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ENUMERATION OF LABELLED CONNECTED GRAPHS AND EULER GRAPHS WITH ONLY ONE CUT VERTEX

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Abstract. This paper is concerned with the enumeration of labelled graphs with only one cut vertex. In this paper we give the exponential generating functions for labelled connected graphs and Euler graphs having exactly n blocks at the cut vertex. The numerical tables are also given.

1. Introduction

In this paper we consider enumeration problems of finite undirected labelled graphs without multiple edges nor loops. The readers are referred to [2] for any terms not defined below. A relation between the generating functions for labelled blocks and labelled connected graphs was investigated in [1] and [4]. The enumerations of labelled graphs with cut vertices may be discussed in this paper. But, the more the number of cut vertices increases, the more it becomes difficult to enumerate such labelled graphs. Therefore, in this paper we shall study the enumeration of labelled connected graphs with only one cut vertex.

The enumeration of labelled connected graphs with one cut vertex will be considered in Section 3. We there state a relation between the exponential generating function $B_n(x)$ for labelled connected graphs which have exactly n blocks incident with the cut vertex and that for labelled connected graphs. We also state relation between $B_n(x)$ and the exponential generating function for labelled blocks.

In Section 4 we shall treat the enumeration of labelled Euler graphs with one cut vertex. The discussion for this enumeration is almost similar to that for the enumeration of labelled connected graphs which is considered in Section 3. Therefore, the similar results to those in Section 3 are obtained. Many researchers have studied the enumerations of Euler graphs. Read [3] achieved a

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counting of labelled Euler graphs. In [5], Tazawa, Jin and Shirakura investigated the enumeration of labelled 2-connected Euler graphs, which do not have any cut vertices.

2. Preliminaries

In this section we shall mention the well-known results with respect to the enumerations for labelled graphs, labelled connected graphs and blocks. They will work powerfully in the following sections.

Noting that the number of labelled graphs of order p is $2^{\binom{p}{2}}$, we consider the exponential generating function having this number as the coefficient of $x^p/p!$, that is,

$$G(x) = \sum_{p=1}^{\infty} \frac{2^{\binom{p}{2}} x^p}{p!}. \tag{2.1}$$

Let

$$C(x) = \sum_{p=1}^{\infty} \frac{C_p x^p}{p!} \tag{2.2}$$

be given the exponential generating function for labelled connected graphs, where C_p is the number of labelled connected graphs of order p. Then Riddell [4] stated the following theorem that relates the above exponential generating functions.

Theorem 1. *The exponential generating functions $G(x)$ and $C(x)$ are related by*

$$G(x) + 1 = e^{C(x)}. \tag{2.3}$$

The first few terms of $C(x)$ are given by

$$\begin{aligned} C(x) &= x + \frac{x^2}{2!} + \frac{4x^3}{3!} + \frac{38x^4}{4!} + \frac{728x^5}{5!} + \frac{26704x^6}{6!} + \frac{1866256x^7}{7!} \\ &\quad + \frac{251548592x^8}{8!} + \frac{66296291072x^9}{9!} + \frac{34496488594816x^{10}}{10!} + \cdots. \end{aligned} \tag{2.4}$$

Let B_p be the number of labelled blocks of order p and let

$$B(x) = \sum_{p=1}^{\infty} \frac{B_p x^p}{p!}, \tag{2.5}$$

where $B_1 = 0$. Then Riddell [4] and Ford and Uhlenbeck [1] established a relation between the exponential generating functions for labelled blocks and labelled connected graphs:

Theorem 2. *The exponential generating functions $B(x)$ and $C(x)$ are related by*

$$C'(x) = \exp\{B'(xC'(x))\}, \tag{2.6}$$

where $C'(x)$ and $B'(x)$ denote derivative of x.

