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ON FORMAL EXTENSIONS OF NEAR-FIELDS

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Abstract. In this note, the notion of a formal limit "Lim" is introduced. A nonprincipal ultrafilter associated with "Lim" is constructed. A characterization of ultrapower extensions of near-fields in terms of formal limits is given. In fact, we show that a near-field Ω is an ultrapower extension of a near-field K if and only if Ω is a formal extension of K.

1. Formal Limits

Let K be a near-field and I an infinite set. An I-sequence $\langle a_i : i \in I\rangle$ of elements of K is a member of the set K^I. These sequences form a (right) near-ring with identity, where the sums and products are defined pointwise,

$$(\text{i.e. } \langle a_i\rangle + \langle b_i\rangle = \langle a_i + b_i\rangle \text{ and } \langle a_i\rangle\langle b_i\rangle = \langle a_i b_i\rangle).$$

For the theory of near-rings and near-fields we refer to [3].

Definition 1.1. Let Ω be a near-field, K be a subnear-field of Ω and I be an infinite set. An Ω valued function "Lim" with domain K^I will be called a formal limit of the I-sequences of elements of K if "Lim" satisties the following properties:

(a) $\mathrm{Lim}\ \langle a_i + b_i\rangle = \mathrm{Lim}\ \langle a_i\rangle + \mathrm{Lim}\ \langle b_i\rangle$.
(b) $\mathrm{Lim}\ \langle a_i b_i\rangle = \mathrm{Lim}\ \langle a_i\rangle \mathrm{Lim}\ \langle b_i\rangle$.
(c) If $a_i = a$ for every $i \in I$, $\mathrm{Lim}\ \langle a_i\rangle = a$.
(d) If $a_i = 0$ for almost all $i \in I$, $\mathrm{Lim}\ \langle a_i\rangle = 0$.

Definition 1.2. An element $b \in \Omega$ is called an approachable element if there exists an I-sequence $\langle b_i\rangle$ of elements of K such that $b = \mathrm{Lim}\ \langle b_i\rangle$.

Definition 1.3. Let K be a subnear-field of a near-filed Ω. The extension Ω/K is said to be formal if there exist an infinite set I and an Ω-valued formal

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limit "Lim" of the I-sequences of elements of K, such that every element of Ω is approachable.

The concept of an ultrapower extension of a "model" is defined in [1]. Let us recall a near-field Ω is an ultrapower extension of a near-field K if $K \subset \Omega$ and for some ultrafilter D, the natural embedding $d : K \to K^I/D$ can be extended to an isomorphism $f : \Omega \cong K^I/D$. For the ultraproducts of near-rings see [2].

Suppose Ω is an ultrapower extension of K. We can define an Ω-valued function "Lim" on K^I by: Lim $\langle x_i \rangle = f^{-1} \circ h\langle x_i \rangle$, where $h : K^I \to K^I/D$, $h\overline{\langle x_i \rangle} = \langle x_i \rangle$, is the canonical projection. Obviously "Lim" satisfies the first two properties of a formal limit. Let $\langle x_i \rangle$ be a constant sequence with value x_i. Then

$$\begin{aligned} \text{Lim } \langle x_i \rangle &= f^{-1} \circ h\langle x_i \rangle = f^{-1}(\overline{\langle \ldots, x, \ldots \rangle}) \\ &= f^{-1}(d(x)) = x, \quad \text{since } f \text{ extends } d. \end{aligned}$$

Now if $\langle x_i \rangle$ is a sequence such that $x_i = 0$ for almost all $i \in I$, then Lim $\langle x_i \rangle = 0$, since $h\langle x_i \rangle = 0$. Finally since h and f^{-1} are surjective, every element of Ω is approachable, i.e. Ω is a formal extension of K. We sum up the result as:

Proposition 1.4. *Every ultrapower extension of a near-field is a formal extension.*

2. Ultrafilters

In this section, we associate with each formal limit of the I-sequences of elements of K, a nonprincipal ultrafilter U on I. Now, for each $J \subseteq I$,let χ_J be the characteristic function of J, i.e. $\chi_J(i) = 1$ for $i \in J$ and $\chi_J(i) = 0$ for $i \in (I - J)$.

