YOKOHAMA MATHEMATICAL
JOURNAL VOL. 45, 1998

DIAGONAL FLIPS IN TRIANGULATIONS ON CLOSED SURFACES, ESTIMATING UPPER BOUNDS

By

SEIYA NEGAMI

(Received December 11, 1997)

Abstract. Negami has already shown that there is a natural number $N(F^2)$ for any closed surface F^2 such that two triangulations on F^2 with n vertices can be transformed into each other by a sequence of diagonal flips if $n \geq N(F^2)$. We shall show a cubic upper bound for $N(F^2)$ with respect to the genus g of F^2 and a quadratic upper bound for the number of diagonal flips in the sequence with respect to n.

Introduction

A triangulation G on a closed surface F^2 is a simple graph embedded on F^2 so that each face is triangular and that any two faces share at most one edge. Two triangulations G_1 and G_2 on F^2 are said to be *equivalent* (or *homeomorphic*) to each other if there is a homeomorphism $h : F^2 \to F^2$ with $h(G_1) = G_2$.

Let ac be an edge of G and let abc and adc be the two faces incident to the edge ac in G. The *diagonal flip* of ac is to replace ac with the other diagonal bd in the quadrilateral $abcd$. Flipping the diagonal ac is however forbidden when there is an edge joining b and d in G. For, if we were not subject to this rule, such a diagonal flip would result in a nonsimple graph with multiple edges between b and d. We always have to keep any triangulation simple.

Classically, Wagner [22] proved that any two triangulations on the sphere can be transformed into each other, up to equivalence, by a sequence of diagonal flips if they have the same number of vertices. Dewdney [5], Negami and Watanabe [15] have shown the same facts for the torus, the projective plane and the Klein bottle.

Such a fact does not hold as it is for other surfaces in general. However, Negami [17] has proved the following theorem for general surfaces, devising some good tricks to connect the diagonal flips to graph minors.

1991 Mathematics Subject Classification: 05C10
Key words and phrases: triangulations, diagonal flips

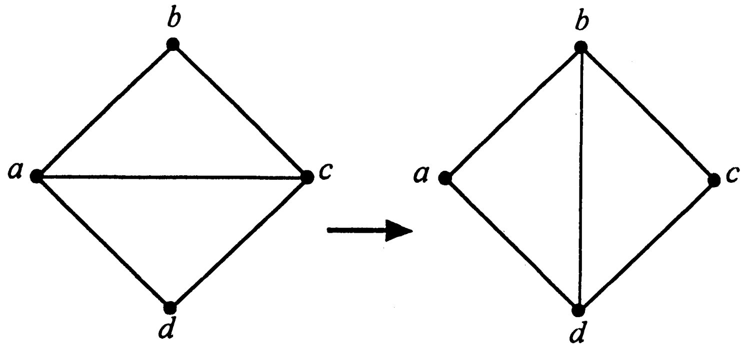

Figure 1 Diagonal flip

Theorem 1. (Negami [17]) *For any closed surface F^2, there exists a natural number $N(F^2)$ such that two triangulations G_1 and G_2 can be transformed into each other, up to equivalence, by a finite sequence of diagonal flips if $|V(G_1)| = |V(G_2)| \geq N(F^2)$.* ■

This theorem is the starting point of many recent works on diagonal flips of triangulations; [4], [6], [11], [12], [13], [14], [16]. This paper also presents related topics and answers more basic problems. Estimate an upper bound for the value of $N(F^2)$ and the length of a sequence of diagonal flips which transforms G_1 into G_2.

Hereafter, let $N(F^2)$ denote its minimum value which makes the theorem valid and $\chi(F^2)$ the Euler characteristic of a closed surface F^2. The *genus* g of an orientable closed surface F^2 is the number of its handles, or holes and $\chi(F^2) = 2 - 2g$. If F^2 is nonorientable, the genus g is the number of crosscaps and $\chi(F^2) = 2 - g$.

