Big)^\delta dx\\ &\leq C\omega(I)2^{-kn(\theta-\delta)}\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)}\end{aligned}$$

Letting $\epsilon = n(\theta-\delta)$, we complete the proof of Lemma 3. □

Proof of Theorem 1. For $I \equiv I(a,r)$, we write f as in (6). It is easy to deduce that

$$\int_{I(a,r)} |T f_0(x)|^p\,dx \leq C\|f_0\|^p_{L^p(\mathbb{R}^n)} \leq C\omega(2I)\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)} \leq C\omega(I)\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)}.$$

For $k>0$, it follows from Lemma 3 that

$$\int_{I(a,r)} |T f_k(x)|^p\,dx \leq C\int_{I(a,r)} |M_\Omega f_k(x)|^p\,dx \leq C\omega(I)2^{-k\epsilon}\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)},$$

where C and ϵ are independent of f and k. Thus,

$$\left(\frac{1}{\omega(I)}\int_{I(a,r)} |T f(x)|^p\,dx\right)^{1/p} \leq C\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)}\Big(1+\sum_{k=1}^\infty 2^{-k\epsilon/p}\Big) \leq C\|f\|_{L^{p,\omega}(\mathbb{R}^n)}.$$

The desirable conclusion can be easily deduced from this now.

This finishes the proof of Theorem 1. □

Proof of Theorem 2. For any $I \equiv I(a,r)$, we write f as in (6). By the L^p-boundedness of $[a,T]$, we obtain

$$\begin{aligned}\int_{I(a,r)} |[a,T]\,f_0(x)|^p\,dx &\leq C\|f_0\|^p_{L^p(\mathbb{R}^n)}\\ &\leq C\omega(2I)\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)} \leq C\omega(I)\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)}.\end{aligned}$$

For $k>0$ and $x \in I(a,r)$, we write

$$|[a,T]\,f_k(x)| \leq \frac{C}{(2^kr)^n}\int_{2^{k+1}I} |a(x)-a_r||\Omega(x-y)\,f_k(y)|\,dy$$

$$+\frac{C}{(2^k r)^n}\int_{2^{k+1}I}|a_r-a_{2^{k+1}r}||\Omega(x-y)\,f_k(y)|\,dy$$
$$+\frac{C}{(2^k r)^n}\int_{2^{k+1}I}|a(y)-a_{2^{k+1}r}||\Omega(x-y)\,f_k(y)|\,dy$$
$$\equiv I_1(x)+I_2(x)+I_3(x),$$

where, and in what follows, a_δ with $\delta>0$ is defined by

$$a_\delta=\frac{1}{|I(a,\delta)|}\int_{I(a,\delta)}a(y)\,dy.$$

By the well-known fact that for any $r>0$ and $k\in\mathbb{N}$,

$$\left|a_{2^{k+1}r}-a_r\right|\le C(n)(k+1)\|a\|_*\quad\text{(see [9])},$$

we obtain

$$I_2(x)\le C(k+1)\|a\|_*\,M_\Omega f_k(x).$$

From Lemma 3 it follows that there exists an $\epsilon_1>0$, which is independent of f, r and k such that

$$\int_{I(a,r)}I_2(x)^p\,dx\le C(k+1)^p\|a\|_*^p\,\omega(I)2^{-k\epsilon_1}\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)}.$$

For $I_3(x)$, we choose $1<u<\min\{p,q,qp/(q+p)\}$. Then we have

$$I_3(x)\le C\left[\frac{1}{(2^k r)^n}\int_{2^{k+1}I}|a(y)-a_{2^{k+1}r}|^{u'}\,dy\right]^{1/u'}$$
$$\times\left[\frac{1}{(2^k r)^n}\int_{2^{k+1}I}|\Omega(x-y)|^u\,|f_k(y)|^u\,dy\right]^{1/u}$$
$$\le C\|a\|_*\left(M_{|\Omega|^u}(|f_k|^u)(x)\right)^{1/u},$$

where we have used Hölder's inequality and the John-Nirenberg Lemma on BMO function (see [9]). Noting that $|\Omega|^u\in L^{q/u}(\Sigma)$, by Lemma 3 we have

$$\int_{I(a,r)}I_3(x)^p\,dx\le C\|a\|_*^p\int_{I(a,r)}\left(M_{|\Omega|^u}(|f_k|^u)(x)\right)^{p/u}dx$$
$$\le C\|a\|_*^p\,2^{-k\epsilon_2}\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)}\,\omega(I),$$

where ϵ_2 is independent of f, $I(a,r)$ and k. Combining the above estimates we have

$$(7)\quad\sum_{k=1}^{\infty}\Big(\int_I I_2(x)^p\,dx\Big)^{1/p}+\sum_{k=1}^{\infty}\Big(\int_I I_3(x)^p\,dx\Big)^{1/p}\le C\|a\|_*\,\omega(I)\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)}.$$