The first few terms of $B(x)$ are given by

$$\begin{aligned} B(x) = {} & \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{10x^4}{4!} + \frac{238x^5}{5!} + \frac{11368x^6}{6!} + \frac{1014888x^7}{7!} \\ & + \frac{166537616x^8}{8!} + \frac{50680432112x^9}{9!} + \frac{29107809374336x^{10}}{10!} + \cdots . \end{aligned} \tag{2.7}$$

3. Enumeration of labelled connected graphs with one cut vertex

Let n be an integer with $n \geq 2$ and let $B_{n,p}$ be the number of labelled connected graphs of order p which have one cut vertex and do exactly n blocks incident with the cut vertex, and consider the exponential generating function

$$B_n(x) = \sum_{p=1}^{\infty} \frac{B_{n,p} x^p}{p!}. \tag{3.1}$$

It is clear that $B_{n,p} = 0$ for $n \geq p$. Then we have the following two theorems, which will be proved along the proof of the theorem in ([2], p. 12). A labelled *un*-rooted graph which is seen in this section is a labelled rooted graph in which only the root is *unlabelled.*

Theorem 3. *Let n be an integer with $n \geq 2$. The exponential generating functions $B_n(x)$ and $C(x)$ are related by*

$$B_n(xC'(x)) = xC'(x)\frac{(\log C'(x))^n}{n!}. \tag{3.2}$$

Proof. Let R_p denote the number of labelled rooted connected graphs of order p. For a labelled connected graph of order p we have p kinds of labelled rooted connected graphs by being rooted at each of its verteces. Therefore, $R_p = P \cdot C_p$ holds. This implies that two exponential generating functions $R(x) = \sum_{p=1}^{\infty} R_p \frac{x^p}{p!}$ and $C(x)$ are related by

$$R(x) = x\frac{dC(x)}{dx}. \tag{3.3}$$

Let $R_m(x)$ be an exponential generating function having the number of labelled rooted connected graphs of order p as the coefficient of $x^p/p!$ in which exactly m blocks contain the root. Then it is easy to see that

$$(3.4) \qquad R(x) = \sum_{m=0}^{\infty} R_m(x),$$

is obtained, where $R_0(x) = x$.

Consider labelled *un*-rooted connected graphs. Then the coefficient of $x^p/p!$ in $R_1(x)/x$ is the number of labelled *un*-rooted connected graphs of order $p+1$ in which exactly one block contains the root. Therefore, it is observed that $(\frac{R_1(x)}{x})^m/m!$ enumerates m-sets of labelled *un*-rooted connected graphs in each of which exactly one block contains the root. If these m roots are identified and a single label is introduced for them, then each m-set corresponds to a labelled rooted connected graph having exactly m blocks at the identified root. Accordingly $R_m(x)$ can be written as follows:

$$(3.5) \qquad R_m(x) = x\frac{(\frac{R_1(x)}{x})^m}{m!}.$$

The application of this expression to (3.4) yields

$$(3.6) \qquad R(x) = \sum_{m=0}^{\infty} x\frac{(\frac{R_1(x)}{x})^m}{m!} = x\exp\left\{\frac{R_1(x)}{x}\right\}$$

and it follows from (3.3) that

$$(3.7) \qquad R_m(x) = x\frac{(\log C'(x))^m}{m!}$$

for $m = 0, 1, 2, \cdots$.

We now see that $R_n(x)$ can be expressed in terms of $B_n(x)$ and $R(x)$. As seen in the above, the coefficient, $a_{k,p}$, of $x^p/p!$ in $(R(x)/x)^{k-1}$ is the number of $(k-1)$ – tuples (the elements in each tuple are orderd) of labelled *un*-rooted connected graphs with order $p+k-1$ vertices including the $k-1$ vertices, where p is the number of labelled vertices in each tuple. Add an unlabelled vertex to each $(k-1)$-tuple, consider the blocks of order k on the set of $(k-1)$ *un*-rooted vertices and the added vertex, and finally, label all vertices except for the added vertex with 1 through $p+k-1$. Then this procedure tells us that $\frac{B_{n,k}}{k}a_{k,p}\binom{p+k-1}{k-1}$ is the number of labelled *un*-rooted connected graphs of order $p+k$ having only one cut vertex and having exactly n blocks at the cut vertex, where the root is the cut vertex. Note here that $\frac{B_{n,k}}{k}$ is the number of labelled *un*-rooted connected graphs of order k which have only one cut vertex and do exactly n blocks at the cut vertex, where the root is the cut vertex. This number equals the coefficient of $x^{p+k-1}/(p+k-1)!$ in the series accomplished by multiplying $(R(x)/x)^{k-1}$ by $\frac{B_{n,k}}{k}\frac{x^{k-1}}{(k-1)!}$. Consequently, we get