Theorem 2. *There is a cubic function of the genus g which gives an upper bound for $N(F^2)$. That is, $N(F^2) = O(g^3)$.*

We shall prove this theorem in Section 1, introducting some related topics; the irreducible triangulations and the crossing number of two graphs embedded on one closed surface. Our proof is based on Negami's theory presented in [17]. Recently, he has succeeded in showing that $N(F^2) = O(g)$ with a new idea [19].

Let $d(G_1, G_2)$ denote the minimum length taken over all the sequences of diagonal flips which transform G_1 into G_2. It is clear that $d(G_1, G_2)$ defines a distance over the triangulations with a fixed number of vertices, say n. For example, Komuro [6] has shown that $d(G_1, G_2) \leq 8n - 48$ for any pair of triangulations G_1 and G_2 on the sphere if $n \geq 7$. The following result presents a quadratic upper bound for $d(G_1, G_2)$, but is for triangulations on general surfaces. We shall prove it in Section 2.

Theorem 3. *Given a closed surface F^2, there are two constants α_1 and α_0, depending only on F^2, such that*

$$d(G_1, G_2) \leq 2n^2 + \alpha_1 n + \alpha_0$$

for any pair of triangulations G_1 and G_2 on F^2 with precisely n vertices.

The coefficients α_1 and α_0 are functions of the genus g of F^2, but we have known nothing about their orders with respect to g, yet. It is not difficult to construct a sequence of pairs of triangulations (G_1, G_2) such that $d(G_1, G_2)$ has a lower bound of order n, modifying the examples given in [6]. Is there a linear upper bound for $d(G_1, G_2)$ with respect to the number of vertices in general, as well as in the spherical case?

1. Bounding the number of vertices

Let G be a triangulation on a closed surface F^2. The *contraction* of an edge ac in G is to contract the edge ac after deleting bc and cd, where abc and adc are the two faces incident to ac. (See Figure 2.) If its contraction yields another triangulation on F^2, denoted by G/ac, then the edge ac is said to be *contractible* in G. It is easy to see that an edge ac is contractible in G if and only if there are precisely two cycles of length 3 containing ac, unless G is isomorphic to K_4. It is forbidden to contract any noncontractible edge since we always have to keep a triangulation simple.

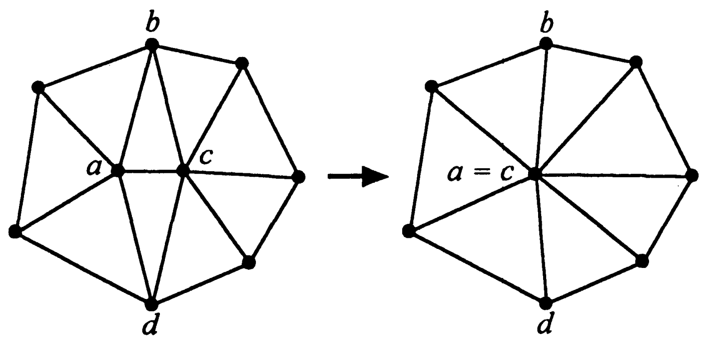

Figure 2 Contracting an edge

A triangulation G is said to be *contractible* to another trinagulation G' if G' can be obtained from G by contracting edges. In particular, if G is not contactible to any other triangulation, that is, if G has no contractible edge, then G is called an *irreducible triangulation* of the surface F^2.

The only irreducible triangulation of the sphere is the tetrahedron, that is, the unique embedding of K_4 on the sphere [21]. Barnette [2] has shown that there are precisely two irreducible triangulations of the projective plane. Lowrencenko [7] also has identified all of the 21 irreducible triangulations of the torus. Recently, Lowrencenko and Negami [8] have classified those of the Klein bottle, which are 25 in number.

In general, the number of inequivalent irreducible triangulations of each closed surface is finite, which has been shown in [3] and also in [17] as an application of Wagner's conjecture [20]. Moreover, Nakamoto and Ota [10] have given a linear upper bound for the number of their vertices with respect to the genus of a closed surface, as follows. This is one of what we need to prove Theorem 2.