It remains to estimate $I_1(x)$. Since $p/q' > 1$, we choose $u > 1$, $1 < v < p$ such that $(v-1)/u < p/q' - 1$. From this we get

$$1/q + v/(pu) + 1/(pu') < 1.$$

Now we take s satisfying $1/q < s < 1$ and $1/(pu') + 1/(sq) = 1$. Thus, we have

$$1/(pu') + 1/(sq) > 1/q + v/(pu) + 1/(pu'),$$

i.e.,

$$1/s > 1 + qv/(pu) = (pu + qv)/(pu).$$

Hence, we obtain

$$pu/(sqv) > 1 \quad \text{and} \quad 1/s > (pu/(sqv))'.$$

Since $1/(pu') + 1/(sqv) + 1/(sqv') = 1$, for $x \in I$, by Hölder's inequality we have

$$\begin{aligned} I_1(x) &\le C|a(x) - a_r| \left[\frac{1}{(2^k r)^n} \int_{2^{k+1}I} |f_k(y)|^p \, dy\right]^{1/(pu')} \\ &\quad \times \left[\frac{1}{(2^k r)^n} \int_{2^{k+1}I} |\Omega(x-y)|^{sq} \, |f_k(y)|^{sqv/u} \, dy\right]^{1/(sqv)} \\ &\quad \times \left[\frac{1}{(2^k r)^n} \int_{2^{k+1}I} |\Omega(x-y)|^{sq} \, dy\right]^{1/(sqv')} \\ &\le \frac{C|a(x) - a_r|}{(2^k r)^{n/(pu')}} \, \omega^{1/(pu')}(2^{k+1}I) \|f\|^{1/u'}_{L^{p,\omega}(\mathbb{R}^n)} \big(M_{|\Omega|^{sq}}(|f_k|^{sqv/u})(x)\big)^{1/(sqv)}, \end{aligned}$$

where, and in what follows, for brevity, we denote $[\omega(2^{k+1}I)]^{1/(pu')}$ by $\omega^{1/(pu')}(2^{k+1}I)$. By $|\Omega|^{sq} \in L^{1/s}(\Sigma)$, $pu/(sqv) > 1$, and $1/s > (pu/(sqv))'$, applying Hölder's inequality and Lemma 3, we obtain

$$\begin{aligned} &\int_I I_1(x)^p \, dx \\ &\le \frac{C}{(2^k r)^{n/u'}} \, \omega^{1/u'}(2^{k+1}I) \|f\|^{p/u'}_{L^{p,\omega}(\mathbb{R}^n)} \\ &\quad \times \int_I |a(x) - a_r|^p \, \big(M_{|\Omega|^{sq}}(|f_k|^{sqv/u})(x)\big)^{p/(sqv)} \, dx \\ &\le \frac{C}{(2^k r)^{n/u'}} \, \omega^{1/u'}(2^{k+1}I) \|f\|^{p/u'}_{L^{p,\omega}(\mathbb{R}^n)} \left[\int_I |a(x) - a_r|^{pu'} \, dx\right]^{1/u'} \\ &\quad \times \left[\int_I \big(M_{|\Omega|^{sq}}(|f_k|^{sqv/u})(x)\big)^{pu/(sqv)} \, dx\right]^{1/u} \\ &\le \frac{C\|a\|_*^p}{2^{kn/u'}} \, \omega^{1/u'}(2^{k+1}I) \|f\|^{p/u'}_{L^{p,\omega}(\mathbb{R}^n)} \left[\int_I \big(M_{|\Omega|^{sq}}(|f_k|^{sqv/u})(x)\big)^{pu/(sqv)} \, dx\right]^{1/u} \\ &\le \frac{C\|a\|_*^p}{2^{kn/u'}} \, \omega^{1/u'}(2^{k+1}I) \|f\|^{p/u'}_{L^{p,\omega}(\mathbb{R}^n)} \omega^{1/u}(I) 2^{-k\epsilon_0} \| \, |f|^{sqv/u} \|^{p/(sqv)}_{L^{pu/(sqv),\omega}(\mathbb{R}^n)}, \end{aligned}$$

