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A MINIMAL RIGID GRAPH IN THE PLANE THAT IS NOT CONSTRUCTIBLE BY RULER AND COMPASS

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Abstract. We present a minimal rigid graph in the plane whose vertex set cannot be constructed by ruler and compass from the data on the distances between adjacent vertices.

We consider those graphs whose vertices are points in the plane. A graph is called *flexible* if it admits a continuous deformation, that is, if its vertices can be continuously moved in the plane in such a way that the distances between adjacent vertices are unchanged, but at least a pair of nonadjacent vertices change their mutual distance. If a graph is not flexible then it is called *rigid*.

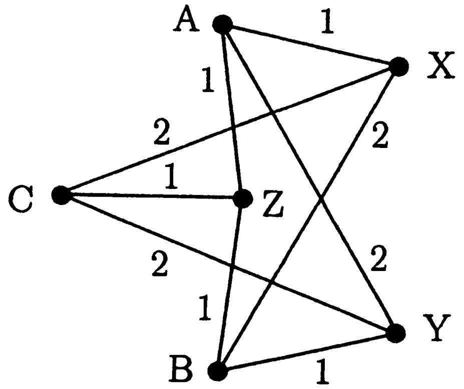

Figure 1 A minimal non-constructible rigid graph

A rigid graph is called *constructible* if a congruent copy of its vertex set can be constructed by ruler and compass constructions, given the graph-structure and the data on the distances between adjacent vertices. A non-constructible, rigid graph of order 8 was given in [2]. It was then asked if there is a non-constructible rigid graph of lesser order.

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Let G be the graph shown in Figure 1. We prove that G is a minimal rigid graph that is not constructible.

(1) It is known [1] that a complete bipartite graph $K_{3,3}$ is rigid unless its six vertices lie on a conic. Hence G is rigid.

(2) Put $C = (0,0)$, $Z = (1,0)$, $A = (x,y)$, $X = (u,v)$, $B = (x,-y)$, $Y = (u,-v)$. Then, from the data on the edge-lengths, we have the following simultaneous equation

$$\begin{aligned}(x-1)^2+y^2&=1\\ u^2+v^2&=4\\ (x-u)^2+(y-v)^2&=1\\ (x-u)^2+(y+v)^2&=4,\end{aligned}$$

which has two real solutions

$$(u,v,x,y)\approx(1.41,\ 1.42,\ 1.85,\ 0.53),\ (1.85,\ 0.76,\ 0.88,\ 0.99).$$

Hence G indeed exists. From the above simultaneous equation, we have

$$8x^3-20x^2+8x+3=0.$$

Since the left-hand-side cubic polynomial is irreducible in $\mathbf{Z}[x]$, it is irreducible in $\mathbf{Q}[x]$. Hence none of its root is algebraic of degree a power of 2 over the field $\mathbf{Q}$ of rationals. Therefore $\overline{AC}$ is not constructible (see, e.g. [3]), and hence G is not constructible.

(3) Let H be a rigid graph of order 5. Then H is 2-connected. Let γ denote the order of minimal cycle in H. If $\gamma = 5$, then H is a 5-cycle, and to be a rigid graph, all vertices of H must lie on a line in such a way that the two farthest vertices are adjacent. In this case H is clearly constructible. If $\gamma = 4$, then H is a complete bipartite graph $K_{2,3}$, and to be rigid, one of its 4-cycle must lies on a line. Hence H is also constructible in this case. Suppose now H has 3-cycle $p_1p_2p_3$, and let p_4, p_5 be the remaining vertices. If one of p_4, p_5 is adjacent to at least two of p_1, p_2, p_3, then H is clearly constructible. If p_4, p_5 are adjacent to each other and $\deg p_4 = \deg p_5 = 2$, then p_4, p_5 are contained in a chordless 4-cycle, and this 4-cycle must lie on a line. Hence H is constructible. Since any non-constructibe rigid graph of order n can be extended, by adding a vertex of degree two, to a non-constructible rigid graph of order $n+1$, it follows that every rigid graph of order ≤ 5 is constructible.

References

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