Theorem 4. (Nakamoto and Ota [10]) *Every irreducible triangulation of a closed surface F^2 with Euler characteristic $\chi(F^2) \leq 1$ has at most $171(2-\chi(F^2)) - 72$ vertices.*

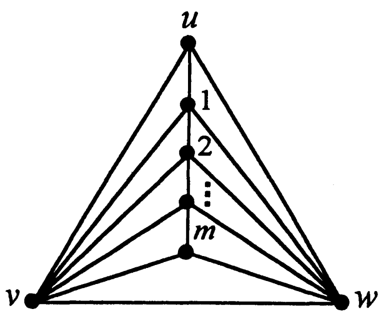

Figure 3 The standard spherical triangulation Δ_m with $m+3$ vertices

The finiteness of those irreducible triangulations works essentially in Negami's proof of Theorem 1 given in [17]. He denotes a triangulation T with one face subdivided as given in Figure 3, by $T + \Delta_m$, where m stands for the number of vertices inside the big triangle. Following his proof, we can bound the value of $N(F^2)$ by the number $N_0 = N_0(F^2)$ such that:

For any two irreducible triangulations T_i and T_j of F^2, $T_i + \Delta_{m_i}$ and $T_j + \Delta_{m_j}$ can be transformed into each other, up to equivalence, by diagonal flips if

$$|V(T_i)| + m_i = |V(T_j)| + m_j \geq N_0.$$

If there were infinitely many irreducible triangulations $T_1, T_2, \ldots$, then there would not be a finite number N_0 with the above condition.

For example, we have the following table for the sphere, the projective plane, the torus and the Klein bottle. The number $N(F^2)$ in the table coincides with the number of vertices of minimal triangulations on each of these surfaces while $N_0(F^2)$ is equal to the maximun number of vertices of their irreducible triangulations. However, these facts do not hold in general.

F^2	$N(F^2)$	$N_0(F^2)$
the sphere	4	4
the projective plane	6	7
the torus	7	10
the Klein bottle	8	11

Another notion we need is the *crossing number* of a graph embedding pair, which was introduced first by Negami [18]. Let G_1 and G_2 be two graphs embedded separately on closed surfaces F_1^2 and F_2^2, each of which is homeomorphic to a common closed surface F^2. Consider homeomorphisms $h_1 : F_1^2 \to F^2$ and $h_2 : F_2^2 \to F^2$ and count the crossing points of $h_1(G_1)$ and $h_2(G_2)$ embedded on F^2. The *crossing number* $\mathrm{cr}(G_1, G_2)$ is defined as the minimum number of crossing points when h_1 and h_2 range over those homeomorphisms such that $h_1(G_1)$ and $h_2(G_2)$ intersect each other only in their edges transversely.

Let $\beta(G)$ denote the *Betti number*, or the *cycle rank* of a graph G, which is equal to $|E(G)| - |V(G)| + 1$ if G is connected.

Theorem 5. (Negami [18]) *Let G_1 and G_2 be two graphs embedded on a closed surface F^2 of genus g, orientable or nonorientable. Then we have the following inequality:*

$$\mathrm{cr}(G_1, G_2) \leq 4g \cdot \beta(G_1) \cdot \beta(G_2)$$

Negami has defined in [18] the *diagonal crossing number* $\mathrm{cr}_\Delta(G_1, G_2)$ as the minimum number of crossing points evaluated under the following conditions.

(i) Any vertex does not lie on the interior of edges.
(ii) A pair of edges coincide fully or cross each other in a finite number of points transversely with or without common ends if they intersect.

He suggested that this is preferable, rather than $\mathrm{cr}(G_1, G_2)$, when we discuss about diagonal flips in triangulations on a closed surface. By their definition, it is clear that:

$$\mathrm{cr}_\Delta(G_1, G_2) \leq \mathrm{cr}(G_1, G_2)$$

Although the former will be smaller than the latter so much, we have never had a good bound for $\mathrm{cr}_\Delta(G_1, G_2)$ yet.

The following two lemmas also are the key facts to prove Theorem 2, which are refered as Lemmas 6 and 8 respectively in [17]. A *refinement* G of a triangulation T is a triangulation which contains a subdivision of T as its subgraph.