where ε_0 that is independent of k, $I(a,r)$ and f, can be choosen to be very small by Lemma 3. Noting that

$$\||f|^{sqv/u}\|^{p/(sqv)}_{L^{pu/(sqv)},\omega(\mathbb{R}^n)} = \|f\|^{p/u}_{L^{p,\omega}(\mathbb{R}^n)},$$

we have

$$\int_I I_1(x)^p\,dx \le C\|a\|_*^p\,\|f\|^p_{L^{p,\omega}(\mathbb{R}^n)}\,\omega^{1/u'}(2^{k+1}I)\,\omega^{1/u}(I)2^{-kn/u'-k\epsilon_0}.$$

Therefore,

$$\begin{aligned}
&\Big(\int_I I_1(x)^p\,dx\Big)^{1/p}\\
&\quad\le C\|a\|_*\,\|f\|_{L^{p,\omega}(\mathbb{R}^n)}\,\omega^{1/(pu')}(2^{k+1}I)\omega^{1/(pu)}(I)2^{-kn/(pu')-k\epsilon_0/p}\\
&\quad\le C\|a\|_*\,\|f\|_{L^{p,\omega}(\mathbb{R}^n)}\,\omega^{1/(pu)}(I)r^{n/(pu')+\epsilon_0/p}\int_{2^k r}^{2^{k+1}r}\frac{\omega^{1/(pu')}(a,t)}{t^{n/(pu')+\epsilon_0/p+1}}\,dt.
\end{aligned}$$

By Lemma 1, there is an $\eta\in(0,1)$ such that

$$\int_r^\infty \frac{\omega(a,t)}{t^{n+\eta}}\,dt \le C\frac{\omega(a,r)}{r^{n+\eta-1}}.$$

Noting that $1/(pu')+1/(sq)=1$, we have

$$\begin{aligned}
&\sum_{k=1}^\infty\Big(\int_I I_1(x)^p\,dx\Big)^{1/p}\\
&\quad\le C\|a\|_*\,\|f\|_{L^{p,\omega}(\mathbb{R}^n)}\,\omega^{1/(pu)}(I)r^{n/(pu')+\epsilon_0/p}\int_r^\infty\frac{\omega^{1/(pu')}(a,t)}{t^{n/(pu')+\epsilon_0/p+1}}\,dt\\
&\quad\le C\|a\|_*\,\|f\|_{L^{p,\omega}(\mathbb{R}^n)}\,\omega^{1/(pu)}(I)r^{n/(pu')+\epsilon_0/p}\\
&\qquad\times\Big(\int_r^\infty\frac{\omega(a,t)}{t^{n+\eta}}\,dt\Big)^{1/(pu')}\Big(\int_r^\infty\frac{dt}{t^{(1+\epsilon_0/p-\eta/(pu'))sq}}\Big)^{1/(sq)}\\
&\quad\le C\|a\|_*\,\|f\|_{L^{p,\omega}(\mathbb{R}^n)}\,\omega^{1/(pu)}(I)r^{n/(pu')+\epsilon_0/p}\\
&\qquad\times\omega^{1/(pu')}(I)r^{-(n+\eta-1)/(pu')}r^{1/(sq)+\eta/(pu')-\epsilon_0/p-1}\\
&\quad\le C\|a\|_*\,\|f\|_{L^{p,\omega}(\mathbb{R}^n)}\,\omega^{1/p}(I),
\end{aligned}$$

which is a desirable estimate for $I_1(x)$.

This finishes the proof of Theorem 2. □

Proof of Theorem 3. For any $I\equiv I(a,r)$, write f as in (6). By the boundedness from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ of $\widetilde{T}$, we are easily deduce that

$$\int_{I(a,r)}|\widetilde{T}f_0(x)|^q\,dx \le C\|f_0\|^q_{L^p(\mathbb{R}^n)}$$

$$\leq C\Big(\int_{2I} |f(x)|^p\, dx\Big)^{q/p} \leq C\omega^{q/p}(I)\|f\|^q_{L^{p,\omega}(\mathbb{R}^n)},$$

Which is desirable.