$$\frac{R_n(x)}{x} = \sum_{k=1}^{\infty} \frac{B_{n,k}(R(x))^{k-1}}{k!}. \tag{3.8}$$

Hence the combination of (3.7) and (3.8) gives

$$\begin{aligned}\frac{(\log(C'(x)))^n}{n!} &= \sum_{k=1}^{\infty} \frac{B_{n,k}(xC'(x))^{k-1}}{k!} \\ &= \frac{B_n(xC'(x))}{xC'(x)}\end{aligned}$$

which implies that (3.2) holds. This completes the proof of Theorem 3. ■

Theorem 4. *Let n be an integer with $n \geq 2$. The exponential generating functions $B_n(x)$ and $B(x)$ are related by*

$$B_n(x) = x\frac{(B'(x))^n}{n!}, \tag{3.9}$$

where $B(x)$ is given in (2.5).

Proof. Let $B_{1,p}$ denote the number of labelled rooted blocks of order p and put $B_1(x) = \sum_{p=1}^{\infty} B_{1,p}\frac{x^p}{p!}$. Then as seen at the beginning part of the proof of Theorem 3, we have $B_{1,p} = pB_p$ and

$$B_1(x) = x\frac{dB(x)}{dx}. \tag{3.10}$$

Then the coefficient of $x^p/p!$ in $B_1(x)/x$ is the number of labelled *un*-rooted blocks of order $p+1$. Therefore, it is observed that $(\frac{B_1(x)}{x})^n/n!$ enumerates n-sets of labelled *un*-rooted blocks. If these n roots are identified and a single label is introduced for them, then each n-set corresponds to a labelled rooted connected graph having exactly n blocks at the identified root. Accordingly $B_n(x)$ can be written as follows:

$$\begin{aligned}B_n(x) &= x\frac{(\frac{B_1(x)}{x})^n}{n!} \\ &= x\frac{(B'(x))^n}{n!},\end{aligned}$$

which is just (3.9). This completes the proof of Theorem 4. ■

A numerical example of $B_{n,p}$ is given for $2 \le n \le 6$ and $1 \le p \le 7$ in Table 1.

Table 1 The number of connected graphs with one cut vertex where they have exactly n blocks

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
$n=2$	0	0	3	12	215	7740	509446
$n=3$	0	0	0	4	30	690	29295
$n=4$	0	0	0	0	5	60	1715
$n=5$	0	0	0	0	0	6	105
$n=6$	0	0	0	0	0	0	7

To find the numerical value of $B_{n,p}$ for each pair of n and p, much calculation may be required. But the value of the sum $\sum_{n=2}^{\infty} B_{n,p}$ for each p can be obtained with less calculation. We shall see it. Put

$$V_p = \sum_{n=2}^{\infty} B_{n,p}, \tag{3.11}$$

which is the number of labelled connected graphs of order p with one cut vertex, and consider

$$V(x) = \sum_{p=1}^{\infty} \frac{V_p x^p}{p!}. \tag{3.12}$$

Then it follows from (3.1) that

$$V(x) = \sum_{n=2}^{\infty} B_n(x). \tag{3.13}$$

By applying Theorem 4 to (3.13), $V(x)$ becomes

$$\begin{aligned} V(x) &= \sum_{n=2}^{\infty} x \frac{(B'(x))^n}{n!} \\ &= x\Big\{\sum_{n=0}^{\infty} \frac{(B'(x))^n}{n!} - (1 + B'(x))\Big\} \\ &= x\Big\{e^{B'(x)} - (1 + B'(x))\Big\}. \end{aligned} \tag{3.14}$$

Consequently, we have a simple expression for $V(x)$ in terms of $B'(x)$.