Lemma 6. *Let G and T be two triangulations of a closed surface F^2. If G is contractible to T, then G is equivalent to $T + \Delta_m$ with $m = |V(G)| - |V(T)|$.*

Lemma 7. *Any refinement T' of a triangulation T is contractible to T.*

Proof of Theorem 2. First, we shall prove the theorem for the orientable closed surfaces. The argument below however works for the nonorientable ones in parallel with suitable modification.

Let T_i and T_j be any two irreducible triangulations of the orientable closed surface F^2 of genus g. Embed T_i and T_j together on F^2 so that they attain their diagonal crossing number $\mathrm{cr}_\Delta(T_i, T_j)$, and construct their common refinement T_{ij}, adding new edges to $T_i \cup T_j$. The vertex set of this refinement consists of the vertices of T_i and T_j and their crossing points. Then, T_{ij} is contractible to each of T_i and T_j by Lemma 6, and is equivalent to $T_i + \Delta_{m_i}$ and to $T_j + \Delta_{m_j}$ by Lemma 7. Thus, $T_i + \Delta_{m_i}$ and $T_j + \Delta_{m_j}$ are equivalent to each other via T_{ij}. By Theorem 5, we have:

$$\begin{aligned} |V(T_{ij})| &\leq |V(T_i)| + |V(T_j)| + \mathrm{cr}_\Delta(T_i, T_j) \\ &\leq |V(T_i)| + |V(T_j)| + 4g \cdot \beta(T_i) \cdot \beta(T_j) \end{aligned}$$

From Euler's formula, it follows that:

$$\chi(F^2) = 2 - 2g, \quad \beta(T_i) = 2|V(T_i)| - 3\chi(F^2) + 1$$

Now let $V_{\max}$ be the maxmun number of vertices taken over all the irreducible triangulations of F^2 and define N_0 by:

$$\begin{aligned} N_0 &= 2V_{\max} + 4g(2V_{\max} - 3\chi(F^2) + 1)^2 \\ &= 2V_{\max} + 4g(2V_{\max} - 3(2 - 2g) + 1)^2 \\ &= 144g^3 + 48(2V_{\max} - 5)g^2 + 4(2V_{\max} - 5)^2 g + 2V_{\max} \end{aligned}$$

Since $|V(T_{ij})| \leq N_0$, we can construct a common refinement of T_i and T_j with n vertices for any $n \geq N_0$, adding new vertices to T_{ij}. Thus, the same argument as above concludes that N_0 satisfies the condition quoted in the previous.

By Theorem 4, we have the following bound for $V_{\max}$:

$$V_{\max} \leq 171(2 - \chi(F^2)) - 72 = 342g - 72$$

Assigning this bound to the above cubic function of g, we obtain an upper bound for N_0 as follows:

$$N_0 \leq 1,904,400g^3 - 822,480g^2 + 89,488g - 144$$

This is also an upper bound for $N(F^2)$.

For the nonorientable closed surfaces of genus g, we should substitute $\chi(F^2) = 2 - g$ and $V_{\max} \leq 171g - 72$ in our previous arguments to obtain a similar upper bound of N_0. Then we have:

$$N_0 \leq 484,416g^3 - 414,816g^2 + 89,146g - 144$$

Since this bound is smaller than the previous, the previous should be chosen as the cubic function of g in the theorem. ■

2. Bounding the number of diagonal flips

In this section, we shall give an upper bound for $d(G_1, G_2)$, that is, for the number of diagonal flips included in a shortest sequence which transforms G_1 into G_2. To do so, we need to give an algorithm to transform G_1 into G_2, guaranteed by Theorem 1.

Negami's proof of Theorem 1 in [17] suggests the following algorithm. Its consistency and details of each step will be shown later in our proof of Theorem 3.

Let G_1 and G_2 be two triangulations on a closed surface F^2 with n vertices. Let N_0 denote the same number as given in the previous section and suppose that $n \geq N_0$.

STEP 1. Let T_1 be a triangulation on F^2 with precisely N_0 vertices which is obtained from G_1 by contracting edges. Translate the sequence of edge contractions from G_1 to T_1 into that of diagonal flips from G_1 through triangulations with n vertices. (We call the latter "the sequence of diagonal flips corresponding to edge contractions".) Let G_1' be the triangulation with n vertices obtained from G_1 by applying this sequence.