Let us now consider the case $k > 0$. Since $\beta' < p$, we can choose $u \in (1,p)$ closed enough to 1 such that $\beta' < p/u$. Thus, by $\beta > (p/u)'$ we have $\Omega \in L^{(p/u)'}(S^{n-1})$. If we denote $\theta = p/q$, then $0 < \theta < 1$ and $\alpha = n(1-\theta)/p$. By Hölder's inequality, we have

$$\begin{aligned}|\widetilde{T} f_k(x)| &\leq C(2^k r)^{\alpha} \left[\frac{1}{(2^k r)^n}\int_{2^{k+1}I\backslash 2^k I} |\Omega(x-y)|\, dy\right]^{1/u'} \\ &\quad\times \left[\frac{1}{(2^k r)^n}\int_{2^{k+1}I\backslash 2^k I} |\Omega(x-y)|\,|f_k(y)|^u\, dy\right]^{1/u} \\ &\leq C(2^k r)^{\alpha}\|\Omega\|^{1/u'}_{L^1(\Sigma)} \left[\frac{1}{(2^k r)^n}\int_{2^{k+1}I\backslash 2^k I} |\Omega(x-y)|\,|f_k(y)|^u\, dy\right]^{\theta/u} \\ &\quad\times \left[\frac{1}{(2^k r)^n}\int_{2^{k+1}I\backslash 2^k I} |\Omega(x-y)|\,|f_k(y)|^u\, dy\right]^{(1-\theta)/u}\end{aligned}$$

Applying Hölder's inequality again, we obtain

$$\begin{aligned}&\left[\frac{1}{(2^k r)^n}\int_{2^{k+1}I\backslash 2^k I} |\Omega(x-y)|\,|f_k(y)|^u\, dy\right]^{(1-\theta)/u} \\ &\quad\leq \left[\frac{1}{(2^k r)^n}\int_{2^{k+1}I\backslash 2^k I} |\Omega(x-y)|^{(p/u)'}\, dy\right]^{1/(p/u)'\cdot(1-\theta)/u} \\ &\quad\times \left[\frac{1}{(2^k r)^n}\int_{2^{k+1}I\backslash 2^k I} |f_k(y)|^p\, dy\right]^{(1-\theta)/p} \\ &\quad\leq C(2^k r)^{-n(1-\theta)/p}\|\Omega\|^{(1-\theta)/u}_{L^{(p/u)'}(\Sigma)}\, \omega^{(1-\theta)/p}(2^{k+1}I)\|f\|^{1-\theta}_{L^{p,\omega}(\mathbb{R}^n)}.\end{aligned}$$

Noting that $\alpha = n(1-\theta)/p$ and

$$\|\Omega\|^{1/u'}_{L^1(\Sigma)}\cdot\|\Omega\|^{(1-\theta)/u}_{L^{(p/u)'}(\Sigma)} \leq C\|\Omega\|^{1-\theta/u}_{L^{\beta}(\Sigma)} \leq C,$$

we have

$$|\widetilde{T} f_k(x)| \leq C\|f\|^{1-\theta}_{L^{p,\omega}(\mathbb{R}^n)}\, \omega^{(1-\theta)/p}(2^{k+1}I)(M_\Omega(|f_k|^u)(x))^{\theta/u}.$$

On the other hand, by $\theta q/u = p/u > 1$, Lemma 2 and the results in [10], then for any $0 < \delta < 1$, we obtain

$$\int_I (M_\Omega(|f_k|^u)(x))^{q\theta/u}\, dx \leq C\int_I (M_\Omega(|f_k|^u)(x))^{q\theta/u}(M\chi_I(x))^{\delta}\, dx$$

$$\le C \int_{\mathbb{R}^n} |f_k(x)|^{q\theta} (M\chi_I(x))^\delta \, dx$$
$$\le C(2^k)^{-n\delta} \|f\|^p_{L^{p,\omega}(\mathbb{R}^n)} \, \omega(2^{k+1}I).$$

Thus, we have

$$\left[\int_I |\widetilde{T} f_k(x)|^q \, dx\right]^{1/q}$$
$$\le C\|f\|^{1-\theta}_{L^{p,\omega}(\mathbb{R}^n)} \, \omega^{(1-\theta)/p}(2^{k+1}I)(2^k)^{-n\delta/q}\|f\|^{p/q}_{L^{p,\omega}(\mathbb{R}^n)} \, \omega^{1/q}(2^{k+1}I)$$
$$\le C(2^k)^{-n\delta/q} \, \|f\|_{L^{p,\omega}(\mathbb{R}^n)} \, \omega^{1/p}(2^k I).$$