Theorem 5. *The exponential generating functions $V(x)$ and $B(x)$ are related by*

$$V(x) = x\Big\{e^{B'(x)} - (1 + B'(x))\Big\}. \tag{3.15}$$

(3.15) can be rewritten as

$$\sum_{p=0}^{\infty} \frac{V_{p+1} + (p+1)B_{p+1}}{(p+1)!} x^p = \exp\Big\{\sum_{p=1}^{\infty} \frac{B_{p+1}}{p!} x^p\Big\}, \tag{3.16}$$

where $V_1 = 1$ and $V_2 = 0$.

The following lemma plays a major part in yielding a recursive formula for V_p.

Lemma 6. *If*

$$\sum_{m=0}^{\infty} A_m x^m = \exp\Big\{\sum_{m=1}^{\infty} a_m x^m\Big\}$$

holds, then

$$a_m = A_m - m^{-1}\Big(\sum_{k=1}^{m-1} k a_k A_{m-k}\Big) \qquad \textit{for} \quad m \geq 1.$$

Applying this lemma into (3.16) we get the recursive formula

$$V_p = \frac{p}{p-1} \sum_{k=1}^{p-2} \binom{p-1}{k-1} B_{k+1} \Big\{V_{p-k} + (p-k)B_{p-k}\Big\}, \tag{3.17}$$

where $p \geq 3$.

For the first few terms of $V(x)$, we have

$$\begin{aligned} V(x) = {} & \frac{3x^3}{3!} + \frac{16x^4}{4!} + \frac{250x^5}{5!} + \frac{8496x^6}{6!} + \frac{540568x^7}{7!} + \frac{61672192x^8}{8!} \\ & + \frac{12608406288x^9}{9!} + \frac{4697459302400x^{10}}{10!} + \cdots \end{aligned}$$

and the values of V_p are listed below for $p = 3, 4, \cdots, 10$.

The number of labelled connected graphs of order p with one cut vertex

p	3	4	5	6	7	8	9	10
V_p	3	16	250	8496	540568	61672192	12608406288	4697459302400

4. Enumeration of labelled Euler graphs with one cut vertex

Let us list the notations used in this section before making into a consideration of the enumeration of labelled Euler graphs. Since a block with more order than two is 2-connected, here, an Euler graph which is a block itself is referred as a 2-connected Euler graph. The symbols on the third column in this notation list are given in Section 2 and 3.

Table 2 Notation list

Symbol	Meaning	Corresponding symbol
W_p	The number of labelled, even graphs of order p	G_p
$W(x)$	$\sum_{p=1}^{\infty} \frac{W_p x^p}{p!}$	$G(x)$
U_p	The number of labelled Euler graphs of order p	C_p
$U(x)$	$\sum_{p=1}^{\infty} \frac{U_p x^p}{p!}$	$C(x)$
B_p^*	The number of labelled 2-connected Euler graphs of order p	B_p
$B^*(x)$	$\sum_{p=1}^{\infty} \frac{B_{n,p}^* x^p}{p!}$	$B(x)$
$B_{n,p}^*$	The number of labelled Euler graphs of order p with one cut vertex where they have exactly n 2-connected Euler graphs	$B_{n,p}$
$B_n^*(x)$	$\sum_{p=1}^{\infty} \frac{B_{n,p}^* x^p}{p!}$	$B_n(x)$
V_p^*	The number of labelled Euler graphs of order p with one cut vertex	V_p
$V^*(x)$	$\sum_{p=1}^{\infty} \frac{V_p^* x^p}{p!}$	$V(x)$