STEP 2. Define T_2 and G_2' for G_2 similarly to the above. Since both T_1 and T_2 have N_0 vertices, they can be transformed into each other by diagonal flips. Translate such a sequence of diagonal flips from T_1 to T_2 into that starting from G_1'. Let G_2'' be the triangulation obtained from G_1' by applying this sequence. In general, G_2'' does not coincide with G_2'.

STEP 3. Each of G_2'' and G_2' includes a subgraph, equivalent to T_2, not subdivided. Transform G_2'' into $T_2 + \Delta_m$ with $m = n - N_0$, moving the vertices not belonging to T_2 into one face of T_2.

STEP 4. Transform $T_2 + \Delta_m$ into G_2' by the inverse of a similar sequence of diagonal flips from G_2'.

STEP 5. Transform G_2' into G_2 by the inverse of a sequence of diagonal flips corresponding to edge contractions from G_2 to T_2.

To give an upper bound for $d(G_1, G_2)$, we shall evaluate the number of diagonal flips in the sequence which this algorithm generates.

The following lemma has been proved in [6] and is rephrased for our purpose. (We don't need it if we don't care about the coefficient of n^2 to be 2.)

Lemma 8. (Komuro [6]) *Let G be a planar triangulation bounded by a peripheral triangle uvw and suppose that there are m vertices inside uvw. Then:*

$$d(G, \Delta_m) \leq 4m + 8 - (3\deg v + \deg w)$$

More precisely speaking, there is a sequence of diagonal flips of such length which transforms G into Δ_m, fixing uvw, so that $\deg u = 3$ and $\deg v = \deg w = m + 2$ afterwards.

Proof of Theorem 3. To describe the above algorithm in more details, we shall look at a partial structure in a triangulation. Let G and T be two triangulations on a closed surface F^2 and suppose that G includes T as its subgraph. Then each vertex of G lies inside a face of T if it does not belong to T and each face of T is subdivided as like a planar triangulation. The subgraph induced by the edges lying in each face of T is often called a *bridge* in G for T.