From Lemma 1 we can deduce

$$\int_r^\infty \frac{\omega(a,t)}{t^{n-\alpha p-\varepsilon+1}} \, dt \le C \frac{\omega(a,r)}{r^{n-\alpha p-\varepsilon}} \tag{8}$$

for some small $\varepsilon > 0$. Taking $\delta = 1 - (\varepsilon q)/(2pn)$ and $s = (2-\varepsilon)/(2p)$, then when ε is small enough we have $0 < \delta < 1$. Moreover, it is easy to see that $(n\delta/q + s)p = n - \alpha p - \varepsilon + 1$ and $(1-s)p' > 1$. Thus, by Hölder's inequality and (8) we get

$$\int_r^\infty \frac{\omega^{1/p}(a,t)}{t^{n\delta/q+1}} \, dt \le \left[\int_r^\infty \frac{\omega(a,t)}{t^{(n\delta/q+s)p}} \, dt\right]^{1/p} \left[\int_r^\infty \frac{1}{t^{(1-s)p'}} \, dt\right]^{1/p'} \le C \frac{\omega^{1/p}(a,r)}{r^{n\delta/q}}.$$

Therefore,

$$\sum_{k=1}^\infty \left[\int_I |\widetilde{T} f_k(x)|^q \, dx\right]^{1/q} \le C r^{n\delta/q} \, \|f\|_{L^{p,\omega}(\mathbb{R}^n)} \sum_{k=1}^\infty \frac{\omega^{1/p}(a, 2^k r)}{(2^k r)^{n\delta/q}}$$
$$\le C r^{n\delta/q} \, \|f\|_{L^{p,\omega}(\mathbb{R}^n)} \int_r^\infty \frac{\omega^{1/p}(a,t)}{t^{n\delta/q+1}} dt$$
$$\le C\|f\|_{L^{p,\omega}(\mathbb{R}^n)} \, \omega^{1/p}(I).$$

Thus, we complete the proof of Theorem 3. □

Proof of Theorem 4. For any $I \equiv I(a,r)$, write f as in (6). The estimate for the case $k = 0$ can be deduced from the boundedness from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ of $[a, \widetilde{T}]$. Hence we only need to estimate for $k \ge 1$. For this purpose, we first point out that in the proof of Theorem 3 we have proved

$$\left(\int_I \left[\int_{2^{k+1}I \setminus 2^k I} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} \, |f(y)| \, dy\right]^q dx\right)^{1/q} \tag{9}$$
$$\le C 2^{-kn\delta/q} \|f\|_{L^{p,\omega}(\mathbb{R}^n)} \, \omega^{1/p}(2^k I),$$

where $0 < \delta < 1$. Now we write

$$J_1 = \left(\int_I \left[|a(x) - a_r| \int_{2^{k+1}I\backslash 2^k I} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\,dy\right]^q dx\right)^{1/q},$$

$$J_2 = \left(\int_I \left[\int_{2^{k+1}I\backslash 2^k I} |a_r - a_{2^{k+1}r}| \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\,dy\right]^q dx\right)^{1/q},$$

and

$$J_3 = \left(\int_I \left[\int_{2^{k+1}I\backslash 2^k I} |a(y) - a_{2^{k+1}r}| \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\,dy\right]^q dx\right)^{1/q}.$$

Thus, we have

$$\left(\int_I |[a, \widetilde{T}]f_k(x)|^q\,dx\right)^{1/q} \leq J_1 + J_2 + J_3.$$

For J_1, by $\beta > p'$ we have $\Omega \in L^{p'}(\Sigma)$. Using Hölder's inequality, we obtain

$$\begin{aligned}
J_1 &\leq \frac{C}{(2^k r)^{n-\alpha}} \left(\int_I |a(x) - a_r|^q \left[\int_{2^{k+1}I\backslash 2^k I} |\Omega(x-y)|\,|f(y)|\,dy\right]^q dx\right)^{1/q} \\
&\leq \frac{C}{(2^k r)^{n-\alpha}} \left(\int_I |a(x) - a_r|^q \left[\int_{2^{k+1}I\backslash 2^k I} |\Omega(x-y)|^{p'}\,dy\right]^{q/p'}\right. \\
&\qquad \times \left.\left[\int_{2^{k+1}I\backslash 2^k I} |f(y)|^p\,dy\right]^{q/p} dx\right)^{1/q} \\
&\leq C(2^k r)^{\alpha-n}(2^k r)^{n/p'} \|\Omega\|_{L^{p'}(\Sigma)} \|f\|_{L^{p,\omega}(\mathbb{R}^n)}\, \omega^{1/p}(2^k I) \left(\int_I |a(x) - a_r|^q\,dx\right)^{1/q} \\
&\leq C 2^{-kn/q} \|a\|_* \, \|f\|_{L^{p,\omega}(\mathbb{R}^n)}\, \omega^{1/p}(2^k I).
\end{aligned}$$