Proposition 2.1. *Let U be the set of all subsets J of I such that* Lim $\chi_J = 1$. *Then U is a nonprincipal ultrafilter on I.*

Proof. Let $A \subseteq I$. Observe that for each $i \in I$, $\chi_A(i)\chi_A(i) = \chi_A(i)$. Hence $(\text{Lim } \chi_A)^2 = \text{Lim } \chi_A$. This means that for every subset A of I, we have either Lim $\chi_A = 1$ or Lim $\chi_A = 0$. To show that U is a filter on I, let $A \in U$ and $B \in U$.

Since $\chi_{A\cap B}(i) = \chi_A(i)\chi_B(i)$, for all $i \in I$, then Lim $\chi_{A\cap B}$ = Lim χ_A Lim $\chi_B = 1$. Hence $A \cap B \in U$. Suppose $A \in U$ and $A \subset M$. It follows from $\chi_A(i) = \chi_A(i)\chi_M(i)$, for all $i \in I$, that Lim χ_A = Lim χ_ALim χ_M. Thus Lim $\chi_M = 1$ and so $M \in U$.

Obviously for the empty set $\emptyset$, Lim $\chi_\emptyset = 0$, hence $\emptyset \notin U$ and U is a proper filter. Now, U is an ultrafilter on I, for if $H \subseteq I$ with Lim $\chi_H = 1$, then $H \in U$.

Otherwise Lim $\chi_H = 0$. In this case $\lim \chi_{I-H} = \lim \chi_I - \lim \chi_H = 1 - 0 = 1$, which means that $(I - H) \in U$. Finally, since for each $j \in J$, Lim $\chi_{\{j\}} = 0$, the ultrafilter U is nonprincipal. □

Proposition 2.2. *Let* $\langle a_i \rangle \in K^I$ *and* $A = \{i \in I/a_i = 0\}$. *Then* $A \in U$ *if and only if* Lim $\langle a_i \rangle = 0$.

Proof. Suppose $A \in U$ (i.e. Lim $\chi_A = 1$). Note that for each $i \in I$, $\chi_A(i)a_i = 0$. Therefore, Lim χ_ALim $\langle a_i \rangle = 0$ and hence Lim $\langle a_i \rangle = 0$. Conversely, let $\langle a_i \rangle \in K^I$ with Lim $\langle a_i \rangle = 0$. We define an I-sequence $\langle b_i \rangle$ by: $b_i = 0$ for $i \in A$ and $b_i = a_i^{-1}$ for $i \in (I - A)$. Obviously, $a_i b_i = 0$ for $i \in A$ and $a_i b_i = 1$ for $i \notin A$. This means the $\chi_{(I-A)} = \langle a_i b_i \rangle$, and hence

$$\text{Lim } \chi_{(I-A)} = \text{Lim } \langle a_i \rangle \text{Lim } \langle b_i \rangle = 0.$$

We conclude that $(I - A) \notin U$ and consequently $A \in U$. □

We will show that the ultrafilter U associated with "Lim" entirely determines the extension.

Proposition 2.3. *Every formal extension of a near-field* K *is an ultrapower extension of* K.

Proof. Suppose Ω/K is a formal extension. Then there exist an infinite set I and an Ω-valued formal limit Lim: $K^I \to \Omega$, such that "Lim" is a near-ring epimorphism. Let U be the non principal ultrafilter associated with "Lim" and consider the natural projection $h : K^I \to K^I/U$. Observe that if $\langle x_i \rangle \in$ Kerh, then $\{i \in I/x_i = 0\} \in U$ and it follows from proposition 2.2, that Lim $\langle x_i \rangle = 0$. Hence Kerh $\subseteq$ ker Lim, and there exists a unique epimorphism f from K^I/U onto Ω such that $foh =$ Lim . Science K^I/U is a near-field, f is an isomorphism. Now for every $a \in K$,

$$\begin{aligned} f^{-1}(a) &= f^{-1}(\text{Lim } \langle \dots, a, \dots \rangle) \\ &= (f^{-1} \circ \text{Lim })(\langle \dots, a, \dots \rangle) \\ &= h(\langle \dots, a, \dots \rangle) = \overline{\langle \dots, a, \dots \rangle} = d(a). \end{aligned}$$

Thus the isomorphism $f^{-1} : \Omega \cong K^I/U$, extends the natural embedding $d : K \to K^I/U$, and so Ω is an ultrapower extension of K. □

Corollary 2.4. *A field* Ω *is an ultrapower extension of a field* K *if and only if* Ω/K *is a formal extension.*

References

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