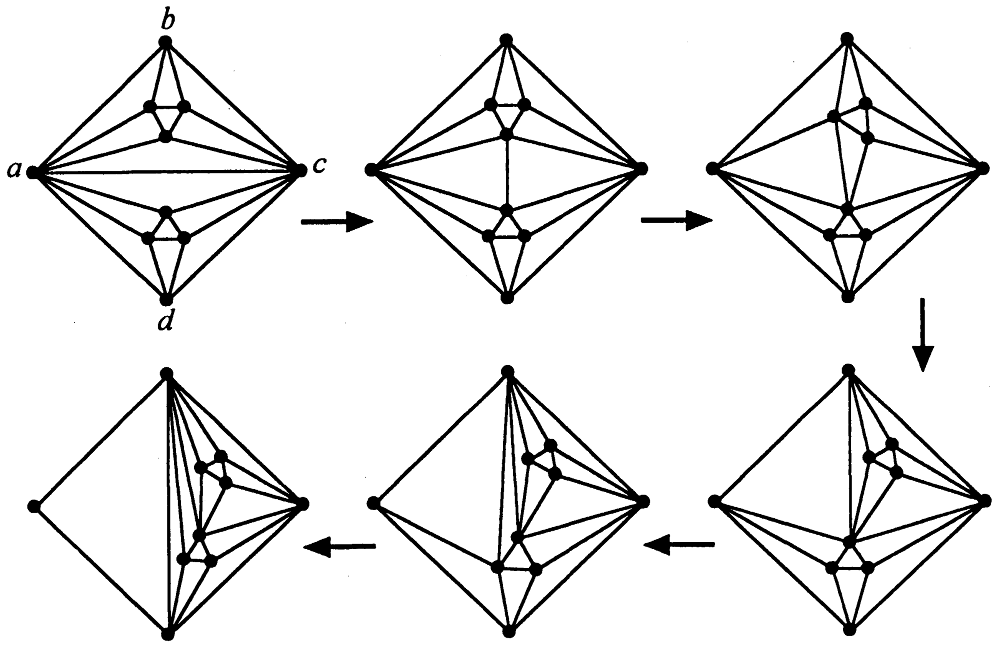

Figure 4 Translating a diagonal flip in T for G

Figure 4 presents a trick to translate a diagonal flip in T into a sequence of diagonal flips in G. We call this *Trick A*. This sequence consists of flipping edges which are incident to the vertex a inside the quadrilateral $abcd$. To simplify figures below, we may omit the vertices inside each face of T and draw only diagonal flips in T. In such a case, we should expand each diagonal flip in the figures by Trick A.

Let G_1 and G_2 be two triangulations on F^2 with n vertices and suppose that $n \geq N_0$. Since the number of vertices of any irreducible triangulation does not exceed N_0, G_i is contractible to a triangulation with precisely N_0 vertices. Thus, we can choose it actually as T_i in the above-mentioned algorithm.

STEP 1. Let G be a triangulation on a closed surface F^2 with n vertices and T another triangulation included in G as a subgraph. Let uv be a contractible edge of T. The contraction of uv in T can be realized as flipping $\deg_T u - 3$ edges incident to u and removing u which has degree 3 after flipping, as shown in Figure 5. Expand the diagonal flips of these edges by Trick A. The resulting sequence is "a sequence corresponding to an edge contraction" and consists of at most $\deg_G u - 3$ diagonal flips in G, transforming T into T/uv. The vertex u of degree 3 will be unifiyed with a bridge in G to be one bridge in the new triangulation which includes T/uv as its subgraph. Since $\deg_G u \leq n - 1$, the number of diagonal flips in the seuquence does not exceed $n - 4$.

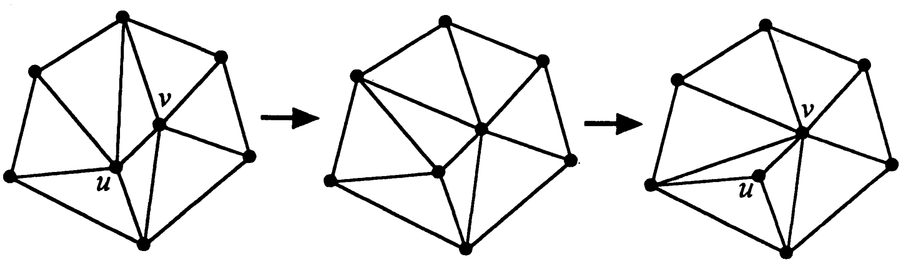

Figure 5 Diagonal flips corresponding to an edge contraction

Apply this argument to the transformation of G_1 into G'_1 in Step 1. At the initial stage, we set $G = T = G_1$ and carry out the sequences of diagonal flips corresponding to edge contractions from G_1 to T_1 in order. Then we have $G = G'_1$ and $T = T_1$ at the final stage. The number of diagonal flips in the total sequence from G_1 to G'_1 is bounded by:

$$(n - N_0)(n - 4) = n^2 - (N_0 + 4)n + 4N_0$$

STEP 2. Now G'_1 includes T_1 as its subgraph and T_1 can be transformed into T_2 by a sequence of diagonal flips since $N_0 \geq N(F^2)$. Expand this sequence from T_1 to T_2 into that in G'_1 by Trick A. Since each diagonal flip in the former corresponds to at most $n - 4$ diagonal flips in the latter, the total length of the sequence from G'_1 to G''_2 is bounded by $(n - 4)d(T_1, T_2)$.

Let $d_0 = d_0(F^2)$ denote the maximum value of $d(T', T'')$ taken over all the pair of triangulations T' and T'' with precisely N_0 vertices. Since there are only a finite number of such triangulations, this maximum value d_0 exists actually as a finite constant, depending only on F^2. Thus, the number of diagonal flips we need in Step 2 is bounded by:

$$(n-4)d_0 = d_0 n - 4d_0$$

STEP 3. Now we transform G_2'' into G_2', preserving the subgraph T_2 in them. Our basic trick is shown in Figure 6 and it carries a bridge in a face of T_2 into a neighboring face. This trick flips the diagonal ac and the edges in the bridge incident to a or b. The number of those edges does not exceed $2m$ if we denote the number of vertices in the bridge by m. Since $m \leq n-4$, this trick consists of at most $2(n-4)+1 = 2n-7$ diagonal flips.