From (9) and the properties of BMO functions, it follows that

$$\begin{aligned}
J_2 &\leq C|a_r - a_{2^{k+1}r}| \left(\int_I \left[\int_{2^{k+1}I\backslash 2^k I} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\,dy\right]^q dx\right)^{1/q} \\
&\leq C(k+1)2^{-kn\delta/q} \|a\|_* \, \|f\|_{L^{p,\omega}(\mathbb{R}^n)}\, \omega^{1/p}(2^k I).
\end{aligned}$$

Now, let us consider J_3. Since $\beta > p'$, we choose $1 < t < \min\{\beta, p\beta/(p+\beta)\}$. Thus we have $|\Omega|^t \in L^{\beta/t}(\Sigma)$ and $\beta/t > (p/t)'$. Noting that $1/(q/t) = 1/(p/t) - \alpha t/n$, using again Hölder's inequality and (9), we obtain

$$J_3 \leq \left(\int_I \left[\int_{2^{k+1}I\backslash 2^k I} |a(y) - a_{2^{k+1}r}|^{t'}\,dy\right]^{q/t'}\right.$$

$$\times \left[\int_{2^{k+1}I\setminus 2^kI} \frac{|\Omega(x-y)|^t\,|f(y)|^t}{|x-y|^{(n-\alpha)t}}\,dy\right]^{q/t} dx\Bigg)^{1/q}$$

$$\le C(2^kr)^{n/t'}\|a\|_*(2^kr)^{-n(t-1)/t}$$

$$\times\left\{\left(\int_I\left[\int_{2^{k+1}I\setminus 2^kI}\frac{|\Omega(x-y)|^t\,|f(y)|^t}{|x-y|^{n-\alpha t}}\,dy\right]^{q/t}dx\right)^{t/q}\right\}^{1/t}$$

$$\le C(2^kr)^{n/t'}\|a\|_*(2^kr)^{-n(t-1)/t}\,2^{-kn\delta/(q/t)\cdot(1/t)}$$

$$\times\,\||f|^t\|^{1/t}_{L^{p/t,\omega}(\mathbb{R}^n)}\,\omega^{1/(p/t)\cdot(1/t)}(2^kI)$$

$$\le C\|a\|_*2^{-kn\delta/q}\,\|f\|_{L^{p,\omega}(\mathbb{R}^n)}\,\omega^{1/p}(2^kI).$$

By the estimates of J_1, J_2, J_3 and the same computational techniques used in the proof of Theorem 3, we obtain

$$\sum_{k=1}^{\infty}\left[\int_I|[a,\widetilde{T}]f_k(x)|^q\,dx\right]^{1/q}\le\sum_{k=1}^{\infty}(J_1+J_2+J_3)\le C\|a\|_*\|f\|_{L^{p,\omega}(\mathbb{R}^n)}\,\omega^{1/p}(I).$$

Thus, we finish the proof of Theorem 4. □

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References

[1] G. Di Fazio and M.A. Ragusa, Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, *J. Funct. Anal.* **112** (1993), 241–256.

[2] J. García-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam New York Oxford, 1985.

[3] Q. Huang, Estimates on the generalized Morrey spaces $L_\varphi^{2,\lambda}$ and BMO_ψ for linear elliptic systems, *Indiana Univ. Math. J.* **45** (1996), 397–439.

[4] S. Lu, D. Yang and Z. Zhou, Sublinear operators with rough kernel on generalized Morrey spaces, *Hokkaido Math. J.* **27** (1998), 219–232.

[5] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*, in *Harmonic Analysis*, ICM-90 Conference Proceedings (Ed. S. Igari), Springer-Verlag, Tokyo, (1991), 183–189.

[6] C.B. Morrey, Jr., On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.* **43** (1938), 126–166.

[7] E. Nakai, Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces, *Math. Nachr.* **166** (1994), 95–103.

[8] F. Soria and G. Weiss, A remark on singular integrals and power weights, *Indiana Univ. Math. J.* **43** (1994), 187–204.

[9] A. Torchinsky, *Real Variable Methods in Harmonic Analysis*, Academic Press, New York, 1988.

[10] D.K. Watson, Vector-valued inequalities, factorization, and extrapolation for a family of rough operators, *J. Funct. Anal.* **121** (1994), 389–415.

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