Read [3] showed that $W_p = 2^{\binom{p-1}{2}}$. The similar expression $W(x)+1 = e^{U(x)}$ to (2.3) is obtained. This expression yields the following recursive formula using Lemma 6:

$$U_p = 2^{\binom{p-1}{2}} - \frac{1}{p}\sum_{k=1}^{p-1} k\binom{p}{k} 2^{\binom{p-k-1}{2}} U_k \tag{4.1}$$

for $p = 1, 2, \cdots$. The first few terms of $U(x)$ are given by

$$\begin{aligned} U(x) = x &+ \frac{x^3}{3!} + \frac{3x^4}{4!} + \frac{38x^5}{5!} + \frac{720x^6}{6!} + \frac{26614x^7}{7!} + \frac{1858122x^8}{8!} \\ &+ \frac{250586792x^9}{9!} + \frac{66121926720x^{10}}{10!} + \cdots . \end{aligned} \tag{4.2}$$

Tazawa, Jin and Shirakura [5] studied a relation between the exponential generating functions for labelled 2-connected Euler graphs and labelled Euler graphs. They obtained the following result:

Theorem 7. *The exponential generating functions $B^*(x)$ and $U(x)$ are related by*

$$U'(x) = \exp\{B^{*\prime}(xU'(x))\}, \tag{4.3}$$

where $U'(x)$ and $B^{*\prime}(x)$ denote derivative of x.

For the first few terms of $B^*(x)$, we have

$$B^*(x) = \frac{x^3}{3!} + \frac{3x^4}{4!} + \frac{23x^5}{5!} + \frac{540x^6}{6!} + \frac{22834x^7}{7!} + \frac{1727922x^8}{8!} + \frac{243177614x^9}{9!} + \frac{65393041920x^{10}}{10!} + \cdots$$

and the values of B_p^* are listed below for $p = 3, 4, \cdots, 10$.

The number of 2-connected Euler graphs of order p

p	3	4	5	6	7	8	9	10
B_p^*	1	3	23	540	22834	1727922	243177614	65393041920

We next consider the enumeration of labelled Euler graphs with one cut vertex. The discussion for this enumeration is almost similar to that for the enumeration of labelled connected graphss which has been considered in Section 3. Therefore, replace the corresponding symbols on the Table 2 in Theorems 3,4,5 and (3.17) by the symbols on the first column of the Table 2, respectively. Then we get the following respective Theorems 8,9,10 and (4.7).

Theorem 8. *Let n be an integer with $n \geq 2$. The exponential generating functions $B_n^*(x)$ and $U(x)$ are related by*

$$B_n^*(xU'(x)) = xU'(x)\frac{(\log U'(x))^n}{n!}. \tag{4.4}$$

Theorem 9. *Let n be an integer with $n \geq 2$. The exponential generating functions $B_n^*(x)$ and $B^*(x)$ are related by*

$$B_n^*(x) = x\frac{(B^{*\prime}(x))^n}{n!}. \tag{4.5}$$

For the first few terms of $B_n^*(x)$, we have Table 3.

Table 3 The number of Euler graphs with one cut vertex where they have exactly n 2-connected Euler graphs

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
$n=2$	0	0	0	0	15	180	3045
$n=3$	0	0	0	0	0	0	105

Theorem 10. *The exponential generating functions $V^*(x)$ and $B^*(x)$ are related by*

$$V^*(x) = x\left\{e^{B^{*\prime}(x)} - (1 + B^{*\prime}(x))\right\}. \tag{4.6}$$

The recursive formula for V_p^*:

$$V_p^* = \frac{p}{p-1}\sum_{k=1}^{p-2}\binom{p-1}{k-1}B_{k+1}^*\left\{V_{p-k}^* + (p-k)B_{p-k}^*\right\}, \tag{4.7}$$

where $p \geq 3$.

For the first few terms of $V^*(x)$, we have

$$V^*(x) = \frac{15x^5}{5!} + \frac{180x^6}{6!} + \frac{3150x^7}{7!} + \frac{112560x^8}{8!} + \frac{6804378x^9}{9!} + \frac{698266800x^{10}}{10!} + \cdots$$

and the values of V_p^* are listed below for $p = 5, 6, \cdots 10$.

The number of labelled Euler graphs of order p with one cut vertex

p	5	6	7	8	9	10
V_p^*	15	180	3150	112560	6804378	698266800

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