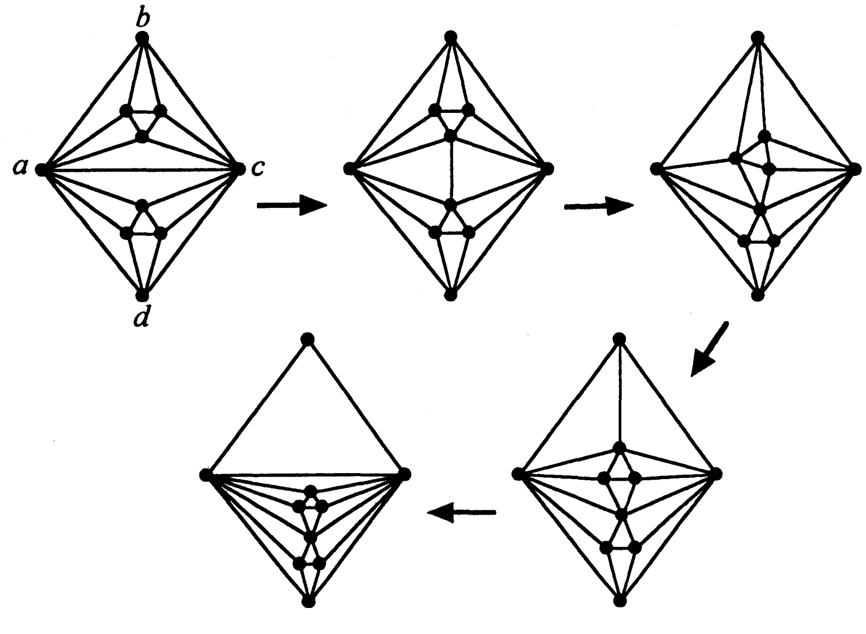

Figure 6 Moving bridges

Let F_0 be the number of faces of T_2, which is equal to $2(N_0 - \chi(F^2))$. Choose one face of T_2, say $A = uvw$, and move all the bridges in G_2'' into the face A. To do this, consider a spanning tree in the dual of T_2 with A as its root. This spanning tree has precisely F_0 vertices and we can define the directions of its edges uniquely so that they induce a path from each vertex to the root A. We move the bridges of G_2'' into the face A by the above trick along such a path system, starting at the ends of the spanning tree.

In this process, we carry out the trick at most $F_0 - 1$ times and bridges are unified together in order since there are precisely $F_0 - 1$ faces of T_2 except the root A. Thus, the number of diagonal flips in such a sequence is bounded by:

$$(2n-7)(F_0-1) = 2(F_0-1)n - 7(F_0-1)$$

We need at most $4m + 8 - (3\deg_B v + \deg_B w)$ diagonal flips, in addition, to transform the inside of A into Δ_m if $m = n - N_0 \geq 2$. In this case, we may assume that $\deg_B v \geq 4$ and $\deg_B w \geq 3$, and hence the number of additional diagonal flips does not exceed $4m - 7 = 4(n - N_0) - 7$. Adding this to the previous, we have the following upper bound for the number of diagonal flips we need in Step 3:

$$(2F_0+2)n - 7F_0 - 4N_0$$

STEPS 4 AND 5. In these cases, we have the same upper bounds as in Steps 3 and 1, respectively.

Sum up the upper bounds given in all of the steps. Then we conclude that:

$$d(G_1, G_2) \leq 2n^2 + (d_0 - 2(N_0+4) + 2(2F_0+2))\,n - 4d_0 - 14F_0$$

Substituting $F_0 = 2(N_0 - \chi(F^2))$ to the above, we obtain the coefficients α_1 and α_2 in the theorem, as follows:

$$\alpha_1 = d_0 + 6N_0 - 8\chi(F^2) - 4, \quad \alpha_2 = -4d_0 - 28\left(N_0 - \chi(F^2)\right) < 0$$

It is clear that they depend only on the surface F^2. ■

The only unknown quantity in the above expressions for α_1 and α_2 is d_0, or the maximum distance over the triangulations with precisely N_0 vertices. This is bounded by the number of inequivalent such triangulations, but the latter will be so big.

References

[1] J.L. Arocha, J. Bracho and V. Neumann-Lara, Tight and untight triangulated surfaces by complete graphs, *J. Combin. Theory, Ser. B* **63** (1995), 185–199.

[2] D.W. Barnette, Generating the triangulations of the projective plane, *J. Combin. Theory Ser. B* **33** (1982), 222–230.

[3] D.W. Barnette and A.L. Edelson, All 2-manifolds have finitely many minimal triangulations, *Isr. J. Math.* **67** (1989), 123–128.

[4] R. Brunet, A. Nakamoto and S. Negami, Diagonal flips of triangulations on closed surfaces preserving specified properties, *J. Combin. Theory, Ser. B* **68** (1996), 295–309.

[5] A.K. Dewdney, Wagner's theorem for the torus graphs, *Discrete Math.* **4** (1973), 139–149.

[6] H. Komuro, The diagonal flips of triangulations on the sphere, *Yokohama Math. J.* **44** (1997), 115–122.

[7] S. Lawrencenko, The irreducible triangulations of the torus, *Ukrain. Geom. Sb.* **30** (1987), 52–62. [In Russian; MR 89c:57002; English translation: J. Soviet Math. 51, No. 5 (1990), 2537–2543.]

[8] S. Lawrencenko and S. Negami, Irreducible triangulations of the Klein bottle, *J. Combin. Theory, Ser. B* **70** (1997), 265–291.

[9] S. Lawrencenko, S. Negami and A.T. White, Three nonisomorphic triangulations of an orientable surface with the same complete graph, *Discrete Math.* **135** (1994), 367–369.

[10] A. Nakamoto and K. Ota, Note on Irreducible triangulations of surfaces, *J. Graph Theory* **20** (1995), 227–233.

[11] A. Nakamoto, "Triangulations and quadrangulations of surfaces", Doctoral thesis, Keio University, 1996.

[12] A. Nakamoto, Diagonal transformations in quadrangulations of surfaces, *J. Graph Theory* **21** (1996), 289–299.

[13] A. Nakamoto, Diagonal transformations and cycle parities of quadrangulations on surfaces, *J. Combin. Theory, Ser. B* **67** (1996), 202–211.

[14] A. Nakamoto and K. Ota, Diagonal transformations of graphs and Dehn twists of surfaces, *J. Combin. Theory, Ser. B* **70** (1997), 292–300.

[15] S. Negami and S. Watanabe, Diagonal transformations of triangulations on surfaces, *Tsukuba J. Math.* **14** (1990), 155–166.

[16] S. Negami and A. Nakamoto, Diagonal transformations of graphs on closed surfaces, *Sci. Rep. Yokohama Nat. Univ., Sec. I* **40** (1993), 71–97.

[17] S. Negami, Diagonal flips in triangulations of surfaces, *Discrete Math* **135** (1994), 225–232.

[18] S. Negami, Crossing numbers of graph embedding pairs on closed surfaces, preprint.

[19] S. Negami, Diagonal flips in pseudo-triangulations on closed surfaces, preprint.

[20] N. Robertson and P. Seymour, Graphs Minors XVI. Wagner's conjecture, preprint.

[21] E. Steinitz and H. Rademacher, "Vorlesungen üder die Theorie der Polyeder", Springer, Berlin, 1934.

[22] K. Wagner, Bemekungen zum Vierfarbenproblem, *J. der Deut. Math.* Ver. 46, Abt. 1, (1936), 26–32.

Department of Mathematics,
Faculty of Education and Human Sciences,
Yokohama National University,
79-2 Tokiwadai, Hodogaya-Ku,
Yokohama 240-8501, Japan.
Email: negami@ms.ed.ynu.ac.jp